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Zero Relaxation Limit for Piecewise Smooth Solutions to a Rate-Type Viscoelastic System in the Presence of Shocks

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We study a rate-type viscoelastic system proposed in I. Suliciu (*Int. J. Engng. Sci.* **28** (1990), 827–841), which is a 3×3 hyperbolic system with relaxation. As the relaxation time tends to zero, this system converges to the well-known p-system formally. In the case that the solutions of the p-system are piecewise smooth, including finitely many noninteracting shock waves, we show that there exist smooth solutions for Suliciu's model which converge to those of the p-system strongly as the relaxation time goes to zero. The method used here is the so-called matched asymptotic analysis suggested in J. Goodman and Z. P. Xin (*Arch. Ration. Mech. Anal.* **121** (1992), 235–265), which includes two parts: the matched asymptotic expansion and stability analysis. © 2000 Academic Press

Key Words: piecewise smooth solution; viscoelasticity; matched asymptotic analysis; zero relaxation limit.

1. INTRODUCTION

We are interested in the asymptotic behavior of the rate-type viscoelastic system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \qquad x \in \mathbf{R}^1, \quad t > 0, \\ (p + Ev)_t &= \frac{p_R(v) - p}{\varepsilon}, \end{aligned} \tag{1.1}$$

as the relaxation time ε goes to zero. Here v and $-p$ denote strain and stress, respectively, u is related to the particle velocity, and E is a positive constant, called the dynamic Young's modulus.

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This system was proposed in [9] to introduce a relaxation approximation to the system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_R(v)_x &= 0. \end{aligned} \tag{1.2}$$

Since the system (1.2) can be obtained from (1.1) by an expansion procedure as the leading order, it is natural to expect that (1.2) governs the evolution of the solutions to (1.1) as $\varepsilon \to 0$. For smooth flow, this statement can be easily verified by Hilbert expansion and a standard energy estimate argument. However, when discontinuities occur in the solutions of (1.2), the analysis is much more complicated and more difficulties appear.

Motivated by [1] and [10], we use the method of matched asymptotic analysis introduced in [1] to overcome the difficulties, i.e., constructing approximate solutions by matched asymptotic expansion and then estimating the error bounds. We show that the piecewise smooth solutions of (1.2) with finitely many noninteracting shocks satisfying the entropy conditions and subcharacteristic condition are strong limits as $\varepsilon \to 0$ of solutions of (1.1). For simplicity of presentation, we only discuss the case in which the solution of (1.2) is a distribution solution smooth up to a single shock.

We make the following assumptions:

(H_1) $p_R'(v) < 0$,

(H_2) $p_R''(v) > 0$,

(H_3) $|p_R'(v)| < E$,

where (H_3) is the so-called subcharacteristic condition (see [6]).

It is easy to know that, under (H_1)–(H_2), (1.2) is strictly hyperbolic and genuinely nonlinear, with eigenvalues

$$\lambda_1 = -\left(-p_R'(v)\right)^{1/2} < 0 < \left(-p_R'(v)\right)^{1/2} = \lambda_2. \tag{1.3}$$

We now give our main result. A function $(v_0(x,t), u_0(x,t))$ is called a single-shock solution of (1.2) up to time $T > 0$ if

(i) $(v_0(x,t), u_0(x,t))$ is a distributional solution of (1.2) in the region $\mathbf{R}^1 \times [0, T]$.

(ii) There is a smooth curve, the shock, $x = s(t)$, $0 \le t \le T$, so that $(v_0(x,t), u_0(x,t))$ is sufficiently smooth at any point $x \ne s(t)$. The left and right limits of $(v_0(x,t), u_0(x,t))$ and its derivatives exist at the shock $x = s(t)$.

(iii) Across the shock $x = s(t)$, the Rankine–Hugoniot conditions hold,

$$\begin{aligned} \dot{s}\left(v_0^l - v_0^r\right) &= u_0^l - u_0^r, \\ \dot{s}\left(u_0^l - u_0^r\right) &= -\left(p_R\left(v_0^l\right) - p_R\left(v_0^r\right)\right). \end{aligned} \tag{1.4}$$

In the following, we will always use the notations $f^l = f(s(t) - 0, t)$ and $f^r = f(s(t) + 0, t)$.

(iv) The Lax-entropy condition

$$\lambda_1^l < \dot{s} < \lambda_1^r \qquad \text{or} \qquad \lambda_2^r < \dot{s} < \lambda_2^l$$

is satisfied. For definiteness, we assume that the shock is in the second family, i.e.,

$$\lambda_2^r(t) < \dot{s}(t) < \lambda_2^l(t). \tag{1.5}$$

THEOREM 1. *Under* (H_1)–(H_3), *setting* $p_0 = p_R(v_0)$, *there exist positive constants* η_0 *and* ε_0, *such that if* $(v_0, u_0)(x,t)$ *is a single-shock solution up to time T with*

$$\sum_{1 \le \alpha \le 6} \left(\int_0^T \int_{x < s(t)} + \int_0^T \int_{x > s(t)} \right) |\partial_x^\alpha (v_0, u_0, p_0)(x,t)|^2 \, dx \, dt < +\infty, \tag{1.6}$$

and

$$|v_0^r - v_0^l| + |u_0^r - u_0^l| + |p_0^r - p_0^l| \le \eta_0, \qquad \forall t \in [0, T], \tag{1.7}$$

then for each $\varepsilon \in (0, \varepsilon_0]$, *there is a smooth solution* $(v^\varepsilon, u^\varepsilon, p^\varepsilon)(x,t)$ *of* (1.1) *with*

$$(v^\varepsilon, u^\varepsilon, p^\varepsilon) - (v_0, u_0, p_0) \in L^\infty([0, T], H^2). \tag{1.8}$$

Moreover, for any given $\alpha \in (0,1)$,

$$\sup_{0 \le t \le T} \int_{\mathbf{R}^1} \left|(v^\varepsilon - v_0, u^\varepsilon - u_0, p^\varepsilon - p_0)(x,t)\right|^2 dx \le C_1 \varepsilon^\alpha \tag{1.9}$$

and

$$\sup_{\substack{0 \le t \le T \\ |x - s(t)| \ge h}} \left|(v^\varepsilon - v_0, u^\varepsilon - u_0, p^\varepsilon - p_0)(x,t)\right| \le C_h \varepsilon, \qquad \forall h > 0, \tag{1.10}$$

where C_1 *and* C_h *are positive constants independent of* ε.

REMARKS. (i) The advantages of the matched asymptotic analysis method are that the structure of the solution $(v^\varepsilon, u^\varepsilon, p^\varepsilon)$ in Theorem 1 will be clear, since it is a perturbation of a formal solution which will be constructed explicitly.

(ii) The solutions $(v^\varepsilon, u^\varepsilon, p^\varepsilon)$ have carefully chosen initial data which are essentially those of the Hilbert expansion and the shock-layer expansion.

(iii) In particular, we have that, away from the shock, $(v^{\varepsilon}, u^{\varepsilon}, p^{\varepsilon})$ approximates (v_0, u_0, p_0) at an optimal rate in ε, i.e., (1.10).

(iv) The same results hold for finite noninteracting shocks solutions of (1.2); this is clear from our analysis.

(v) The technique used here can be extended to deal with the general relaxation systems proposed by [5].

