Stability of Contact Discontinuity for Jin-Xin Relaxation System

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Abstract: In this paper, we first construct a contact wave for 1-dimensional Jin-Xin relaxation system [15]. This wave serves as the relaxation version of contact discontinuity of the corresponding hyperbolic system at equilibrium. Such a contact wave is shown to be nonlinearly stable under small initial perturbation. The time-decay rate is also obtained by weighted energy estimates.

1 Introduction

The relaxation phenomena arise in many physical situations, such as the kinetic theory, non-equilibrium gas dynamics, elasticity with memory, flood flow with friction and magnetohydrodynamics etc; see [29]. Mathematically, the investigation of the behavior of the solutions to the relaxation system is an important subject. It also motivates effective numerical schemes for the systems of nonlinear PDEs. A good survey in this direction is [26].

In this paper, we consider the initial value problem of 1-dimensional Jin-

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Xin relaxation system [15] which reads

$$\begin{cases} u_t + v_x = 0, \\ v_t + a^2 u_x = \dfrac{1}{\varepsilon}(f(u) - v), \quad x \in \mathbb{R}^1, t \geq 0, \\ (v, u)(x, t = 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}^1, \end{cases} \tag{1.1}$$

where $u = u(x,t), v = v(x,t)$ are vector-valued functions in $\mathbb{R}^n$, $f(u)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$, $a > 0$ is a given constant satisfying the sub-characteristic condition (1.5) below, and $\varepsilon > 0$ represents the relaxation time.

Assume that the initial data satisfies

$$(v_0(x), u_0(x)) \to (v_\pm, u_\pm), \qquad \text{as } x \to \pm\infty, \tag{1.2}$$

where $v_\pm, u_\pm$ are given constants subject to the constraints $v_\pm = f(u_\pm)$.

As $\varepsilon \to 0$, formally, the leading order approximation of the system (1.1) is the following conservation laws:

$$u_t + f(u)_x = 0. \tag{1.3}$$

The relaxation system (1.1) is designed by Jin-Xin in [15] to approximate the conservation laws (1.3) for numerical purpose. The main advantage of this scheme is its generality and simplicity since (1.1) is semi-linear.

As usual, we make the following assumptions throughout this paper.

(H): *Assume that the system (1.3) is strictly hyperbolic. Namely, the Jacobian matrix $Df(u)$ of the flux $f(u)$ has real and distinct eigenvalues*

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

with corresponding left and right eigenvectors $l_j(u), r_j(u)$ $(j = 1, 2, \cdots, n)$ satisfying

$$\begin{aligned} &L(u)Df(u)R(u) = diag(\lambda_1(u), \lambda_2(u), \cdots, \lambda_n(u)) \equiv \Lambda(u), \\ &L(u)R(u) = Id., \end{aligned} \tag{1.4}$$

where $L(u) = (l_1(u), \cdots, l_n(u))^t$, $R(u) = (r_1(u), \cdots, r_n(u))$, $Id.$ = Identity matrix. We further assume that each $i-$field is either **genuinely nonlinear**

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0,$$

or **linearly degenerate**

$$\nabla \lambda_i(u) \cdot r_i(u) \equiv 0.$$

Under the above assumptions, it is well-known that the hyperbolic conservation laws (1.3) has rich wave phenomena. In the genuinely nonlinear field, the nonlinear waves (shock waves or rarefaction waves) may appear; and contact discontinuities, which are the linear wave, may occur in the linearly degenerate field.

To ensure the dissipative nature of the system (1.1), it is important to impose the sub-characteristic condition ([1], [2], [15], [18], [29]):

$$-a < \lambda_i(u) < a, \qquad \forall\ u, \forall\ i = 1, 2, \cdots, n. \tag{1.5}$$

Although the dissipation of relaxation is not strong enough to smooth out discontinuities, it does prevent the singularity formation from small smooth initial data. The elementary hyperbolic waves of (1.3), i.e., shock waves, rarefaction waves and contact discontinuities, have smooth correspondences in the system (1.1). It is interesting to investigate the asymptotic stability of these waves (relaxation versions of the hyperbolic waves) and the relations to their correspondences of (1.3).

Liu [18] first considered a general 2×2 relaxation system in one spatial dimension, and gave the stability criteria for the shock waves, rarefaction waves and diffusion waves. Since then, many authors have stuided the stability of shock waves and rarefaction waves to the relaxation system in one or several space dimensions; see [3], [4], [5], [6], [7], [8], [16], [17], [21], [22], [23], [24], [25], [27], [31], [32], [33], [34] etc. However, as far as we know, there is no result corresponding to contact discontinuities for the relaxation system (1.1). We will pursue this issue in current paper.

The investigation of the asymptotic stability of contact discontinuity for the viscous conservation laws dates back to Xin [30], which concerned with the Euler system with uniform viscosity. It was first discovered in [30] that the inviscid contact discontinuity can not be an asymptotic state for the viscous system, but a viscous contact wave, which approximates the contact discontinuity on any finite time interval as the viscosity tends to zero, is nonlinearly stable. This phenomenon is called meta-stability [30].

In this paper, we study the meta-stability of contact discontinuities for the relaxation system (1.1) under the assumptions (H) and (1.5). That is, we construct a contact wave, which approximates the contact discontinuity of the corresponding hyperbolic system (1.3) on any finite time interval as the relaxation time tends to zero, and prove that the contact wave is nonlinear stable.

Our idea is as follows. We observed that the relaxation system (1.1) is equivalent to the following perturbed viscous conservation laws:

$$u_t + f(u)_x = a^2 \varepsilon u_{xx} - \varepsilon u_{tt}. \tag{1.6}$$

The term u_{tt} is treated as a higher-order perturbation term in (1.6). We thus expect that the long time behavior of the solutions to (1.6) is similar to that for the viscous conservation laws

$$u_t + f(u)_x = a^2 \varepsilon u_{xx}. \tag{1.7}$$

We remark that u_{tt} in (1.6) is of the same order as u_{xx} due to the hyperbolic nature of the system. The sub-charateristic condition (1.5) will play an essential role in this paper.

For (1.7), Liu-Xin [20] showed that the inviscid contact discontinuity is meta-stable by the pointwise estimates. Liu-Xin's analysis is based on approximate Green's function, which is very difficult to construct in many physical systems when viscosity matrix is only semi-definite, such as compressible Navier-Stokes and Boltzmann equation. Recently, Huang-Matsumura-Xin [10] and Huang-Xin-Yang [12] develop a new energy method to treat the stability of the contact discontinuity for the compressible Navier-Stokes equations. Such approach admits that the energy estimate involving the lower order grows at the rate $(1+t)^{\frac{1}{2}}$. But it can be compensated by the decay in the energy estimate for derivatives which is of the order of $(1+t)^{-\frac{1}{2}}$ due to the underlying properties of the viscous contact wave. Thus, these reciprocal order of decay rates for the time evolution can close the a priori estimate containing the uniform bounds of the L^∞ norm in the lower order estimate due to Hölder inequality. This method can be widely applied to many physical systems, see [13] and [11]. In this paper, we shall adopt the ideas of [10] and [12] to investigate the stability of the contact discontinuity for the relaxation system (1.1).

Assume that $p-$field of system (1.3) is linearly degenerate for some p such that $1 \leq p \leq n$. Therefore,

$$\nabla \lambda_p(u) \cdot r_p(u) \equiv 0.$$

Consider the hyperbolic system (1.3) with the following Reimann initial data

$$u(x,0)=\begin{cases} u_-, & x<0, \\ u_+, & x>0, \end{cases} \tag{1.8}$$

where u_+ and u_- are chosen such that

$$f(u_+)-f(u_-)=s(u_+-u_-), \quad s=\lambda_p(u_+)=\lambda_p(u_-). \tag{1.9}$$

Then (1.3) and (1.8) admit a $p-$contact discontinuity solution

$$\hat{U}(x,t)=\begin{cases} u_-, & x<st, \\ u_+, & x>st. \end{cases} \tag{1.10}$$

Without loss of generality, we assume that $s\equiv 0$ in (1.9).

