MATH 2551, Fall 2018
Practice Exam 2, Solutions

Problem 1. Calculations.

(a) Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Solution:

$$\nabla f = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (y + x)\mathbf{k},$$

$\nabla f(1, -1, 1) = 2\mathbf{j}$. $\mathbf{u} = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, so

$$f'_{\mathbf{u}}(1, -1, 1) = \nabla f(1, -1, 1) \bullet \mathbf{u} = \frac{2}{3}\sqrt{6}.$$

(b) Find the rate of change of $f(x, y) = xe^y + ye^{-x}$ along the curve $\mathbf{r}(t) = (lnt)\mathbf{i} + t(lnt)\mathbf{j}$.

Solution:

$$\nabla f = (e^y - ye^{-x})\mathbf{i} + (xe^y + e^{-x})\mathbf{j},$$

$$\nabla f(\mathbf{r}(t)) = (t^t - lnt)\mathbf{i} + (t^t lnt + \frac{1}{t})\mathbf{j},$$

$$\frac{df}{dt} = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = t^t(\frac{1}{t} + lnt + (lnt)^2) + \frac{1}{t}.$$

(c) Find $\frac{\partial u}{\partial s}$ for $u = x^2 - xy$, $x = scost$, $y = tsins$.

Solution:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x-y)(cost)+(-x)(tcoss) = 2scos^2t-t\ sins\ cost-st\ coss\ cost.$$

(d) Find $\frac{dy}{dx}$ if $xcos(xy) + ycos(x) = 2$.

Solution: Set $u = xcos(xy) + ycos(x) - 2$,

$$\frac{\partial u}{\partial x} = cos(xy) - xysin(xy) - ysin(x).$$

$$\frac{\partial u}{\partial y} = -x^2 sin(xy) + cos(x).$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{cos(xy) - xysin(xy) - ysin(x)}{x^2 sin(xy) - cos(x)}.$$

(e) Is $\mathbf{F}(x, y) = (x + siny)\mathbf{i} + (xcosy - 2y)\mathbf{j}$ a gradient of a function $f(x, y)$? If yes, find the general form of $f(x, y)$.

Solution: Set $P = x + siny$, $Q = xcosy - 2y$. $\frac{\partial P}{\partial y} = cosy = \frac{\partial Q}{\partial x} = cosy.$

Thus, $\mathbf{F}$ is a gradient of a function. For $f(x, y)$, we have from $\frac{\partial f}{\partial x} = P$ that

$$f(x, y) = \frac{1}{2}x^2 + xsin(y) + g(y).$$

To determine $g(y)$, we have

$$Q = \frac{\partial f}{\partial y} = xcos(y) + g'(y),$$

which implies that $g'(y) = -2y$, thus $g(y) = -y^2 + C$, with C a constant. So, $f(x, y) = \frac{1}{2}x^2 + xsin(y) - y^2 + C$.

(f) Set $f(x, y) = \frac{x^2 - y^4}{x^3 - y^4}$. Determine whether or not f has a limit at $(1, 1)$.

solution: Along $x = 1$, the limit is 1, while along $y = 1$, the limit is 2/3. So it has no limit at $(1, 1)$.

Problem 2 Consider the function $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

(a) Find the equation for the tangent plane to the level surface $f = 4$ at the point $P(1, 4, 1)$.

Solution:

$$\nabla f = \frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2\sqrt{y}}\mathbf{j} + \frac{1}{2\sqrt{z}}\mathbf{k},$$

$$\nabla f(1, 4, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}.$$

Tangent plane: $\frac{1}{2}(x - 1) + \frac{1}{4}(y - 4) + \frac{1}{2}(z - 1) = 0$.

(b) Find the equation for the normal line to $f = 4$ at $P(1, 4, 1)$.

Solution: The normal line: $x = 1 + \frac{1}{2}t$, $y = 4 + \frac{1}{4}t$, $z = 1 + \frac{1}{2}t$.

(c) Use differentials to estimate $f(0.9, 4.1, 1.1)$.

Solution: $f(0.9, 4.1, 1.1)\tilde{-}f(1, 4, 1) + df$.

$$df = \frac{1}{2} \times (-0.1) + \frac{1}{4} \times 0.1 + \frac{1}{2} \times 0.1 = 0.025.$$

Thus, the estimate is 4.025.

Problem 3. Find the area of the largest rectangle with edges parrallel to the coordinate axes that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Solution: Use Lagrange multiplies method. Set the coordiates of the coner points of the rectangle to be (x, y), $(-x, y)$, $(-x, -y)$, $(x, -y)$. We need to

maximize $f(x, y) = 4xy$ with the side condition $g(x, y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$. $\nabla f = 4y\mathbf{i} + 4x\mathbf{j}$, $\nabla g = \frac{2}{9}x\mathbf{i} + \frac{1}{2}y\mathbf{j}$. Solve the following system:

$$\begin{cases} 4y = \lambda\frac{2}{9}x \\ 4x = \lambda\frac{1}{2}y \\ g(x, y) = 0. \end{cases}$$

We have

$\lambda = 12$, $x = \frac{3}{2}\sqrt{2}$, $y = \sqrt{2}$, and the area is 12.

Problem 4 Find the absolute extreme values taken on $f(x, y) = \frac{-2y}{x^2+y^2+1}$ on

the set $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

Solution: $\nabla f = \frac{4xy}{(x^2+y^2+1)^2}\mathbf{i} + \frac{2y^2-2x^2-2}{(x^2+y^2+1)^2}\mathbf{j} = \mathbf{0}$ at $P_1 = (0,1)$ and $P_2 = (0,-1)$ in D.

Next we consider the boundary of D. We parametrize the circle by

$$C : \mathbf{r}(t) = 2cos(t)\mathbf{i} + 2sin(t)\mathbf{j},\ t \in [0, 2\pi].$$

The values of f on the boundary are given by the function:

$$F(t) = f(\mathbf{r}(t)) = -\frac{4}{5}sin(t),\ t \in [0, 2\pi].$$

$F'(t) = -\frac{4}{5}cos(t) = 0$ at $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$. Thus the critical points on boundary of D are $P_3 = \mathbf{r}(0) = \mathbf{r}(2\pi) = (2,0)$, $P_4 = \mathbf{r}(\frac{1}{2}\pi) = (0,2)$, and $P_5 = \mathbf{r}(\frac{3}{2}\pi) = (0,-2)$. Evaluate f at all critical points:

$$f(0,1) = -1,\ f(0,-1) = 1,\ f(2,0) = 0,$$

$$f(0,2) = -\frac{4}{5},\ f(0,-2) = \frac{4}{5}.$$

So, f takes on its absolute maximum of 1 at $(0,-1)$ and its absolute minimum of -1 at $(0,1)$.