It is known (see [1]) that if we can construct a formal solution for (1.1) by matching the truncated Hilbert expansion (outer expansion) and shock-layer expansion (inner expansion), then the existence of solutions to (1.1) and its convergence to the solutions of (1.2) can be reduced to the stability analysis for the approximate solution. Since the dissipation of relaxation is much weaker than viscosity, the limit here is more singular than those in [1], and we need to use the higher-order corrections to weaken the nonlinearity in the error equations between (1.1) and (1.2). Comparing with [10], where the smooth steady shock profile can be constructed explicitly, we only have an abstract result for the existence of shock profiles for (1.1). Due to the fact that the leading order, the time-dependent shock profile, has exactly the shape of a steady shock profile with parameters varying with time, this becomes crucial. However, we can get enough information on shock profiles of (1.1), and then a modified energy estimate method as used in [2] and [4] gives the result.

For the stability analysis for the elementary waves of (1.1), we refer to [2–4, 7].

In the next section, we construct the approximate solutions by use of the matched asymptotic expansion method. The existence and asymptotic behavior of the solutions to (1.1) are proved in Section 3.

2. CONSTRUCTION OF APPROXIMATE SOLUTION

In this section we will construct an approximate solution for (1.1) by using the method of matched asymptotic expansions. The outer solutions come from the Hilbert expansion and the inner solutions are found by shock-layer expansion. By matching the outer and inner solutions on an appropriate "matching zone," we will obtain the various outer and inner functions and form a formal approximate solution for (1.1). For convenience, instead of (1.1), we will use the following equivalent form of (1.1), namely,

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ p_t + Eu_x &= \frac{1}{\varepsilon}\left(p_R(v) - p\right). \end{aligned} \tag{2.1}$$

2.1. *Outer Expansion*

Let $\chi^0(x,t) = (v^0, u^0, p^0)(x,t)$ and $\chi_i(x,t) = (v_i, u_i, p_i)(x,t)$, $i = 0, 1, 2, \dots$. In the zone away from the shock $x = s(t)$, solutions of (2.1) may be approximated by

$$\chi^0(x,t) + \varepsilon\chi_1(x,t) + \varepsilon^2\chi_2(x,t) + \cdots, \qquad x \neq s(t). \tag{2.2}$$

Substituting (2.2) into (2.1) and comparing the coefficients of power ε leads to

$$O(\varepsilon^{-1}) \qquad p_R\big(v^0(x,t)\big) = p^0, \tag{2.3}$$

$$O(1) \qquad \begin{aligned} v_t^0 - u_x^0 &= 0, \\ u_t^0 + p_x^0 &= 0, \\ p_t^0 + Eu_x^0 &= -p_1 + p_R'(v^0)v_1, \end{aligned} \tag{2.4}$$

$$O(\varepsilon) \qquad \begin{aligned} v_{1t} - u_{1x} &= 0, \\ u_{1t} + p_{1x} &= 0, \\ p_{1t} + Eu_{1x} &= -p_2 + p_R'(v^0)v_2 + \tfrac{1}{2}p_R''(v^0)v_1^2, \end{aligned} \tag{2.5}$$

$$O(\varepsilon^2) \qquad \begin{aligned} v_{2t} - u_{2x} &= 0, \\ u_{2t} + p_{2x} &= 0, \end{aligned} \tag{2.6}$$

etc.

From (2.3), $(2.4)_{1,2}$ becomes a closed system for (v^0, u^0), which is identical with (1.2), and its solution can be taken as the piecewise smooth functions described in Section 1. Thus we set

$$\chi^0 = (v_0, u_0, p_0)(x,t) \equiv \chi_0(x,t), \qquad x \neq s(t). \tag{2.7}$$

Next, we can get from $(2.4)_3$ that

$$\begin{aligned} p_1 &= -(p_{0t} + Eu_{0x}) + p_R'(v_0)v_1 \\ &= -(E + p_R'(v_0))u_{0x} + p_R'(v_0)v_1, \qquad x \neq s(t). \end{aligned} \tag{2.8}$$

We conclude from (2.8) that $(2.5)_{1,2}$ becomes the following closed system for $(v_1, u_1)(x,t)$,

$$\begin{aligned} v_{1t} - u_{1x} &= 0, \\ u_{1t} + \big(p_R'(v_0)v_1\big)_x &= \big((E + p_R'(v_0))u_{0x}\big)_x, \qquad x \neq s(t), \end{aligned} \tag{2.9}$$

and (2.8)–(2.9) are equations for $\chi_1(x,t)$.

Similarly, $\chi_2(x,t)$ satisfies

$$p_2 = -(p_{1t} + Eu_{1x}) + p_R'(v_0)v_2 + \tfrac{1}{2}p_R''(v_0)v_1^2, \tag{2.10}$$

$$\begin{aligned} &v_{2t} - u_{2x} = 0, \\ &u_{2t} + \left(p_R'(v_0)v_2\right)_x = \left(p_{1t} + Eu_{1x} - \tfrac{1}{2}p_R''(v_0)v_1^2\right)_x, \qquad x \neq s(t). \end{aligned} \tag{2.11}$$

This process can be continued to find higher-order outer functions $\chi_i(x,t)$, $i \geq 3$. χ_i may be discontinuous at $x = s(t)$ but are expected to be smooth away from the shock uniformly up to $x = s(t)$.

2.2. *Inner Expansion and Matching Conditions*

Near the shock, the solution of (2.1) will be represented by a shock-layer expansion of the form

$$X_0(\xi,t) + \varepsilon X_1(\xi,t) + \varepsilon^2 X_2(\xi,t) + \cdots, \tag{2.12}$$

where $X_i = (V_i, U_i, P_i)$ and ξ is given by

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t,\varepsilon), \tag{2.13}$$

in which $\delta(t,\varepsilon)$ is the perturbation of the shock position which is to be determined later. Assume

$$\delta(t,\varepsilon) = \delta_0(t) + \varepsilon\delta_1(t) + \varepsilon^2\delta_2(t) + \cdots. \tag{2.14}$$

Substituting (2.12)–(2.14) into (2.1) and matching powers of ε, we have

$$O(\varepsilon^{-1}) \qquad \begin{aligned} &-\dot{s}V_{0\xi} - U_{0\xi} = 0, \\ &-\dot{s}U_{0\xi} + P_{0\xi} = 0, \\ &-\dot{s}P_{0\xi} + EU_{0\xi} = p_R(V_0) - P_0, \end{aligned} \tag{2.15}$$

$$O(1) \qquad \begin{aligned} &-\dot{s}V_{1\xi} - U_{1\xi} = -\left(V_{0t} + \dot{\delta}_0(t)V_{0\xi}\right), \\ &-\dot{s}U_{1\xi} + P_{1\xi} = -\left(U_{0t} + \dot{\delta}_0(t)U_{0\xi}\right), \\ &-\dot{s}P_{1\xi} + EU_{1\xi} = -\left(P_{0t} + \dot{\delta}_0(t)P_{0\xi}\right) + p_R'(V_0)V_1 - P_1, \end{aligned} \tag{2.16}$$

$$O(\varepsilon) \qquad \begin{aligned} &-\dot{s}V_{2\xi} - U_{2\xi} = -\left(V_{1t} + \dot{\delta}_0(t)V_{1\xi}\right) + \dot{\delta}_1(t)V_{0\xi}\Big), \\ &-\dot{s}U_{2\xi} + P_{2\xi} = -\left(U_{1t} + \dot{\delta}_0(t)U_{1\xi}\right) + \dot{\delta}_1(t)U_{0\xi}\Big), \\ &-\dot{s}P_{2\xi} + EU_{2\xi} = -\left(P_{1t} + \dot{\delta}_0(t)P_{1\xi} + \dot{\delta}_1(t)P_{0\xi}\right) + \Psi, \end{aligned} \tag{2.17}$$

etc., with

$$\Psi = p_R'(V_0)V_2 - P_2 + \tfrac{1}{2}p_R''(V_0)V_1^2.$$

The inner expansion is assumed to be true in a zone of size $O(\varepsilon)$ around $x = s(t)$.