Now we construct the $p-$contact wave for (1.1). Eliminating $v(x,t)$, we obtain from (1.1) a system for $u(x,t)$ itself

$$\begin{cases} u_t+f(u)_x=a^2\varepsilon u_{xx}-\varepsilon u_{tt} \\ u(x,0)=u_0(x), \\ u_t(x,0)=-v_{0x}(x). \end{cases} \tag{1.11}$$

We first construct the contact wave for (1.11) following [20], and the v component will be recovered through (1.1). With a non-singular parameter ρ chosen below in (1.14), we define the $p-$contact wave curve by

$$C_p(u_-)=\{u|u=u(\rho), \frac{du}{d\rho}=r_p(u(\rho)), u(\rho_-)=u_-\}. \tag{1.12}$$

Along the curve $C_p(u_-)$, it holds that

$$\frac{d\lambda_p(u(\rho))}{d\rho}=\nabla\lambda_p(u(\rho))\cdot\frac{du(\rho)}{d\rho}=\nabla\lambda_p\cdot r_p\equiv 0.$$

So we have

$$\lambda_p(u(\rho))=\lambda_p(u_+)=\lambda_p(u_-)\equiv 0. \tag{1.13}$$

This means that the $p-$eigenvalue $\lambda_p(u)$ is zero along the curve $C_p(u_-)$. The non-singular parameter ρ is defined by

$$u(\rho_-)=u_-,\ u(\rho_+)=u_+,$$

and

$$\begin{cases} \rho_t - a^2\varepsilon\rho_{xx} = 0, & x \in \mathbb{R}^1, t \geq -1, \\ \rho(x, t=-1) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0. \end{cases} \end{cases} \tag{1.14}$$

Without loss of generality, we assume that $0 < \rho_- < \rho_+$. With $\rho(x,t)$ defined in (1.14), we define the viscous $p-$contact wave $\bar{U}(x,t)$ by

$$\bar{U}(x,t) \in C_p(u_-), \qquad \bar{U}(x,t) \equiv u(\rho(x,t)). \tag{1.15}$$

From the construction of $\bar{U}(x,t)$, we have

$$\begin{aligned} &\bar{U}_t(x,t) = r_p(u(\rho))\rho_t, \qquad \bar{U}_x(x,t) = r_p(u(\rho))\rho_x, \\ &\bar{U}_{xx}(x,t) = r_p(u(\rho))\rho_{xx} + \nabla r_p(u(\rho)) \cdot r_p(u(\rho))(\rho_x)^2. \end{aligned} \tag{1.16}$$

Following [20], we impose the following structure condition

$$\nabla l_p(u(\rho)) \cdot r_p(u(\rho)) \equiv 0, \ \nabla r_p(u(\rho)) \cdot r_p(u(\rho)) \equiv 0, \qquad \forall\ u \in C_p(u_-). \tag{1.17}$$

We remark that physical systems such as compressible Euler equations do satisfy (1.17). Under (1.17), the viscous contact wave $\bar{U}(x,t)$ defined in (1.15) satisfies the system

$$\bar{U}_t + f(\bar{U})_x - a^2\varepsilon\bar{U}_{xx} = 0. \tag{1.18}$$

The parameter $\rho(x,t)$ in (1.14) has the following properties as $x \to \pm\infty$:

$$\begin{aligned} &|\rho - \rho_\pm| = O(1)(\rho_+ - \rho_-)e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}, \\ &|\rho_x| = O(1)(\rho_+ - \rho_-)[\varepsilon(1+t)]^{-\frac{1}{2}}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}, \\ &|\rho_t, \varepsilon\rho_{xx}| = O(1)(\rho_+ - \rho_-)(1+t)^{-1}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}. \end{aligned} \tag{1.19}$$

Consequently the contact wave $\bar{U}(x,t)$ satisfies the properties:

$$\begin{aligned} &|\bar{U} - u_\pm| = O(\delta)e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}, \\ &|\bar{U}_x| = O(\delta)[\varepsilon(1+t)]^{-\frac{1}{2}}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}, \\ &|\bar{U}_t, \varepsilon\bar{U}_{xx}| = O(\delta)(1+t)^{-1}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}, \end{aligned} \tag{1.20}$$

as $x \to \pm\infty$, where $\delta = |u_+ - u_-| = O(1)(\rho_+ - \rho_-)$.

Direct computations show that

$$\|\bar{U}-\hat{U}\|_{L^q(\mathbb{R})} = O(1)\varepsilon^{\frac{1}{2q}}(1+t)^{\frac{1}{2q}}, \qquad q \geq 1.$$

where $\hat{U}$ is the inviscid contact discontinuity defined in (1.10). The above property means that the viscous contact wave $\bar{U}(x,t)$ for (1.1) approximates the inviscid contact discontinuity $\hat{U}(x,t)$ to the system (1.3) in L^q norm, $q \geq 1$, on any finite time interval as the relaxation time $\varepsilon \to 0$.

In the following, we only consider the asymptotic behavior of the solutions of the system (1.1) for fixed relaxation time ε with initial data that is a small perturbation near $\bar{U}$. Without loss of generality, we fix $\varepsilon = 1$.

Usually the integral

$$\int_{-\infty}^{+\infty} (u(x,0) - \bar{U}(x,0))dx$$

does not equal to zero. We shall introduce some linear diffusion waves to carry the excessive initial mass. We remark that the nonlinear diffusion waves are first introduced by [19] for the study on the nonlinear stability of the viscous shock wave to. But in our case, as in [10], it is sufficient to use the linear diffusion waves due to the different stability analysis.

For weak contact discontinuity, i.e. $\delta \ll 1$, the vectors $r_1(u_-), \cdots, r_{p-1}(u_-)$, $u_+ - u_-, r_{p+1}(u_+), \cdots, r_n(u_+)$ form a basis of $\mathbb{R}^n$. We thus decompose the excessive initial mass as

$$\int_{-\infty}^{+\infty} (u(x,0) - \bar{U}(x,0))dx = \sum_{i \neq p} \alpha_i r_i(\hat{u}_i) + x_0(u_+ - u_-), \tag{1.21}$$

with the uniquely determined constants α_i $(i \neq p)$ and x_0, where and in the sequel, we use the notation

$$\hat{u}_i = \begin{cases} u_-, & i < p, \\ u_+, & i > p. \end{cases}$$

Define the linear diffusion waves by

$$\begin{cases} \theta_{it} + \lambda_i(\hat{u}_i)\theta_{ix} = a^2\theta_{ixx}, & x \in \mathbb{R}, t \geq -1, \quad i \neq p, \\ \theta_i(x, t=-1) = \alpha_i \delta(x), \end{cases}$$

where $\delta(x)$ is the Dirac function satisfying

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1.$$

Then we have

$$\theta_i(x,t) = \frac{\alpha_i}{\sqrt{4\pi a^2(1+t)}} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{4a^2(1+t)}}, \qquad \int_{-\infty}^{+\infty} \theta_i(x,t)dx = \alpha_i. \tag{1.22}$$

Now we define the ansantz $\tilde{U}(x,t)$ by

$$\tilde{U}(x,t) = \bar{U}(x+x_0,t) + \theta(x,t) \tag{1.23}$$

with $\theta(x,t) = \sum_{i\neq p} \theta_i(x,t) r_i(\hat{u}_i)$. Thus we have

$$\int_{-\infty}^{+\infty} (u(x,0) - \tilde{U}(x,0))dx = 0. \tag{1.24}$$

A direct computation gives

$$\tilde{U}_t + \tilde{U}_{tt} + f(\tilde{U})_x - a^2\tilde{U}_{xx} = R_x, \tag{1.25}$$

with the error term

$$\begin{aligned} R(x,t) &= [f(\tilde{U}) - f(\bar{U}) - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i)] \\ &\quad +[-f(\bar{U})_t + a^2\tilde{U}_{xt} - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_{it} r_i(\hat{u}_i)] \\ &= O(\bar{\delta})(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}}, \end{aligned} \tag{1.26}$$

where we have used the fact

$$\begin{aligned} & f(\tilde{U}) - f(\bar{U}) - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i) \\ &= Df(\bar{U})\theta - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i) + O(1)|\theta|^2 \\ &= \sum_{i\neq p} [Df(\bar{U}) - Df(\hat{u}_i)]\theta_i r_i(\hat{u}_i) + O(1)|\theta|^2 \\ &= O(1)(\delta|\alpha| + |\alpha|^2)(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}} \\ &= O(\bar{\delta})(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}}, \end{aligned}$$

with the diffusion wave strength $|\alpha| = \sum_{i\neq p} |\alpha_i|$ and $\bar{\delta} = \delta + |\alpha|$.