The outer expansion and the inner expansion are expected to be valid in the "matching zone," in which $|\xi| \to \infty$ and $|x - s(t)|$ is small. Therefore, they must agree there. We can express the outer solutions in terms of ξ and use Taylor's series to find the following "matching conditions" as $\xi \to \mp\infty$,

$$X_0(\xi, t) = \chi_0(s(t) \mp 0, t) + o(1), \tag{2.18}$$

$$X_1(\xi, t) = \chi_1(s(t) \mp 0, t) + (\xi - \delta_0)\chi_{0x}(s(t) \mp 0, t) + o(1), \tag{2.19}$$

$$\begin{aligned} X_2(\xi, t) = {} & \chi_2(s(t) \mp 0, t) + (\xi - \delta_0)\chi_{1x}(s(t) \mp 0, t) \\ & - \delta_1 \chi_{0x}(s(t) \mp 0, t) \\ & + \tfrac{1}{2}(\xi - \delta_0)^2 \chi_{0xx}(s(t) \mp 0, t) + o(1), \end{aligned} \tag{2.20}$$

etc.

Equations (2.18)–(2.20) require that inner functions have algebraic growth rates at both infinities.

2.3. *Constructions of the Outer and Inner Functions*

We construct the outer and inner functions order by order. Simultaneously, the matching conditions will be satisfied, and $\delta(t, \varepsilon)$ will be determined.

The leading order of outer functions, $\chi_0(x, t)$, is the single-shock solution of (1.2) in Theorem 1. For fixed t (taken as a parameter), $X_0(\xi, t)$, determined by (2.15), is exactly the traveling wave solution of (1.1) with the boundary conditions (2.18). Up to phase shift, $X_0(\xi, t)$ can be uniquely determined (see [2] and [4]). Here, since the shift can be absorbed by $\delta(t, \varepsilon)$, we can take it as zero. Although we could not get the explicit formula for $X_0(\xi, t)$ as in [10], we have the following (see [4]).

LEMMA 2.1 (Shock profile). *Under the entropy condition and the subcharacteristic condition*, (2.15) *and* (2.18) *have a smooth solution* $\Phi(v_0^l, \xi, \dot{s}) = (V_0, U_0, P_0)$ *which is unique up to a shift in* ξ *and satisfies* $V_{0\xi} > 0$, *and*

$$|V_{0\xi}| + |U_{0\xi}| + |P_{0\xi}| \le O(1)|v_0^r - v_0^l|,$$

$$|\Phi_\xi| < C_1\big(|v_0^r - v_0^l|\big)\exp(-C_2|\xi|),$$

where C_i, $i = 1, 2$, is a positive constant. Furthermore, as $\xi \to -\infty$, it holds that, for some positive constants C_3, C_4, and C_5,

$$
\begin{aligned}
|\Phi - (v_0^l, u_0^l, p_0^l)| &\le O(1)|v_0^r - v_0^l| \exp(-C_3|\xi|),\\
\left|\frac{\partial \Phi}{\partial v_0^l} - I\right| &\le O(1)\exp(-C_4|\xi|),\\
\left|\frac{\partial \Phi}{\partial \dot{s}}\right| &\le O(1)\exp(-C_5|\xi|).
\end{aligned}
$$

Similar results hold as $\xi \to +\infty$ if we substitute v_0^r for v_0^l and make some revisions.

We now turn to the first-order functions $\chi_1(x,t)$ and $X_1(\xi,t)$. χ_1, X_1, and δ_0 will be determined at the same time.

Integrating $(2.16)_{1,2}$ over $[0, \xi]$, we have

$$
\begin{aligned}
\dot{s}V_1 + U_1 &= \dot{\delta}_0 V_0 + \int_0^\xi V_{0\xi}\, d\xi + c_1(t),\\
\dot{s}U_1 - P_1 &= \dot{\delta}_0 U_0 + \int_0^\xi U_{0\xi}\, d\xi + c_2(t),
\end{aligned}
\tag{2.21}
$$

where $c_1(t)$ and $c_2(t)$ are integration constants to be determined. Equations (2.16) and (2.21) give

$$
\begin{aligned}
P_{1\xi} &= f_1 P_1 - f_2 h_1 + f_3 h_2 + f_4\\
&\equiv f_1 P_1 + Q,
\end{aligned}
\tag{2.22}
$$

where

$$
f_1 = \frac{\dot{s}^2 + p_R'(V_0)}{\dot{s}(\dot{s}^2 - E)}, \qquad f_2 = \frac{p_R'(V_0)}{\dot{s}^2 - E}, \qquad f_3 = \frac{1}{\dot{s}} f_2,
$$

$$
f_4 = \frac{E}{\dot{s}^2 - E}\left(\dot{\delta}_0 U_{0\xi} + U_{0t}\right) + \frac{\dot{s}}{\dot{s}^2 - E}\left(\dot{\delta}_0 P_{0\xi} + P_{0t}\right),
$$

$$
h_1 = \dot{\delta}_0 V_0 + \int_0^\xi V_{0t}\, d\xi + c_1(t),
$$

$$
h_2 = \dot{\delta}_0 U_0 + \int_0^\xi U_{0t}\, d\xi + c_2(t).
$$

From (2.22) we obtain

$$P_1(\xi,t) = \left(\exp\left\{\int_0^{\xi} f_1(\eta,t)\,d\eta\right\}\right) \times\left(\int_0^{\xi}\exp\left\{\int_0^{\eta} -f_1(\lambda,t)\,d\lambda\right\}Q(\eta,t)\,d\eta\right), \tag{2.23}$$

and then X_1 can be determined provided that $c_1(t)$, $c_2(t)$, and $\delta_0(t)$ can be determined, since we have

$$\begin{aligned} U_1 &= \frac{1}{\dot{s}}(P_1 + h_2),\\ V_1 &= -\frac{1}{\dot{s}^2}(P_1 + h_2) + \frac{1}{\dot{s}}h_1. \end{aligned} \tag{2.24}$$

$c_1(t)$, $c_2(t)$, and $\delta_0(t)$ will be determined in such a way that $X_1(\xi,t)$ constructed above satisfies the matching conditions (2.19). Similar to [10], we observe that

LEMMA 2.2. *Equation* $(2.19)_3$ *will be satisfied if* $(2.19)_{1,2}$ *hold.*

Proof. We need to check that, for $\xi \to \mp\infty$, it holds that

$$\begin{aligned} P_1(\xi,t) &= p_1(s(t)\mp 0,t) + (\xi-\delta_0)p_{0x}(s(t)\mp 0,t) + o(1)\\ &= p_1(s(t)\mp 0,t) + (\xi-\delta_0)p_R'(v_0)v_{0x}(s(t)\mp 0,t) + o(1). \end{aligned} \tag{2.25}$$

We only check the case for $\xi \to +\infty$, since the case for $\xi \to -\infty$ is similar. From (2.16), we see that

$$\begin{aligned} P_1 &= \dot{s}P_{1\xi} - EU_{1\xi} + p_R(V_0)V_1 - \dot{\delta}_0 P_{0\xi} - P_{0t}\\ &= (\dot{s}^2 - E)U_{1\xi} - \dot{s}\dot{\delta}_0 U_{0\xi} - \dot{s}U_{0t} + p_R(V_0)V_1 - \dot{\delta}_0 P_{0\xi} - P_{0t}. \end{aligned}$$

As $\xi \to +\infty$, using $(2.19)_{1,2}$ and (2.18), we have

$$\begin{aligned} P_1(\xi,t) &= (\dot{s}^2 - E)u_{0x}^r(t) - \dot{s}\dot{u}_0^r(t) - \dot{p}_0^r\\ &\qquad + p_R'(v_0^r)\left(v_1^r + (\xi-\delta_0)v_{0x}^r\right) + o(1)\\ &= \left(-Eu_{0x}^r - p_{0t}^r + p_R'(v_0^r)v_1^r\right) + (\xi-\delta_0)p_R'(v_0^r)v_{0x}^r + o(1)\\ &= p_1(s(t)+0,t) + (\xi-\delta_0)p_R'(v_0)v_{0x}(s(t)+0,t) + o(1), \end{aligned}$$

where we have used (2.8) and the relations $\dot{f}^r = f_t^r + \dot{s} f_x^r$. The proof is completed.

In the following, we will also use the notations $f^r(t) \equiv \lim_{\xi \to +\infty} f(\xi, t)$ and $f^l(t) \equiv \lim_{\xi \to -\infty} f(\xi, t)$. It is clear that these are identical to the original definition. Due to the entropy condition, we see that

$$\dot{s}^2 + p_R'\big(v_0^l\big) < 0, \qquad \dot{s}^2 + p_R'\big(v_0^r\big) > 0,$$

which, with Lemma 2.1 and the subcharacteristic condition, imply that

$$f_1(\xi, t) = \begin{cases} f_1^l(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to -\infty, \\ f_1^r(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to +\infty, \end{cases} \tag{2.26}$$

and

$$f_1^l(t) = \frac{\dot{s}^2 + p_R'\big(v_0^l\big)}{\dot{s}(\dot{s}^2 - E)} > 0, \qquad f_1^r(t) = \frac{\dot{s}^2 + p_R'\big(v_0^r\big)}{\dot{s}(\dot{s}^2 - E)} < 0, \tag{2.27}$$

where $\alpha_0 > 0$ is a suitable constant.

By Lemma 2.1, we see that

$$X_{0t}(\xi, t) = \begin{cases} \dot{\chi}_0^l(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to -\infty, \\ \dot{\chi}_0^r(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to +\infty. \end{cases} \tag{2.28}$$

Then we can get the asymptotic behavior of $V_1(\xi, t)$ and $U_1(\xi, t)$ as follows,

$$\begin{aligned} V_1^l &= \frac{1}{A^l}\Big(\dot{\delta}_0\big(\dot{s}v_0^l - u_0^l\big) + \xi\big(\dot{s}\dot{v}_0^l - \dot{u}_0^l\big) + \dot{s}c_1(t) - c_2(t)\Big) \\ &\quad + O_{11}(t) + O(1)\exp\{-\alpha_0|\xi|\}, \\ V_1^r &= \frac{1}{A^r}\Big(\dot{\delta}_0\big(\dot{s}v_0^r - u_0^r\big) + \xi\big(\dot{s}\dot{v}_0^r - \dot{u}_0^r\big) + \dot{s}c_1(t) - c_2(t)\Big) \\ &\quad + O_{12}(t) + O(1)\exp\{-\alpha_0|\xi|\}, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} U_1^l &= \frac{\dot{s}}{A^l}\Big(\dot{\delta}_0\big(B^l v_0^l + u_0^l\big) + \xi\big(B^l \dot{v}_0^l + \dot{u}_0^l\big) + B^l c_1(t) + c_2(t)\Big) \\ &\quad + O_{21}(t) + O(1)\exp\{-\alpha_0|\xi|\}, \\ U_1^r &= \frac{\dot{s}}{A^r}\Big(\dot{\delta}_0\big(B^r v_0^r + u_0^r\big) + \xi\big(B^r \dot{v}_0^r + \dot{u}_0^r\big) + B^r c_1(t) + c_2(t)\Big) \\ &\quad + O_{22}(t) + O(1)\exp\{-\alpha_0|\xi|\}, \end{aligned} \tag{2.30}$$

where

$$A = \dot{s}^2 + p_R'(V_0), \qquad B = \frac{p_R'(V_0)}{\dot{s}}, \tag{2.31}$$

and O_{11}, O_{12}, O_{21}, and O_{22} are known functions.

From (1.2), we have

$$\begin{aligned} v_{0x}^l &= \frac{\dot{s}\dot{v}_0^l - \dot{u}_0^l}{A^l}, \qquad v_{0x}^r = \frac{\dot{s}\dot{v}_0^r - \dot{u}_0^l}{A^r}, \\ u_{0x}^l &= \frac{\dot{s}}{A^l}\left(B^l\dot{v}_0^l + \dot{u}_0^l\right), \qquad u_{0x}^r = \frac{\dot{s}}{A^r}\left(B^r\dot{v}_0^r + \dot{u}_0^r\right). \end{aligned} \tag{2.32}$$

Now, we set $o(1) = O(1)\exp\{-\alpha_0|\xi|\}$ in (2.19). The matching condition $(2.19)_{1,2}$ will be satisfied provided that we can choose $c_1(t)$ and $c_2(t)$ such that

$$v_1^l - \delta_0 v_{0x}^l = \frac{1}{A^l}\left(\dot{\delta}_0\left(\dot{s}v_0^l - u_0^l\right) + \dot{s}c_1(t) - c_2(t)\right) + O_{11}(t),$$

$$v_1^r - \delta_0 v_{0x}^r = \frac{1}{A^r}\left(\dot{\delta}_0\left(\dot{s}v_0^r - u_0^r\right) + \dot{s}c_1(t) - c_2(t)\right) + O_{12}(t), \tag{2.33}$$

and

$$\begin{aligned} u_1^l - \delta_0 u_{0x}^l &= \frac{\dot{s}}{A^l}\left(\dot{\delta}_0\left(B^l v_0^l + u_0^l\right) + B^l c_1(t) + c_2(t)\right) + O_{21}(t), \\ u_1^r - \delta_0 u_{0x}^r &= \frac{\dot{s}}{A^r}\left(\dot{\delta}_0\left(B^r v_0^r + u_0^r\right) + B^r c_1(t) + c_2(t)\right) + O_{22}(t). \end{aligned} \tag{2.34}$$

By use of (2.32), (2.33)–(2.34) are reduced to

$$\dot{s}c_1(t) - c_2(t) = A^l v_1^l - \delta_0\left(\dot{s}\dot{v}_0^l - \dot{u}_0^l\right) - \dot{\delta}_0\left(\dot{s}v_0^l - u_0^l\right) - A^l O_{11}(t), \tag{2.35}$$

$$\dot{s}c_1(t) - c_2(t) = A^r v_1^r - \delta_0\left(\dot{s}\dot{v}_0^r - \dot{u}_0^r\right) - \dot{\delta}_0\left(\dot{s}v_0^r - u_0^r\right) - A^r O_{12}(t), \tag{2.36}$$

$$B^l c_1(t) + c_2(t) = \frac{A^l}{\dot{s}}u_1^l - \delta_0\left(B^l\dot{v}_0^l + \dot{u}_0^l\right) - \dot{\delta}_0\left(B^l v_0^l + u_0^l\right) - \frac{A^l}{\dot{s}}O_{21}(t), \tag{2.37}$$

$$B^r c_1(t) + c_2(t) = \frac{A^r}{\dot{s}}u_1^r - \delta_0\left(B^r\dot{v}_0^r + \dot{u}_0^r\right) - \dot{\delta}_0\left(B^r v_0^r + u_0^r\right) - \frac{A^r}{\dot{s}}O_{21}(t). \tag{2.38}$$