Without loss of generality, we assume that $x_0 = 0$ from now on. Set

$$H(t) = \int_{-\infty}^{+\infty} (u(x,t) - \tilde{U}(x,t))dx.$$

It follows from the equation (1.11) for $u(x,t)$, and (1.25) for $\tilde{U}(x,t)$, that

$$\frac{d}{dt}H(t) + \frac{d^2}{dt^2}H(t) = 0.$$

We know $H(0) = 0$ from (1.24), and

$$\begin{aligned} H'(0) &= \int_{-\infty}^{+\infty} (u_t(x,0) - \tilde{U}_t(x,0))dx \\ &= \int_{-\infty}^{+\infty} (-v_x(x,0) - \bar{U}_t(x,0) - \sum_{i\neq p} \theta_{it}(x,0) r_i(\hat{u}_i))dx \\ &= -(v_+ - v_-) + (f(u_+) - f(u_-)) = 0. \end{aligned}$$

Thus we have, for all $t \geq 0$,

$$H(t) = \int_{-\infty}^{+\infty} (u(x,t) - \tilde{U}(x,t))dx = 0. \tag{1.27}$$

Set the perturbation by

$$\phi(x,t) = u(x,t) - \tilde{U}(x,t)$$

and introduce the anti-derivative variable

$$\Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy.$$

The equation (1.27) ensures that the anti-derivative variable $\Phi(x,t)$ is well-defined in some Soblev spaces like $L^2(\mathbb{R}^1), H^1(\mathbb{R}^1)$ etc.

Now we construct the ansatz $\tilde{V}(x,t)$ for $v(x,t)$. From the conservative part in (1.1), we set

$$\tilde{V}(x,t) = f(\tilde{U}) - a^2\tilde{U}_x + \int_{-\infty}^{x} \tilde{U}_{tt}dx - R, \tag{1.28}$$

then we have

$$\tilde{U}_t + \tilde{V}_x = 0. \tag{1.29}$$

Set

$$\psi(x,t) = v(x,t) - \tilde{V}(x,t).$$

From (1.1) and (1.29), we have

$$\phi_t + \psi_x = 0$$

and

$$\Phi_t = -\psi, \qquad \Phi_x = \phi. \tag{1.30}$$

We now state our main result:

Theorem 1.1. *Fix $\varepsilon = 1$. Under (H) and the sub-characteristic condition (1.5), assume $p-$characteristic field is linearly degenerate $(1 \le p \le n)$ and the structure condition (1.17) holds. Let $\tilde{U}(x,t)$ be the ansatz in (1.23). Then there exists a small positive constant δ_0 such that if the wave strength $\bar{\delta}$ and the initial values $(v_0(x), u_0(x))$ satisfy*

$$\bar{\delta} + \|\Phi(x,0)\|_{H^3}^2 + \|\psi(x,0)\|_{H^2}^2 \le \delta_0^2, \tag{1.31}$$

then the problem (1.1)-(1.2) admits a unique global solution $(v(x,t), u(x,t))$ satisfying

$$u(x,t) \in C([0,+\infty); H^2) \cap L^2(0,+\infty; H^3),$$
$$v(x,t) \in C([0,+\infty); H^1) \cap L^2(0,+\infty; H^2),$$

and

$$\|(u - \tilde{U}, v - \tilde{V})(\cdot,t)\|_{L^\infty} \le C\delta_0(1+t)^{-\frac{1}{4}}, \tag{1.32}$$

where C is a positive constant independent of t.

This theorem will be proved in next two sections. In Section 2, we will derive the desired a priori energy estimates. The estimates will be closed and decay rates will be given in Sections 3.

2 Energy Estimate

From (1.11) and (1.25), we obtain a system for $\phi(x,t)$

$$\phi_t + \phi_{tt} + (f(u) - f(\tilde{U}))_x - a^2\phi_{xx} = -R_x, \tag{2.1}$$

Integrating the system (2.1) over $(-\infty, x)$, one yields

$$\Phi_t + \Phi_{tt} + (f(u) - f(\tilde{U})) - a^2\Phi_{xx} = R.$$

Linearizing the above system, one has

$$\begin{aligned} &\Phi_t + \Phi_{tt} + Df(\bar{U})\Phi_x - a^2\Phi_{xx} \\ &= -[f(u) - f(\tilde{U}) - Df(\tilde{U})(u - \tilde{U})] - [Df(\tilde{U}) - Df(\bar{U})]\Phi_x + R \\ &=: R_1, \end{aligned} \tag{2.2}$$

where

$$|R_1| \leq O(1)(|\Phi_x|^2 + |\theta|^2 + |R|). \tag{2.3}$$

From now on, we will work on the Cauchy problem of (2.2) with the initial data

$$\Phi(x, 0) = \int_{-\infty}^{x} (u(y, t) - \tilde{U}(y, t))\, dy,\ \Phi_t(x, 0) = v(x, 0) - \tilde{V}(x, 0). \tag{2.4}$$

We note that (2.2) is a system of semi-linear wave equations with damping and source terms. Standard theory gives the local existence and uniqueness of classical solution for a short time T^* for smooth initial data. In order to prove the global existence and further to study the large time asymptotic behavior, we need to derive some uniform estimates under the condition (1.31). For technical reason, we will perform weighted energy estimates under the following a priori assumption:

(H1) *The smooth solution of (2.2)–(2.4) exits on time interval* $[0, T]$ *for some* $T > 0$ *and satisfies*

$$N(T) = \sup_{t\in[0,T]} (\|\Phi\|_{L^\infty} + \|\phi\|_{H^2} + \|\Phi_t\|_{H^1} + (1 + t)^{\frac{1}{4}}\|\phi\|_{L^2}) \leq \varepsilon_0, \tag{2.5}$$

where the small positive constant ε_0 *is only depending on the initial values and the wave strength* $\bar{\delta}$.

Clearly, (H1) is true for a short time if we choose δ_0 small, due to local theory. We will prove that $T = +\infty$ with the help of uniform estimates and continuity argument.

To diagonalize the system (2.2), we introduce the new variable

$$W(x, t) = L(\bar{U})\Phi(x, t), \quad \Phi(x, t) = R(\bar{U})W(x, t), \tag{2.6}$$

where $L(\bar{U})$ and $R(\bar{U})$ are defined in (1.4). Substitute $\Phi = R(\bar{U})W$ into (2.2), we have

$$\begin{aligned}
&R(\bar{U})W_t + R(\bar{U})W_{tt} + Df(\bar{U})R(\bar{U})W_x - a^2R(\bar{U})W_{xx} \\
&= -R(\bar{U})_tW - 2R(\bar{U})_tW_t - R(\bar{U})_{tt}W - Df(\bar{U})R(\bar{U})_xW \\
&\qquad +2a^2R(\bar{U})_xW_x + a^2R(\bar{U})_{xx}W + R_1 \\
&=: -Df(\bar{U})R(\bar{U})_xW + R_2.
\end{aligned}$$

Multiplying the above system by $L(\bar{U})$ to the left, we get

$$W_t + W_{tt} + \Lambda(\bar{U})W_x - a^2W_{xx} = -\Lambda(\bar{U})L(\bar{U})R(\bar{u})_xW + L(\bar{U})R_2, \tag{2.7}$$

where

$$L(\bar{U})R_2 = O(1)(\rho_tW + \rho_tW_t + \rho_{tt}W + \rho_xW_x + \rho_{xx}W + R_1). \tag{2.8}$$

Let

$$W = (W_1, W_2, \cdots, W_{p-1}, W_p, W_{p+1}, \cdots, W_n)^t, \tag{2.9}$$

where and in the sequel the notation $()^t$ represents the matrix transpose.