We can solve (2.35) and (2.37) for $c_1(t)$ and $c_2(t)$ to get

$$\begin{aligned} c_1(t) &= \left(\dot{s}v_1^l + u_1^l\right) - \delta_0\dot{v}_0^l - \dot{\delta}_0 v_0^l - O_{31}(t), \\ c_2(t) &= \dot{s}\left(u_1^l - B^l v_1^l\right) - \delta_0\dot{u}_0^l - \dot{\delta}_0 u_0^l - O_{32}(t), \end{aligned} \tag{2.39}$$

where $O_{31}(t)$ and $O_{32}(t)$ are known functions. Similarly, we know from (2.36) and (2.38) that

$$\begin{aligned} c_1(t) &= \left(\dot{s}v_1^r + u_1^r\right) - \delta_0\dot{v}_0^r - \dot{\delta}_0 v_0^r - O_{41}(t), \\ c_2(t) &= \dot{s}\left(u_1^r - B^r v_1^r\right) - \delta_0\dot{u}_0^r - \dot{\delta}_0 u_0^r - O_{42}(t). \end{aligned} \tag{2.40}$$

Thus, the compatibility condition is

$$\begin{aligned} \dot{s}\left(v_1^l - v_1^r\right) + \left(u_1^l - u_1^r\right) - \dot{\delta}_0\left(v_0^l - v_0^r\right) - \delta_0\left(\dot{v}_0^l - \dot{v}_0^r\right) + O_{51}(t) &= 0, \\ \dot{s}\left(u_1^l - u_1^r\right) - \dot{s}\left(B^l v_1^l - B^r v_1^r\right) - \dot{\delta}_0\left(u_0^l - u_0^r\right) - \delta_0\left(\dot{u}_0^l - \dot{u}_0^r\right) + O_{52}(t) &= 0, \end{aligned} \tag{2.41}$$

where $O_{51}(t)$ and $O_{52}(t)$ are known functions.

Introducing

$$e_{11} \equiv -\left[\dot{s}v_1 + u_1\right] = \dot{s}\left(v_1^l - v_1^r\right) + \left(u_1^l - u_1^r\right), \tag{2.42}$$

$$e_{12} \equiv -\left[\dot{s}u_1 - p_R'(V_0)v_1\right] = \dot{s}\left(u_1^l - u_1^r\right) - \dot{s}\left(B^l v_1^l - B^r v_1^r\right), \tag{2.43}$$

(2.41) becomes

$$\begin{aligned} e_{11} &= \delta_0\left(\dot{v}_0^l - \dot{v}_0^r\right) + \dot{\delta}_0\left(v_0^l - v_0^r\right) - O_{51}(t), \\ e_{12} &= \delta_0\left(\dot{u}_0^l - \dot{u}_0^r\right) + \dot{\delta}_0\left(u_0^l - u_0^r\right) - O_{52}(t) \\ &= -\dot{s}e_{11} - \ddot{s}\left(v_0^l - v_0^r\right)\delta_0 - O_{52}(t). \end{aligned} \tag{2.44}$$

Now we see that the matching conditions $(2.19)_{1,2}$ will be satisfied if the boundary values crossing the shock for $\chi_1(x,t)$ satisfy (2.44).

Next, we will show (2.44) is exactly the relation between the boundary data of $\chi_1(x,t)$ required in solving the initial boundary value problems for linear hyperbolic equations (2.9) on Ω_+ and Ω_-, respectively, with

$$\begin{aligned} \Omega_- &= \{(x,t)\colon x < s(t), 0 \le t \le T\}, \\ \Omega_+ &= \{(x,t)\colon x > s(t), 0 \le t \le T\}. \end{aligned}$$

This is now routine by the standard theory. The system (2.9)

$$\begin{aligned} v_{1t} - u_{1x} &= 0, \\ u_{1t} + \left(p_R'(v_0)v_1\right)_x &= \left(\left(E + p_R'(v_0)\right)u_{0x}\right)_x \end{aligned}$$

has eigenvalues $\lambda_1(v_0) = -\sqrt{-p_R'(v_0)}$ and $\lambda_2(v_0) = \sqrt{-p_R'(v_0)}$, with corresponding right eigenvectors $r_1 = (-1, \lambda_1)^t$ and $r_2 = (-1, \lambda_2)^t$, respectively.

Setting

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = M \begin{pmatrix} n_1 \\ z_1 \end{pmatrix} \tag{2.45}$$

with

$$M = \begin{pmatrix} -1 & -1 \\ \lambda_1 & \lambda_2 \end{pmatrix},$$

we can diagonalize the system to obtain

$$\begin{aligned} &\begin{pmatrix} n_1 \\ z_1 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} n_1 \\ z_1 \end{pmatrix}_x \\ &\qquad = -M^{-1}(M_t + (JM)_x) \begin{pmatrix} n_1 \\ z_1 \end{pmatrix} + M^{-1} \begin{pmatrix} 0 \\ ((E + p_R'(v_0))u_{0x})_x \end{pmatrix}, \end{aligned} \tag{2.46}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ p_R'(v_0) & 0 \end{pmatrix}.$$

By the characteristic method, with the help of the entropy condition, we see that z_1^l, z_1^r, and n_1^r will be determined by integrating along appropriate characteristics, and only n_1^l needs to be specified at $x = s(t)$. This boundary condition can be obtained by (2.44). Rewrite (2.44) in terms of (n_1, z_1) as

$$\begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = \left[(-\dot{s}M + JM) \begin{pmatrix} n_1 \\ z_1 \end{pmatrix} \right], \tag{2.47}$$

for which the solvability condition is

$$\begin{aligned} \lambda^l e_{11} + e_{12} &= z_1^l(\lambda_2^l - \lambda_1^l)(\dot{s} - \lambda_2^l) + n_1^r(\dot{s} - \lambda_1^r)(\lambda_1^l - \lambda_1^r) \\ &\qquad + z_1^r(\dot{s} - \lambda_2^r)(\lambda_1^l - \lambda_2^r) \\ &\equiv F_1(t). \end{aligned} \tag{2.48}$$

If (2.48) is satisfied, one can solve (2.47) to obtain

$$n_1^l = (\dot{s} - \lambda_1^l)^{-2}\left(n_1^r(\dot{s} - \lambda_1^r)^2 + z_1^r(\dot{s} - \lambda_2^r)^2 - z_1^l(\dot{s} - \lambda_2^l)^2\right) + \ddot{s}(v_0^l - v_0^r)\delta_0 + O_{53}(t), \tag{2.49}$$

where we have used (2.44). By (2.44), we also can reduce (2.48) into

$$(\lambda_1^l - \dot{s})\frac{d}{dt}\{\delta_0(v_0^l - v_0^r)\} - \ddot{s}(v_0^l - v_0^r)\delta_0 = F_1(t) + O_{53}(t). \tag{2.50}$$

This is a linear ordinary differential equation on $\delta_0(v_0^l - v_0^r)$, which completely determines δ_0 up to a constant. Then we get n_1^l by (2.49). The standard theory for mixed problems for linear hyperbolic systems gives the smooth solutions to (2.9) over Ω_- and Ω_+, respectively. The outer functions $\chi_1(x, t)$ are determined then.

By the constructions of χ_1 and δ_0, the inner functions $X_1(\xi, t)$ are obtained, and $(2.19)_{1,2}$ are satisfied as well. We collect all the results obtained so far as the following.

THEOREM 2.3. *$\chi_1(x,t)$, $X_1(\xi,t)$, and δ_0 can be determined such that*

(i) *$\chi_1(x,t)$ and its derivatives are uniformly continuous up to $x = s(t)$ and*

$$\sum_{|\alpha|\le 2} \iint_{x \ne s(t)} |\partial_x^\alpha \chi_1(x,t)|^2\, dx\, dt < +\infty, \tag{2.51}$$

(ii) *$X_1(\xi,t)$ is smooth and, for some $\alpha_0 > 0$, we have*

$$X_1(\xi,t) = \chi_1(s(t) \mp 0, t) + (\xi - \delta_0)\chi_{0x}(s(t) \mp 0, t) + O(1)\exp\{-\alpha_0|\xi|\}, \qquad \text{as } \xi \to \mp\infty. \tag{2.52}$$

It is easy to see that the above procedure can be carried out to any order. Especially, we can construct $\chi_2(x,t)$, $X_2(\xi,t)$, and δ_1 such that similar properties as in Theorem 2.3 hold.

2.4. *Approximate Solutions*

Now we construct a smooth approximate solution to (1.1) by patching the inner and outer solutions discussed.

Set

$$I(x,t) = \left(X_0 + \varepsilon X_1 + \varepsilon^2 X_2\right)\left(\frac{x - s(t)}{\varepsilon} + \delta_0 + \varepsilon\delta_1, t\right) \quad (2.53)$$

and

$$O(x,t) = \left(\chi_0 + \varepsilon\chi_1 + \varepsilon^2\chi_2\right)(x,t), \qquad x \neq s(t). \quad (2.54)$$

Let $m(y) \in C_0^\infty(\mathbf{R}^1)$ such that $0 \le m(y) \le 1$ and

$$m(y) = \begin{cases} 1, & |y| < 1, \\ 0, & |y| \ge 2. \end{cases} \quad (2.55)$$

Take $\alpha \in (\frac{2}{3}, 1)$ to be a constant. We define

$$S^\varepsilon(x,t) = m\left(\frac{x - s(t)}{\varepsilon^\alpha}\right) I(x,t) + \left(1 - m\left(\frac{x - s(t)}{\varepsilon^\alpha}\right)\right) O(x,t) + d(x,t), \quad (2.56)$$

where $d(x,t) = (d_1, d_2, d_3)^t(x,t)$ is a higher-order correction to be determined. We use the following notations:

$$S^\varepsilon = (v^\varepsilon, u^\varepsilon, p^\varepsilon)^t, \qquad I = (I_1, I_2, I_3)^t, \qquad O = (O_1, O_2, O_3)^t.$$

Using the structure of the various orders of inner and outer solutions, we have

$$\begin{aligned} v_t^\varepsilon - u_x^\varepsilon &= F_2 + d_{1t} - d_{2x}, \\ u_t^\varepsilon + p_x^\varepsilon &= F_3 + d_{2t} + d_{3x}, \\ p_t^\varepsilon + Eu_x^\varepsilon - q(S^\varepsilon) &= F_4 + d_{3t} + Ed_{2x} + \Delta, \end{aligned} \quad (2.57)$$

where $q(f) = \frac{1}{\varepsilon}(p_R(f_1) - f_3)$ for $f = (f_1, f_2, f_3)^t$, and

$$\begin{aligned} F_2 &= \varepsilon^2 m\left(\dot\delta_1 V_{1\xi} + \left(\dot\delta_0 + \varepsilon\dot\delta_1\right)V_{2\xi} + V_{2t}\right) \\ &\quad + m_t(I_1 - O_1) - m_x(I_2 - O_2), \\ F_3 &= \varepsilon^2 m\left(\dot\delta_1 U_{1\xi} + \left(\dot\delta_0 + \varepsilon\dot\delta_1\right)U_{2\xi} + U_{2t}\right) \\ &\quad + m_t(I_2 - O_2) + m_x(I_3 - O_3), \end{aligned}$$

$$F_4 = \varepsilon^2 m\Big(\dot{\delta}_1 P_{1\xi} + \big(\dot{\delta}_0 + \varepsilon\dot{\delta}_1\big)P_{2\xi} + P_{2t}\Big) + m_t(I_3 - O_3) + Em_x(I_2 - O_2),$$

$$\Delta = -m\Delta_1 - (1-m)\Delta_2 - \Delta_3 + (1-m)(p_{2t} + Eu_{2x}),$$

$$\Delta_1 = \frac{1}{\varepsilon}\Big\{p_R\big(V_0 + \varepsilon V_1 + \varepsilon^2 V_2\big) - \Big(p_R(V_0) + \varepsilon p_R'(V_0)V_1 + \varepsilon^2\big(p_R'(V_0)V_2 + \tfrac{1}{2}p_R''(V_0)V_1^2\big)\Big)\Big\},$$

$$\Delta_2 = \frac{1}{\varepsilon}\Big\{p_R\big(v_0 + \varepsilon v_1 + \varepsilon^2 v_2\big) - \Big(p_R(v_0) + \varepsilon p_R'(v_0)v_1 + \varepsilon^2\big(p_R'(v_0)v_2 + \tfrac{1}{2}p_R''(v_0)v_1^2\big)\Big)\Big\},$$

$$\Delta_3 = \big(q(S^\varepsilon) - q(S^\varepsilon - d)\big) + \big(q(mI + (1-m)O) - (mq(I) + (1-m)q(O))\big).$$

By Taylor's expansion, we have

$$\Delta_1 = O(1)\varepsilon^2, \qquad \Delta_2 = O(1)\varepsilon^2.$$

Due to the construction, we see that

$$\frac{\partial^k}{\partial x^k}(I-O)(x,t) = O(1)\varepsilon^{(3-k)\alpha}, \qquad \text{in } \varepsilon^\alpha < |x - s(t)| < 2\varepsilon^\alpha,$$

which can be obtained from the matching conditions (2.18)–(2.20).

Choose (d_1, d_2, d_3) such that

$$\begin{aligned} d_{1t} - d_{2x} &= -F_2, \\ d_{2t} + d_{3x} &= -F_3, \\ d_3 &= C_0 d_1, \\ d_1(x,0) = d_2(x,0) &= 0, \end{aligned} \tag{2.58}$$

with $C_0 = 2\max_{t\in[0,T]}\lambda_2^l(t)$. In view of the facts

(i) F_2, F_3, and m have their support in $\{(x,t)\colon |x - s(t)| \le 2\varepsilon^\alpha, 0 \le t \le T\}$,

(ii) $|(\partial^k/\partial x^k)(F_2, F_3)(x,t)| \le O(1)\varepsilon^{(2-k)\alpha}$, we have, by a standard characteristic argument, that

Lemma 2.4. *The solution (d_1, d_2) to (2.58) has compact support and satisfies, for integers $2 \ge k \ge 0$,*

$$\left|\frac{\partial^k}{\partial x^k} d\right| \le O(1)\varepsilon^{(3-k)\alpha}, \qquad \forall (x,t) \in \mathbf{R}^1 \times [0,T]. \tag{2.59}$$

Thus, we can estimate Δ_3 by use of the mean value theorem to get

$$\left|\frac{\partial^k}{\partial x^k}\Delta_3\right| \le O(1)\,\varepsilon^{(3-k)\alpha-1}. \tag{2.60}$$

We conclude that

THEOREM 2.5. *Let $S^\varepsilon(x,t)$ be the smooth function defined in* (2.56) *with $d(x,t)$ determined in* (2.58). *Then S^ε satisfies*

$$\begin{aligned} v_t^\varepsilon - u_x^\varepsilon &= 0,\\ u_t^\varepsilon + p_x^\varepsilon &= 0,\\ p_t^\varepsilon + Eu_x^\varepsilon - q(S^\varepsilon) &= R(x,t), \end{aligned} \tag{2.61}$$

with $R(x,t) = F_4 + d_{3t} + Ed_{2x} + \Delta$ and

$$\begin{aligned} \left|\frac{\partial^k}{\partial x^k}R\right| &\le O(1)\,\varepsilon^{(3-k)\alpha-1},\\ \int_0^T\int_{-\infty}^{+\infty}\left|\frac{\partial^k}{\partial x^k}R\right|^2 dx\,dt &\le O(1)\,\varepsilon^{2(3-k)\alpha-2}. \end{aligned} \tag{2.62}$$

This finishes the construction of the formal approximate solution to (1.1).