Introduce a weight function

$$\eta(x,t) = \frac{\rho(x,t)}{\rho_+}, \tag{2.10}$$

where $\rho(x,t), \rho_+$ are defined in (1.12). Clearly $|\eta(x,t) - 1| \le C\delta$. Here and after, we will use C and C_i $(i = 1, 2, \cdots)$ for generic positive constants which are independent of $\bar{\delta}$, ε_0 and time t. Note that $0 < \rho_- < \rho_+$, thus $\rho_x > 0$. Set

$$\bar{W} = (\eta^NW_1, \eta^NW_2, \cdots, \eta^NW_{p-1}, W_p, \eta^{-N}W_{p+1}, \cdots, \eta^{-N}W_n)^t,$$

where $N = \max\{2, \frac{1}{\sqrt{\delta}}\}$. If δ is small enough, we have

$$1 - C_1\sqrt{\delta} \le \eta^N \le 1 \le \eta^{-N} \le 1 + C_1\sqrt{\delta}. \tag{2.11}$$

Multiplying the system (2.7) by $\bar{W}^t$, we have

$$
\begin{aligned}
&(\frac{1}{2}\bar{W}^tW+\bar{W}^tW_t)_t-(\eta^N\sum_{i=1}^{p-1}W_{it}^2+W_{pt}^2+\eta^{-N}\sum_{i=p+1}^{n}W_{it}^2)\\
&+a^2(\eta^N\sum_{i=1}^{p-1}W_{ix}^2+W_{px}^2+\eta^{-N}\sum_{i=p+1}^{n}W_{ix}^2)\\
&-\sum_{i=1}^{p-1}[(\eta^N)_x\lambda_i(\bar{U})+\eta^N\lambda_i(\bar{U})_x]\frac{W_i^2}{2}\\
&-\sum_{i=p+1}^{n}[(\eta^{-N})_x\lambda_i(\bar{U})+\eta^{-N}\lambda_i(\bar{U})_x]\frac{W_i^2}{2}\\
&+a^2(\eta^N)_x\sum_{i=1}^{p-1}W_iW_{ix}+a^2(\eta^{-N})_x\sum_{i=p+1}^{n}W_iW_{ix}\\
&-(\eta^N)_t\sum_{i=1}^{p-1}(\frac{W_i^2}{2}+W_iW_{it})-(\eta^{-N})_t\sum_{i=p+1}^{n}(\frac{W_i^2}{2}+W_iW_{it})\\
&=-\bar{W}^t\Lambda(\bar{U})L(\bar{U})R(\bar{U})_xW+\bar{W}^tL(\bar{U})R_2+(\cdots)_x,
\end{aligned}
\tag{2.12}
$$

which is equivalent to

$$
\begin{aligned}
&(\frac{1}{2}\bar{W}^tW+\bar{W}^tW_t)_t-(\eta^N\sum_{i=1}^{p-1}W_{it}^2+W_{pt}^2+\eta^{-N}\sum_{i=p+1}^{n}W_{it}^2)\\
&+a^2(\eta^N\sum_{i=1}^{p-1}W_{ix}^2+W_{px}^2+\eta^{-N}\sum_{i=p+1}^{n}W_{ix}^2)+Q_1\\
&=R_3+(\cdots)_x,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
Q_1\ &=-\sum_{i=1}^{p-1}[(\eta^N)_x\lambda_i(\bar{U})+\eta^N\lambda_i(\bar{U})_x]\frac{W_i^2}{2}\\
&\quad-\sum_{i=p+1}^{n}[(\eta^{-N})_x\lambda_i(\bar{U})+\eta^{-N}\lambda_i(\bar{U})_x]\frac{W_i^2}{2}\\
&\quad+\bar{W}^t\Lambda(\bar{U})L(\bar{U})R(\bar{U})_xW,\\
R_3\ &=O(1)N\sum_{i\neq p}[\rho_xW_iW_{ix}+\rho_t(W_i^2+W_iW_{it})]+\bar{W}^tL(\bar{U})R_2.
\end{aligned}
\tag{2.14}
$$

We now claim that if δ is small enough, it holds that

$$Q_1 \geq C_2 N \rho_x \sum_{i \neq p} W_i^2. \tag{2.15}$$

To prove this claim, we note that $|\lambda_i(\bar{U})_x| \leq C_3 \rho_x$, and

$$\begin{aligned}
&|\bar{W}^t \Lambda(\bar{U}) L(\bar{U}) R(\bar{U})_x W| \\
&= |\eta^N \sum_{i=1}^{p-1} \lambda_i(\bar{U}) W_i \sum_{j=1}^{n} l_i^t(\bar{U}) r_j(\bar{U})_x W_j + \eta^{-N} \sum_{i=p+1}^{n} \lambda_i(\bar{U}) W_i \sum_{j=1}^{n} l_i^t(\bar{U}) r_j(\bar{U})_x W_j| \\
&= |\eta^N \sum_{i=1}^{p-1} \lambda_i(\bar{U}) W_i \sum_{j \neq p} l_i^t(\bar{U}) r_j(\bar{U})_x W_j + \eta^{-N} \sum_{i=p+1}^{n} \lambda_i(\bar{U}) W_i \sum_{j \neq p} l_i^t(\bar{U}) r_j(\bar{U})_x W_j| \\
&\leq C_4 \rho_x \sum_{i \neq p} W_i^2.
\end{aligned}$$

Here we have used the structure condition (1.17), which implies

$$r_p(\bar{U})_x = \nabla r_p(\bar{U}) \cdot r_p(\bar{U}) \rho_x = 0.$$

Therefore, noting that $N = \frac{1}{\sqrt{\delta}}$,

$$\begin{aligned}
Q_1 \quad &\geq -\eta^{N-1} \sum_{i=1}^{p-1} (N \eta_x \lambda_i(\bar{U}) + \eta \lambda_i(\bar{U})_x) \frac{W_i^2}{2} \\
&\quad + \eta^{-N-1} \sum_{i=p+1}^{n} (N \eta_x \lambda_i(\bar{U}) - \eta \lambda_i(\bar{U})_x) \frac{W_i^2}{2} - C_2 \rho_x \sum_{i \neq p} W_i^2 \\
&\geq (C_5 N - C_4) \rho_x \sum_{i \neq p} W_i^2 \\
&\geq \frac{1}{2} C_5 N \rho_x \sum_{i \neq p} W_i^2,
\end{aligned}$$

if we choose δ small enough. This proves (2.15).

Integrating (2.13) over $(-\infty, \infty)$, we have

$$
\begin{aligned}
&[\int (\frac{1}{2}\bar{W}^t W + \bar{W}^t W_t)\ dx]_t \\
&- \int (\eta^N \sum_{i=1}^{p-1} W_{it}^2 + W_{pt}^2 + \eta^{-N} \sum_{i=p+1}^{n} W_{it}^2)\ dx \\
&+a^2 \int (\eta^N \sum_{i=1}^{p-1} W_{ix}^2 + W_{px}^2 + \eta^{-N} \sum_{i=p+1}^{n} W_{ix}^2 + Q_1)\ dx \\
&\leq \int R_3\ dx.
\end{aligned} \tag{2.16}
$$

Multiplying the system (2.7) by $2(W_t)^t$, then integrating over $(-\infty, \infty)$, we obtain

$$
\begin{aligned}
&[\int (W_t^2 + a^2 W_x^2)\ dx]_t + \int (2W_t^2 + 2\sum_{i=1}^{n} \lambda_i(\bar{U}) W_{ix} W_{it})\ dx \\
&\leq -2\int (W_t)^t \Lambda(\bar{U}) L(\bar{U}) R(\bar{U})_x W\ dx + 2\int (W_t)^t L(\bar{U}) R_2\ dx.
\end{aligned} \tag{2.17}
$$

Adding (2.16) to (2.17), one has

$$
\begin{aligned}
&[\int (\frac{1}{2}\bar{W}^t W + \bar{W}^t W_t + W_t^2) + a^2 W_x^2\ dx]_t \\
&+ \int \sum_{i=1}^{p-1} (a^2 \eta^N W_{ix}^2 + 2\lambda_i(\bar{U}) W_{ix} W_{it} + (2 - \eta^N) W_{it}^2)\ dx \\
&+ \int \sum_{i=p+1}^{n} (a^2 \eta^{-N} W_{ix}^2 + 2\lambda_i(\bar{U}) W_{ix} W_{it} + (2 - \eta^{-N}) W_{it}^2)\ dx \\
&+ \int (a^2 W_{px}^2 + W_{pt}^2)\ dx + \int Q_1\ dx \\
&\leq \int [-2(W_t)^t \Lambda(\bar{U}) L(\bar{U}) R(\bar{U})_x W + 2(W_t)^t L(\bar{U}) R_2 + R_3]\ dx.
\end{aligned} \tag{2.18}
$$