3. STABILITY ANALYSIS

We now prove that there exists an exact solution to (1.1) in a neighborhood of our approximate solution $S^\varepsilon(x,t)$, and, for ε sufficiently small, the asymptotic behavior of the solution to (1.1) is governed by $S^\varepsilon(x,t)$.

Let $\overline{S}^\varepsilon$ be an exact solution to (1.1) with initial data $\overline{S}^\varepsilon(x,0) = S^\varepsilon(x,0)$. We decompose the solution as

$$\overline{S}^\varepsilon(x,t) = S^\varepsilon(x,t) + \left(\overline{\phi},\overline{\psi},\overline{w}\right)^t(x,t), \qquad (x,t)\in \mathbf{R}^1\times[0,T]. \tag{3.1}$$

It is easy to show that

$$\begin{aligned} \overline{\phi}_t - \overline{\psi}_x &= 0,\\ \overline{\psi}_t + \overline{w}_x &= 0,\\ \overline{w}_t + E\overline{\psi}_x &= \tfrac{1}{\varepsilon}\left(p_R(v^\varepsilon+\overline{\phi}) - p_R(v^\varepsilon) - \overline{w}\right) - R(x,t),\\ \overline{\phi}(x,0) = \overline{\psi}(x,0) &= \overline{w}(x,0) = 0. \end{aligned} \tag{3.2}$$

Setting

$$\overline{\phi} = \tilde{\phi}_x, \qquad \overline{\psi} = \tilde{\psi}_x, \qquad \overline{w} = \tilde{w}, \tag{3.3}$$

we have

$$\begin{aligned}
\tilde{\phi}_t - \tilde{\psi}_x &= 0, \\
\tilde{\psi}_t + \tilde{w} &= 0, \\
\tilde{w}_t + E\tilde{\psi}_{xx} &= \tfrac{1}{\varepsilon}\Big(p_R\big(v^\varepsilon + \tilde{\phi}_x\big) - p_R(v^\varepsilon) - \tilde{w}\Big) - R(x,t), \\
\tilde{\phi}(x,0) &= \tilde{\psi}(x,0) = \tilde{w}(x,0) = 0,
\end{aligned}$$

which implies

$$\begin{aligned}
\tilde{\phi}_t - \tilde{\psi}_x &= 0, \\
\tilde{\psi}_{tt} - E\tilde{\psi}_{xx} &= -\tfrac{1}{\varepsilon}\Big(p_R\big(v^\varepsilon + \tilde{\phi}_x\big) - p_R(v^\varepsilon) + \tilde{\psi}_t\Big) + R(x,t), \\
\tilde{\phi}(x,0) &= \tilde{\psi}(x,0) = \tilde{\psi}_t(x,0) = 0.
\end{aligned} \tag{3.4}$$

Using the scalings

$$\tilde{\phi} = \varepsilon\phi, \qquad \tilde{\psi} = \varepsilon\psi, \tag{3.5}$$

and

$$y = \frac{x - s(t)}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon}, \tag{3.6}$$

we simplify (3.4) into

$$\begin{aligned}
L_1(\phi,\psi) &= 0, \\
L_2(\phi,\psi) &= F(x,t), \\
\phi(y,0) &= \psi(y,0) = \psi_\tau(y,0) = 0,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
L_1 &= \phi_\tau - \dot{s}\phi_y - \psi_y, \\
L_2 &= \big(\psi_\tau - \dot{s}\psi_y\big)_\tau - \dot{s}\big(\psi_\tau - \dot{s}\psi_y\big)_y - E\psi_{yy} + \big(\psi_\tau - \dot{s}\psi_y\big) - D\phi_y, \\
D &= -p_R'(v^\varepsilon), \\
F &= \varepsilon R - \Big(p_R\big(v^\varepsilon + \phi_y\big) - p_R(v^\varepsilon) - p_R'(v^\varepsilon)\phi_y\Big).
\end{aligned} \tag{3.8}$$

To study the existence and the asymptotic behavior of $\bar{S}^{\varepsilon}(x,t)$ for ε small, it is sufficient to show that, for ε sufficiently small, (3.7) has a smooth "small" solution up to $\tau = \frac{T}{\varepsilon}$. This will be carried out by an argument similar to the stability anlaysis for the shock profiles of (1.1) (see [2] and [8]). The different part is that S^{ε} depends on ε and t here.

In what follows, we use H^l $(l \geq 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$, and $\|\cdot\|$ denotes the usual L^2-norm. We also use the following notation for simplicity:

$$\|(g_1, g_2, \dots, g_k)\|_m^2 \equiv \sum_{i=1}^{k} \|g_i\|_m^2.$$

Let us define the solution space of (3.7) by

$$X(0,\tau_0) = \left\{(\phi,\psi) \in C^0(0,T;H^2),\ \psi_\tau \in C^0(0,T;H^1)\right\}, \tag{3.9}$$

with $0 < \tau_0 \leq \frac{T}{\varepsilon}$. Suppose that for some $0 < \tau_0 \leq \frac{T}{\varepsilon}$, there exists a solution (ϕ, ψ) to (3.7), such that $(\phi,\psi) \in X(0,\tau_0)$. Denote the norm for (ϕ,ψ) by

$$N^2(\tau) = \sup_{0 \leq s \leq \tau} \left(\|(\phi,\psi)(s)\|_2^2 + \|\psi_t(s)\|_1^2\right). \tag{3.10}$$

The main result in this section is the following a priori estimate.

THEOREM 3.1. *Suppose* (H_1)–(H_3) *are satisfied. There exist positive constants* ε_0, η_0, δ_0, *and* K_0 *which are independent of* ε *and* τ_0 *such that, if*

(i) $0 < \varepsilon \leq \varepsilon_0$,
(ii) $|v_0^r - v_0^l| + |u_0^r - u_0^l| + |p_0^r - p_0^l| \leq \eta_0$,
(iii) $N(\tau_0) \leq \delta_0$,

then it holds that

$$\sup_{0 \leq \tau \leq \tau_0} \left(\|(\phi,\psi)(\tau)\|_2^2 + \|\psi_\tau(\tau)\|_1^2\right) + \int_0^{\tau_0} \left(\|(\phi_y,\psi_y)(\tau)\|_1^2 + \|\psi_\tau(\tau)\|_1^2\right) d\tau \leq K_0 \varepsilon^{6\alpha - 3}, \tag{3.11}$$

for $(\phi,\psi) \in X(0,\tau_0)$.

Before making the energy estimate, we derive some properties of S^{ε}.

LEMMA 3.2. *Let $S^{\varepsilon}(x,t)$ be defined as in* (2.56). *Then it holds that*

$$\text{(i)}\quad S^{\varepsilon}(x,t) = \begin{cases} \chi_0 + O(1)\,\varepsilon^{\min(3\alpha,1)}, & |x-s(t)| \ge \varepsilon^{\alpha}, \\ X_0 + O(1)\,\varepsilon^{\alpha}, & |x-s(t)| \le 2\varepsilon^{\alpha}, \end{cases}$$

$$\text{(ii)}\qquad S_y^{\varepsilon}(y,t) = mX_{0y} + O(1)\,\varepsilon, \qquad S_{\tau}^{\varepsilon} = O(1)\,\varepsilon.$$

Proof. This lemma can be shown by the construction of S^{ε}. Similar lemmas can be found in [1] and [10]. The proof is the same (see Lemma 4.2 in [10], for instance) and the details will be omitted.