Using the sub-characteristic condition (1.5), the estimate (2.11) and the smallness of δ, it is clear that there is C_6 such that

$$\min\{a^2, 1\} > C_6 > 0$$

and the following hold

$$(\frac{1}{2}\bar{W}^tW + \bar{W}^tW_t + W_t^2) \geq C_6(W^2 + W_t^2),$$
$$\sum_{i=1}^{p-1}(a^2\eta^N W_{ix}^2 + 2\lambda_i(\bar{U})W_{ix}W_{it} + (2-\eta^N)W_{it}^2) \geq C_6\sum_{i=1}^{p-1}(W_{ix}^2 + W_{it}^2),$$
$$\sum_{i=p+1}^{n}(a^2\eta^N W_{ix}^2 + 2\lambda_i(\bar{U})W_{ix}W_{it} + (2-\eta^N)W_{it}^2) \geq C_6\sum_{i=p+1}^{n}(W_{ix}^2 + W_{it}^2).$$

Define

$$\begin{aligned} E_1(t) &= \int (\frac{1}{2}\bar{W}^tW + \bar{W}^tW_t + W_t^2) + a^2W_x^2\ dx, \\ K_1(t) &= C_6\int (W_x^2 + W_t^2)\ dx. \end{aligned} \tag{2.19}$$

(2.18) reduces to

$$\begin{aligned} &E_{1t} + K_1 + \int Q_1\ dx \\ &\leq \int [-2(W_t)^t\Lambda(\bar{U})L(\bar{U})R(\bar{U})_xW + 2(W_t)^tL(\bar{U})R_2 + R_3]\ dx. \end{aligned} \tag{2.20}$$

We now work on the right hand side of (2.20). First, we observe that

$$\begin{aligned} &\int [-2(W_t)^t\Lambda(\bar{U})L(\bar{U})R(\bar{U})_xW]\ dx \\ &\leq C_7\int |\rho_x(W_t)^tW|\ dx \\ &\leq C_7\delta\int W_t^2\ dx + C_7(1+t)^{-1}\delta\int W^2\ dx, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &\int 2(W_t)^tL(\bar{U})R_2\ dx \\ &\leq \frac{1}{8}C_6\int W_t^2\ dx + C_8\int (L(\bar{U})R_2)^2\ dx \\ &\leq C_9(1+t)^{-\frac{3}{2}} + \frac{1}{4}C_6\int (W_t^2 + W_x^2)\ dx \\ &\qquad +C_8(1+t)^{-1}\delta^2\int W^2\ dx, \end{aligned} \tag{2.22}$$

where we have used the a priori assumption (H1) and Cauchy-Shwartz inequality. We now work on R_3. In view of (2.14) and (2.15), we have

$$
\begin{aligned}
&O(1)N\int\sum_{i\neq p}[\rho_x W_iW_{ix}+\rho_t(W_i^2+W_iW_{it})]\ dx\\
&\le\frac{1}{2}\int Q_1\ dx+C_{10}\sqrt{\delta}\int(W_x^2+W_t^2)\ dx\\
&\quad+C_{10}\sqrt{\delta}(1+t)^{-1}\int W^2\ dx,
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
&\textstyle\int\bar{W}^tL(\bar{U})R_2\ dx\\
&\le C_{11}\delta(1+t)^{-1}\int(W^2+W_t^2)\ dx+C_{11}\delta\int W_x^2+C_{11}\int|\bar{W}^tR_1|\ dx.
\end{aligned}
\tag{2.24}
$$

However, using (2.3), (H1) and (1.26), we have

$$
\begin{aligned}
&\int|\bar{W}^tR_1|\ dx\\
&\le C\int(|W^t|(\Phi_x^2+\theta^2+|R|)\ dx\\
&\le C_{12}\int|W^t|(\rho_x^2W^2+W_x^2)\ dx+C_{12}\int|W^t|(\theta^2+|R|)\ dx\\
&\le C_{12}\bar{\delta}(1+t)^{-1}\int W^2\ dx+C_{12}\varepsilon_0\int W_x^2\ dx+C_{12}\bar{\delta}(1+t)^{-\frac{1}{2}}.
\end{aligned}
\tag{2.25}
$$

Choosing $\bar{\delta}$ and ε_0 small, we conclude from (2.20)–(2.25) the following lower order estimate:

Lemma 2.1. Let E_1 and K_1 be defined in (2.19). Assume (H1) holds. There exits a small positive constant δ_1 such that if $\bar{\delta}<\delta_1$ and $\varepsilon_0\le\delta_1$ then

$$
E_{1t}+\frac{1}{2}K_1\le C_{13}\sqrt{\bar{\delta}}(1+t)^{-1}E_1+C_{13}\bar{\delta}(1+t)^{-\frac{1}{2}},
\tag{2.26}
$$

holds for any $t\in[0,T]$.

Now we estimate $\Phi_x=\phi$. Let

$$
Z(x,t)=L(\bar{U})\phi(x,t),
$$

then

$$
\phi(x,t)=R(\bar{U})Z(x,t).
$$

Applying ∂_x to the system (2.2), we have the system for $\phi(x,t)$

$$\phi_t + \phi_{tt} + (Df(\bar{U})\phi)_x - a^2\phi_{xx} = R_{1x}. \tag{2.27}$$

Substituting $\phi(x,t) = R(\bar{U})Z(x,t)$ into (2.27), then multiplying by $L(\bar{U})$ to the left, we get the following system for $Z(x,t)$,

$$\begin{aligned} &Z_t + Z_{tt} + (\Lambda(\bar{U})Z)_x - a^2 Z_{xx} - L(\bar{U})_x R(\bar{U})\Lambda(\bar{U})Z \\ &= [-L(\bar{U})R(\bar{U})_t + L(\bar{U})R(\bar{U})_{tt} + a^2 L(\bar{U})R(\bar{U})_{xx}]Z \\ &\quad -2L(\bar{U})R(\bar{U})_t Z_t + 2a^2 L(\bar{U})R(\bar{U})_x Z_x + L(\bar{U})R_{1x}. \end{aligned} \tag{2.28}$$

Let

$$\bar{Z} = (\eta^N Z_1, \cdots, \eta^N Z_{p-1}, Z_p, \eta^{-N} Z_{p+1}, \cdots, \eta^{-N} Z_n)^t,$$

with $\eta(x,t)$ defined in (2.10) and $N = \max\{2, \frac{1}{\sqrt{\delta}}\}$. Multiplying $(\bar{Z})^t$ to (2.28), similar to (2.13), one has

$$\begin{aligned} &(\frac{1}{2}\bar{Z}^t Z + \bar{Z}^t Z_t)_t - (\eta^N \sum_{i=1}^{p-1} Z_{it}^2 + Z_{pt}^2 + \eta^{-N} \sum_{i=p+1}^{n} Z_{it}^2) \\ &+a^2(\eta^N \sum_{i=1}^{p-1} Z_{ix}^2 + Z_{px}^2 + \eta^{-N} \sum_{i=p+1}^{n} Z_{ix}^2) + Q_2 \\ &= R_4 + (\cdots)_x, \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} Q_2 &= -\sum_{i=1}^{p-1} [(\eta^N)_x \lambda_i(\bar{U}) - \eta^N \lambda_i(\bar{U})_x] \frac{Z_i^2}{2} \\ &\quad - \sum_{i=p+1}^{n} [(\eta^{-N})_x \lambda_i(\bar{U}) - \eta^{-N} \lambda_i(\bar{U})_x] \frac{Z_i^2}{2} \\ &\quad +\bar{Z}^t L(\bar{U})_x R(\bar{U})\Lambda(\bar{U})Z, \\ R_4 &= O(1)N \sum_{i \neq p} [\rho_x Z_i Z_{ix} + \rho_t (Z_i^2 + Z_i Z_{it})] + \bar{Z}^t L(\bar{U}) R_{1x} \\ &\quad +O(1)\delta(1+t)^{-1}(Z^2 + Z^t Z_t + Z_t^2). \end{aligned} \tag{2.30}$$

Here, we have used the following fact

$$2a^2 \bar{Z}^t L(\bar{U})R(\bar{U})_x Z_x = (\cdots)_x + O(1)(N\rho_x^2 + |\rho_{xx}|)Z^2.$$

Due to the structure condition (1.17), we know

$$l_p(\bar U)_x = \nabla l_p(\bar U)\cdot r_p(\bar U)\rho_x = 0.$$

Therefore,

$$\begin{aligned}
&|\bar Z^t L(\bar U)_x R(\bar U)\Lambda(\bar U)Z| \\
&= |\eta^N \sum_{i=1}^{p-1}\lambda_i(\bar U)Z_i \sum_{j\neq p}^{n} l_i^t(\bar U)_x r_j(\bar U)Z_j + \eta^{-N}\sum_{i=p+1}^{n}\lambda_i(\bar U)Z_i\sum_{j\neq p}^{n} l_i^t(\bar U)_x r_j(\bar U)Z_j| \\
&\leq O(1)\rho_x \sum_{i\neq p} Z_i^2.
\end{aligned}$$

Similar to (2.15), if δ is small, there is a positive constant C_{14} such that

$$Q_2 \geq C_{14} N\rho_x \sum_{i\neq p} Z_i^2. \tag{2.31}$$

We now integrate (2.29) over $(-\infty,+\infty)$ to obtain

$$\begin{aligned}
&[\int(\frac{1}{2}\bar Z^t Z + \bar Z^t Z_t)\ dx]_t - \int(\eta^N\sum_{i=1}^{p-1} Z_{it}^2 + Z_{pt}^2 + \eta^{-N}\sum_{i=p+1}^{n} Z_{it}^2)\ dx \\
&+\int a^2(\eta^N \sum_{i=1}^{p-1} Z_{ix}^2 + Z_{px}^2 + \eta^{-N}\sum_{i=p+1}^{n} Z_{ix}^2)\ dx + \int Q_2\ dx \\
&\qquad \leq \int R_4\ dx.
\end{aligned}\tag{2.32}$$

Multiplying (2.28) by $2(Z_t)^t$, it gives

$$(Z_t^2 + a^2 Z_x^2)_t + 2Z_t^2 + \sum_{i=1}^{n}\lambda_i(\bar U)Z_{ix}Z_{it} = R_5 + (\cdots)_x, \tag{2.33}$$

where

$$R_5 = O(1)(\rho_x+\rho_t+\rho_{tt}+\rho_{xx})(Z_t^t Z + Z_t^2 + Z_t^t Z_x) + Z_t^t L(\bar U)R_{1x}. \tag{2.34}$$

Integrating (2.33) over $(-\infty,+\infty)$ yields

$$\begin{aligned}
&[\int(Z_t^2 + a^2 Z_x^2)\ dx]_t + \int(2Z_t^2 + \sum_{i=1}^{n}\lambda_i(\bar U)Z_{ix}Z_{it})\ dx \\
&\qquad \leq \int R_5\ dx,
\end{aligned}\tag{2.35}$$

Add (2.35) to (2.32), we have

$$
\begin{aligned}
&[\int (\frac{1}{2}\bar{Z}^t Z + \bar{Z}^t Z_t + Z_t^2) + a^2 Z_x^2 \, dx]_t \\
&+ \int \sum_{i=1}^{p-1} (a^2\eta^N Z_{ix}^2 + 2\lambda_i(\bar{U}) Z_{ix} Z_{it} + (2-\eta^N) Z_{it}^2) \, dx \\
&+ \int \sum_{i=p+1}^{n} (a^2\eta^{-N} Z_{ix}^2 + 2\lambda_i(\bar{U}) Z_{ix} Z_{it} + (2-\eta^{-N}) Z_{it}^2) \, dx \\
&+ \int (a^2 Z_{px}^2 + Z_{pt}^2) \, dx + \int Q_2 \, dx \\
&\leq \int [R_4 + R_5] \, dx.
\end{aligned} \tag{2.36}
$$

Define

$$
\begin{aligned}
E_2(t) &= \int (\frac{1}{2}\bar{Z}^t Z + \bar{Z}^t Z_t + Z_t^2) + a^2 Z_x^2 \, dx, \\
K_2(t) &= C_6 \int (Z_x^2 + Z_t^2) \, dx.
\end{aligned} \tag{2.37}
$$

(2.36) reduces to

$$
E_{2t} + K_2 + \int Q_2 \, dx \leq \int [R_4 + R_5] \, dx, \tag{2.38}
$$

where we have used the following fact

$$
\begin{aligned}
&(\frac{1}{2}\bar{Z}^t Z + \bar{Z}^t Z_t + Z_t^2) \geq C_6 (Z^2 + Z_t^2), \\
&\sum_{i=1}^{p-1} (a^2\eta^N Z_{ix}^2 + 2\lambda_i(\bar{U}) Z_{ix} Z_{it} + (2-\eta^N) Z_{it}^2) \geq C_6 \sum_{i=1}^{p-1} (Z_{ix}^2 + Z_{it}^2), \\
&\sum_{i=p+1}^{n} (a^2\eta^N Z_{ix}^2 + 2\lambda_i(\bar{U}) Z_{ix} Z_{it} + (2-\eta^N) Z_{it}^2) \geq C_6 \sum_{i=p+1}^{n} (Z_{ix}^2 + Z_{it}^2).
\end{aligned}
$$

We now estimate the right hand side of (2.38). By (2.30) and (2.34), it is clear that

$$
R_4 + R_5 \leq C_{15}\sqrt{\delta}(1+t)^{-1} Z^2 + \sqrt{\delta}(Z_x^2 + Z_t^2) + (\bar{Z}^t + Z_t^t) L(\bar{U}) R_{1x}. \tag{2.39}
$$

On the other hand, we have

$$\begin{aligned}
&\int \bar{Z}^t L(\bar{U}) R_{1x}\ dx = -\int (\bar{Z}_x^t R_1 + \bar{Z}^t L(\bar{U})_x R_1)\ dx \\
&\leq C \int R_1^2\ dx + C\delta(1+t)^{-1} \int Z^2\ dx + \frac{1}{4} C_6 \int Z_x^2\ dx.
\end{aligned} \tag{2.40}$$

and

$$\int {Z_t}^t L(\bar{U}) R_{1x}\ dx \leq \frac{1}{4} C_6 \int Z_t^2\ dx + C \int R_{1x}^2\ dx \tag{2.41}$$

It remains to estimate the terms with R_1. From (2.3) we know that

$$\begin{aligned}
&\int R_1^2\ dx \leq C \int (Z^4 + \theta^4 + R^2)\ dx \\
&\leq C\bar{\delta}^2 (1+t)^{-\frac{3}{2}} + C\|Z\|_{L^\infty}^2 \|Z\|_{L^2}^2 \\
&\leq C\bar{\delta}^2 (1+t)^{-\frac{3}{2}} + C\|Z\|_{L^2} \|Z_x\|_{L^2} \|Z\|_{L^2}^2 \\
&\leq C\bar{\delta}^2 (1+t)^{-\frac{3}{2}} + C_{16}\|Z\|_{L^2}^6 + \frac{1}{8} C_6 \|Z_x\|_{L^2}^2.
\end{aligned} \tag{2.42}$$

Recall

$$R_1 = -[f(u) - f(\tilde{U}) - Df(\tilde{U})(u - \tilde{U})] + [Df(\tilde{U}) - Df(\bar{U})]\Phi_x + R.$$

We have

$$\int R_x^2\ dx \leq C\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.43}$$

and

$$\begin{aligned}
&|[f(u) - f(\bar{U}) - Df(\bar{U})(u - \tilde{U})]_x| \\
&\quad = |[\frac{1}{2}\phi^t \nabla^2 f(\tilde{U}, \phi)\phi]_x| \\
&\quad = O(1)(|\phi_x| + \rho_x + |\theta_x|)\phi^2 + O(1)|\phi||\phi_x|, \\
&|[Df(\tilde{U}) - Df(\bar{U})]\Phi_x]_x| = |[\theta^t \nabla^2 f(\bar{U}, \theta)\phi]_x| \\
&\quad = O(1)(\rho_x + |\theta_x|)|\theta||\phi| + O(1)(|\theta_x||\phi| + |\theta||\phi_x|).
\end{aligned} \tag{2.44}$$

Hence,

$$\begin{aligned}
&\int R_{1x}^2\ dx \\
&\leq C\bar{\delta}(1+t)^{-\frac{3}{2}} + C \int (\rho_x^2 + \theta_x^2)\phi^2 + (\phi^2 + \theta_x^2)\phi_x^2)\ dx \\
&\leq C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\bar{\delta}(1+t)^{-1} \int Z^2\ dx + C(\bar{\delta} + \varepsilon_0) \int Z_x^2\ dx.
\end{aligned} \tag{2.45}$$

Due to a priori assumption (H1), one has

$$\|Z\|_{L2}^6 \le \varepsilon_0^4 \int Z^2 \; dx.$$

We thus conclude from (2.38)—(2.45) the following estimates

Lemma 2.2. Let E_2 and K_2 be defined in (2.37). Assume (H1) holds. There is small positive constant δ_1 such that if $\bar{\delta} < \delta_1$ and $\varepsilon_0 < \delta_1$ then

$$E_{2t} + \frac{1}{2} K_2 \le C_{17}(\sqrt{\bar{\delta}} + \varepsilon_0^4)(1+t)^{-1}\|Z\|^2 + C_{17}\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.46}$$

holds for any $t \in [0, T]$.

In order to obtain the uniform estimates and close the argument with the a priori assumption (H1), we need to work on higher order derivatives. For this purpose, we apply ∂_x to the system (2.28) to obtain

$$Z_{xt} + Z_{xtt} + (\Lambda(\bar{U})Z)_{xx} - a^2 Z_{xxx} = R_{6x} \tag{2.47}$$

where

$$\begin{aligned} R_6 = {} & L(\bar{U})R_{1x} + [L(\bar{U})_x R(\bar{U})\Lambda(\bar{U}) - L(\bar{U})R(\bar{U})_t + L(\bar{U})R(\bar{U})_{tt} \\ & + a^2 L(\bar{U})R(\bar{U})_{xx}]Z - 2L(\bar{U})R(\bar{U})_t Z_t + 2a^2 L(\bar{U})R(\bar{U})_x Z_x. \end{aligned} \tag{2.48}$$

Multiplying (2.47) with $(Z_x^t + 2Z_{xt}^t)$ and integrating over $(-\infty, +\infty)$, integrating by parts, we arrive at

$$E_{3t} + K_3 \le \int R_7 \; dx, \tag{2.49}$$

where

$$\begin{aligned} E_3(t) &= \int (\frac{1}{2} Z_x^2 + Z_x Z_{xt} + Z_{xt}^2) + a^2 Z_{xx}^2 \; dx, \\ K_3(t) &= C_6 \int (Z_{xx}^2 + Z_{xt}^2) \; dx, \end{aligned} \tag{2.50}$$

$$R_7 = (Z_x^t + 2Z_{xt}^t)R_{6x} - Z_x^t(\Lambda(\bar{U})Z)_{xx} - 2Z_{xt}^t(\Lambda(\bar{U})_{xx}Z + 2\Lambda(\bar{U})_x Z_x), \tag{2.51}$$

and we have used the following fact

$$\begin{aligned} &(\frac{1}{2} Z_x^2 + Z_x Z_{xt} + Z_{xt}^2) \ge C_6(Z_x^2 + Z_{xt}^2), \\ &(a^2 Z_{xx}^2 + 2Z_{xt}^t \Lambda(\bar{U}) Z_{xx} + Z_{xt}^2) \ge C_6(Z_{xx}^2 + Z_{xt}^2). \end{aligned}$$

We now estimate $\int R_7\ dx$ term by term. First of all, one has

$$
\begin{aligned}
&|\int -2Z_{xt}^t(\Lambda(\bar{U})_{xx}Z + 2\Lambda(\bar{U})_x Z_x)\ dx| \\
&\leq C\delta(1+t)^{-1}\int (Z^2+Z_x^2)\ dx + C\delta\int Z_{xt}^2\ dx,
\end{aligned}
\tag{2.52}
$$

and

$$
\begin{aligned}
&|\int Z_x^t(\Lambda(\bar{U})Z)_{xx}\ dx| = |\int Z_{xx}^t(\Lambda(\bar{U})Z)_x\ dx| \\
&= |\int Z_{xx}^t(\Lambda(\bar{U})_x Z + \Lambda(\bar{U})Z_x)\ dx| \\
&\leq C\delta(1+t)^{-1}\int Z^2\ dx + C\delta\int Z_{xx}^2\ dx + |\int \frac{1}{2}Z_x^t\Lambda(\bar{U})_x Z_x\ dx| \\
&\leq C\delta(1+t)^{-1}\int Z^2\ dx + C\delta\int (Z_{xx}^2+Z_x^2)\ dx.
\end{aligned}
\tag{2.53}
$$

Then, we note that

$$
\begin{aligned}
&|\int Z_x^t R_{6x}\ dx| = |\int Z_{xx}^t R_6\ dx| \\
&\leq \frac{1}{4}C_6\int Z_{xx}^2\ dx + C\int R_6^2\ dx \\
&\leq C\bar{\delta}(1+t)^{-1}\int (Z^2+Z_x^2+Z_t^2)\ dx \\
&\qquad + C\bar{\delta}(1+t)^{-\frac{3}{2}} + \frac{1}{4}C_6\int Z_{xx}^2\ dx.
\end{aligned}
\tag{2.54}
$$

Finally, we have

$$
\begin{aligned}
&|\int Z_{xt}^t R_{6x}\ dx| \leq C|\int Z_{xt}^t R_{1xx}\ dx| \\
&+ C\bar{\delta}(1+t)^{-\frac{1}{2}}\int (|Z|+|Z_x|+|Z_t|+|Z_{xt}|+|Z_{xx}|+|R_{1x}|)|Z_{xt}|\ dx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&|\int Z_{xt}^t R_{6x}\ dx| \\
&\le C|\int Z_{xt}^t R_{1xx}\ dx| + C\bar\delta(1+t)^{-1}\int (Z^2+Z_x^2+Z_t^2)\ dx \\
&\qquad + C\bar\delta \int (Z_{xt}^2+Z_{xx}^2)+R_{1x}^2\ dx \\
&\le C\bar\delta(1+t)^{-1}\int Z^2\ dx + C\bar\delta(1+t)^{-\frac{3}{2}} \\
&+ C(\bar\delta+\varepsilon_0)\int (Z_x^2+Z_t^2+Z_{xt}^2+Z_{xx}^2)\ dx + C|\int Z_{xt}^t R_{1xx}\ dx|.
\end{aligned}
\tag{2.55}
$$

Furthermore, from the expression of R_1, a standard calculation gives

$$
\begin{aligned}
&|\int Z_{xt}^t R_{1xx}\ dx| \le C\bar\delta(1+t)^{-\frac{3}{2}} \\
&\qquad + C(\bar\delta+\varepsilon_0)\int (Z_x^2+Z_{xx}^2+Z_{xt}^2)\ dx + C\bar\delta(1+t)^{-1}\int Z^2\ dx.
\end{aligned}
\tag{2.56}
$$

With the help of Lemma 2, we collect the estimates in (2.49)–(2.56) and conclude the following lemma.

Lemma 2.3. Let E_3 and K_3 be defined in (2.50). Assume (H1) holds. There is a small positive constant δ_1 such that if $\bar\delta < \delta_1$ and $\varepsilon_0 < \delta_1$ then

$$
E_{3t} + \frac{1}{4}K_3 \le C_{18}(\sqrt{\bar\delta}+\varepsilon_0^4)(1+t)^{-1}\|Z\|^2 + C_{18}\bar\delta(1+t)^{-\frac{3}{2}}, \tag{2.57}
$$

holds for any $t \in [0,T]$.

In order to get the decay rate of $\psi = -\Phi_t$, we need the estimate on Φ_{tt}. To this end, we apply ∂_t to the system (2.2),

$$
\Phi_{tt} + \Phi_{ttt} + (f(\bar U))_t\Phi_x + Df(\bar U)\Phi_{xt} - a^2\Phi_{xxt} = R_{1t}. \tag{2.58}
$$

Multiplying (2.58) by $2\Phi_{tt}^t$, integrating it over $(-\infty,+\infty)$ and integrating by parts, we have

$$
\begin{aligned}
&(\int (\Phi_{tt}^2 + a^2\Phi_{xt}^2)\ dx)_t + 2\int \Phi_{tt}^2\ dx \\
&\le 2\int (|\Phi_{tt}^t (Df(\bar U))_t\Phi_x| + |\Phi_{tt}^t Df(\bar U)\Phi_{xt}| + 2|\Phi_{tt}^t R_{1t}|)\ dx \\
&\le C\bar\delta(1+t)^{-1}\|\Phi_x\|^2 + \int \Phi_{tt}^2\ dx + \bar C\int \Phi_{xt}^2\ dx + \int |R_{1t}|^2\ dx.
\end{aligned}
\tag{2.59}
$$

Recall $\Phi_x = \phi$. With the help of the expression of R_1 and a straightforward computation, (2.59) and Lemma 2.2, lead to

$$\begin{aligned}&(\int (\Phi_{tt}^2 + a^2\Phi_{xt}^2)\ dx)_t + \int \Phi_{tt}^2\ dx \\ &\leq C(\sqrt{\bar{\delta}} + \varepsilon_0^4)(1+t)^{-1}\|Z\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}},\end{aligned} \tag{2.60}$$

We now set

$$E_4 = E_3 + \int (\Phi_{tt}^2 + a^2\Phi_{xt}^2)\ dx, \qquad K_4 = \frac{1}{4}K_3 + \int \Phi_{tt}^2\ dx. \tag{2.61}$$

Note that

$$\begin{aligned}\int |Z|^2 dx &= \int |L(\bar{U})\Phi_x|^2 dx = \int |L(\bar{U})(R(\bar{U})W)_x|^2 dx \\ &\leq C\bar{\delta}(1+t)^{-1}\int |W|^2 dx + C\int |W_x|^2 dx \\ &\leq C\bar{\delta}(1+t)^{-1}E_1 + CK_1.\end{aligned} \tag{2.62}$$

We conclude from Lemma 2.3, and (2.60)–(2.62) our desired estimate

Lemma 2.4. Let E_4 and K_4 be defined in (2.61). Assume (H1) holds. There is a small positive constant δ_1 such that if $\bar{\delta} < \delta_1$ and $\varepsilon_0 < \delta_1$ then

$$E_{4t}+K_4 \leq C_{19}\bar{\delta}(1+t)^{-2}E_1+C_{19}(\sqrt{\bar{\delta}}+\varepsilon_0^4)(1+t)^{-1}K_1+C_{19}\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.63}$$

holds for any $t \in [0, T]$.

3 Time decay rate

In this section, we are going to complete the proof of Theorem 1 based on estimates stated in Lemma 2.1–2.4. Lemma 2.1 implies that

$$E_1 \leq C(E_1(0) + \bar{\delta})(1+t)^{\frac{1}{2}}. \tag{3.1}$$

Set

$$E_5 = E_1 + E_2 + E_4, \qquad K_5 = K_1 + \frac{1}{2}K_2 + K_4. \tag{3.2}$$

For $\bar{\delta}$ and ε_0 small, we conclude from Lemmas 2.1–2.4 and (2.62) that

$$E_{5t} + K_5 \leq C\bar{\delta}(1+t)^{-1}E_5 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.3}$$

Since $K_5 \geq 0$, we thus have

$$\frac{d}{dt}[(1+t)^{-C\bar{\delta}}E_5] + (1+t)^{-C\bar{\delta}}K_5 \leq C\bar{\delta}(1+t)^{-\frac{1}{2}-C\bar{\delta}}, \tag{3.4}$$

which gives

$$E_5 \leq C(E_5(0)+\bar{\delta})(1+t)^{\frac{1}{2}}, \tag{3.5}$$

and

$$\int_0^t K_5 d\tau \leq C(E_5(0)+\bar{\delta})(1+t)^{\frac{1}{2}}. \tag{3.6}$$

Now set

$$E_6 = E_2 + E_4,\ K_6 = \frac{1}{2}K_2 + K_4. \tag{3.7}$$

By Lemma 2.2, Lemma 2.4 and (2.62), we have

$$E_{6t}+K_6 \leq C_{20}\bar{\delta}(1+t)^{-2}E_1 + C_{20}(\sqrt{\bar{\delta}}+\varepsilon_0^4)(1+t)^{-1}K_1 + C_{20}\bar{\delta}(1+t)^{-\frac{3}{2}}. \tag{3.8}$$

Then we compute

$$\begin{aligned} [(1+t)E_6]_t &= E_6 + (1+t)E_{6t} \\ &\leq E_6 + C\bar{\delta}(1+t)^{-1}E_1 + C(\bar{\delta}+\varepsilon_0^4)K_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \\ &\leq CK_5 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{3.9}$$

Integrating the above inequality in t yields

$$E_6 \leq C(E_5(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}. \tag{3.10}$$

We collect all estimates we obtained in (3.1)–(3.10) in the following Lemma.

Lemma 3.1. Let E_5 be defined in (3.2) and E_6 in (3.7). Assume (H1) holds. There is a small positive constant δ_1 such that if $\bar{\delta} < \delta_1$ and $\varepsilon_0 < \delta_1$ then

$$E_5 \leq C(E_5(0)+\bar{\delta})(1+t)^{\frac{1}{2}}, \quad E_6 \leq C(E_5(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}, \tag{3.11}$$

hold for any $t \in [0,T]$.

We remark that all estimates we obtained up to now are based on the a priori assumption (H1). Now, we are able to show that if we choose δ_0

(in Theorem 1) small, (H1) is true in the time range where smooth solution exists.

First of all, since $\phi = R(\bar{U})Z$, we have

$$\|\phi\|_{H^2}^2 \leq CE_6 \leq C(E_5(0) + \bar{\delta})(1+t)^{-\frac{1}{2}}, \tag{3.12}$$

which also implies

$$(1+t)^{\frac{1}{4}}\|\phi\|_{L^2} \leq C\sqrt{(E_5(0) + \bar{\delta})}. \tag{3.13}$$

Furthermore,

$$\begin{aligned} \|\Phi\|_{L_\infty}^2 &\leq C\|\Phi\|_{L_2}\|\phi\|_{L_2} \leq C\|W\|_{L_2}\|Z\|_{L_2} \\ &\leq CE_1^{\frac{1}{2}}E_6^{\frac{1}{2}} \leq C(E_5(0) + \bar{\delta}), \end{aligned} \tag{3.14}$$

For $\psi = -\Phi_t$, we know from the system (2.2) that

$$\Phi_t = -\Phi_{tt} - Df(\bar{U})\Phi_x + a^2\Phi_{xx} + R. \tag{3.15}$$

Thus

$$\begin{aligned} \|\Phi_t\|_{L_2} &\leq C(\|\Phi_{tt}\|_{L_2} + \|\Phi_x\|_{L_2} + \|\Phi_{xx}\|_{L_2} + \|R\|_{L_2}) \\ &\leq CE_4^{\frac{1}{2}} + C\bar{\delta}(1+t)^{-\frac{1}{4}} \\ &\leq C(E_5(0) + \bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}. \end{aligned} \tag{3.16}$$

We conclude from (3.10)–(3.16) that, there is a positive constants C_{21}, and C_{22} such that

$$N(T) \leq C_{21}(E_5(0) + \bar{\delta})^{\frac{1}{2}} \leq C_{22}\delta_0. \tag{3.17}$$

Therefore, if we choose δ_0 small enough such that

$$C_{22}\delta_0 \leq \frac{1}{2}\varepsilon_0, \tag{3.18}$$

the a priori assumption (H1) is true as long as the smooth solution exists. On the other hand, the uniform estimate (3.17), together with the local well-posedness theory, gives the global existence of unique smooth solution. We thus proved the first part of Theorem 1.

We now show that (3.11)–(3.16) also give the decay estimate (1.32). In fact, we have

$$\|(\phi, \phi_x)\|_{L_\infty} \le C\|\phi\|_{H^2} \le C(E_5(0) + \bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}. \tag{3.19}$$

and

$$\|\psi\|_{L_\infty} = \|\Phi_t\|_{L_\infty} \le C\|\Phi_t\|_{L_2}^{\frac{1}{2}}\|\Phi_{xt}\|_{L_2}^{\frac{1}{2}} \le C(E_5(0) + \bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}. \tag{3.20}$$

Therefore, the proof of Theorem 1 is complete.

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