Now we proceed with the energy estimates. First, we establish the following basic energy estimate.

LEMMA 3.3. *Suppose the conditions in Theorem* 3.1 *are satisfied*; *then*, *for all $\tau \in [0, \tau_0]$, we have*

$$\begin{aligned}\|(\phi,\psi)(\tau)\|_1^2 &+ \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2 + \int_0^\tau \|(\phi_y,\psi_y,\psi_\tau - \dot{s}\psi_y)(s)\|^2\,ds \\ &+ \int_0^\tau\int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\,dy\,ds \\ &\le O(1)\,\varepsilon^{6\alpha-3}.\end{aligned}$$

Proof. By the Sobolev embedding theorem, we know that $v^{\varepsilon} + \phi_y$ is bounded. Thus we can choose $E_1 = \sup|p_R'(v^{\varepsilon} + \phi_y)| < E$ in view of (H_2). We consider the equality

$$\begin{aligned}(\phi + \mu\psi_y)L_1 &+ D^{-1}\big(\mu(\psi_\tau - \dot{s}\psi_y) + \psi\big)L_2 \\ &= D^{-1}\big(\psi + \mu(\psi_\tau - \dot{s}\psi_y)\big)F, \end{aligned} \tag{3.12}$$

with $1 < \mu = (E+E_1)/2E_1 < E/E_1$. Equation (3.12) can be reduced into

$$[G_1 + G_2]_\tau + G_3 + G_4 = D^{-1}\big(\psi + \mu(\psi_\tau - \dot{s}\psi_y)\big)F, \tag{3.13}$$

with

$$G_1 = \frac{1}{2}\phi^2 + \mu\phi\psi_y + \frac{\mu E}{2}D^{-1}\psi_y^2,$$

$$G_2 = \frac{1}{2}\big(D^{-1} + \dot{s}D_y^{-1}\big)\psi^2 + \frac{\mu}{2}D^{-1}\big(\psi_\tau - \dot{s}\psi_y\big)^2 + D^{-1}\psi\big(\psi_\tau - \dot{s}\psi_y\big),$$

$$G_3 = \left[(\mu-1)D^{-1} - \frac{1}{2}\mu D_\tau^{-1} + \frac{1}{2}\dot{s}D_y^{-1}\right]\big(\psi_\tau - \dot{s}\psi_y\big)^2$$

$$+\left[ED^{-1} - \frac{1}{2}\mu ED_\tau^{-1} + \frac{1}{2}\mu ED_y^{-1} - \mu\right]\psi_y^2$$
$$+ \mu ED_y^{-1}\left(\psi_\tau - \dot{s}\psi_y\right)\psi_y,$$
$$G_4 = \frac{1}{2}\dot{s}D_y^{-1}\psi^2 - \frac{1}{2}D_\tau^{-1}\psi^2 - D_\tau^{-1}\psi\left(\psi_\tau - \dot{s}\psi_y\right) + \dot{s}D_\tau^{-1}\psi\psi_y + \{\cdots\}_y,$$

where $\{\cdots\}_y$ denotes the terms which disappear after integrations with respect to y.

We see from Lemma 3.2 that

$$\begin{aligned} D_y^{-1} &= D^{-2}p_R''(v^\varepsilon)\left(mV_{0y} + O(1)\varepsilon\right), \\ D_\tau^{-1} &= O(1)\varepsilon. \end{aligned} \tag{3.14}$$

Thus

$$D_y^{-1} = O(1)(\eta_0 + \varepsilon). \tag{3.15}$$

For η_0 and ε_0 suitably small, there exist positive constants a_1, a_2, b_1, b_2, and b_3 such that

$$\begin{gathered} a_1\left(\phi^2 + \psi_y^2\right) \le G_1 \le b_1\left(\phi^2 + \psi_y^2\right), \\ a_2\left(\psi^2 + \left(\psi_\tau - \dot{s}\psi_y\right)^2\right) \le G_2 \le b_2\left(\psi^2 + \left(\psi_\tau - \dot{s}\psi_y\right)^2\right), \\ G_3 \ge b_3\left(\psi_y^2 + \left(\psi_\tau - \dot{s}\psi_y\right)^2\right). \end{gathered} \tag{3.16}$$

We now estimate G_4. By (3.14)–(3.15), we have

$$\tfrac{1}{2}\dot{s}D_y^{-1}\psi^2 \ge b_4 mV_{0y}\psi^2 + O(1)\varepsilon\psi^2, \tag{3.17}$$

$$\begin{aligned} &\left| -\tfrac{1}{2}D_\tau^{-1}\psi^2 - D_\tau^{-1}\psi\left(\psi_\tau - \dot{s}\psi_y\right) + \dot{s}D_\tau^{-1}\psi\psi_y\right| \\ &\quad \le O(1)\varepsilon\left(\psi^2 + \psi_y^2 + \left(\psi_\tau - \dot{s}\psi_y\right)^2\right) \end{aligned} \tag{3.18}$$

for a positive constant b_4.

We integrate (3.13) over $[0, \tau] \times (-\infty, +\infty)$ to obtain, with the help of (3.16)–(3.18), that

$$\begin{aligned} &\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|\left(\psi_\tau - \dot{s}\psi_y\right)(\tau)\|^2 \\ &\quad + \int_0^\tau \|\left(\psi_y, \psi_\tau - \dot{s}\psi_y\right)(s)\|^2\, ds + \int_0^\tau\int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\, dy\, ds \\ &\quad \le O(1)\varepsilon\int_0^\tau\left(\|\psi(s)\|_1^2 + \|\left(\psi_\tau - \dot{s}\psi_y\right)(s)\|^2\right) ds \\ &\quad\quad + O(1)\left|\int_0^\tau\int_{-\infty}^{+\infty} D^{-1}\left(\psi + \mu\left(\psi_\tau - \dot{s}\psi_y\right)\right)F\, dy\, ds\right|. \end{aligned} \tag{3.19}$$

By using the facts

$$F = \varepsilon R + O(1)\,\phi_y^2,$$

$$\int_0^\tau \int_{-\infty}^{+\infty} |D^{-1}\big(\psi + \mu(\psi_\tau - \dot{s}\psi_y)\big)|\phi_y^2\,dy\,ds \le O(1)N(\tau)\int_0^\tau \|\phi_y(s)\|^2\,ds,$$

$$\int_0^\tau \int_{-\infty}^{+\infty} |D^{-1}\big(\psi + \mu(\psi_\tau - \dot{s}\psi_y)\big)\varepsilon R|\,dy\,ds$$
$$\le O(1)\,\varepsilon\int_0^\tau \Big(\|\psi(s)\|^2 + \|\big(\psi_\tau - \dot{s}\psi_y\big)(s)\|^2\Big)\,ds + \varepsilon\int_0^\tau \int_{-\infty}^{+\infty} R^2\,dy\,ds,$$

and

$$\varepsilon\int_0^\tau \int_{-\infty}^{+\infty} R^2\,dy\,ds = \varepsilon^{-1}\int_0^{\varepsilon\tau} \int_{-\infty}^{+\infty} R^2(x,\eta)\,dx\,d\eta$$
$$\le O(1)\,\varepsilon^{6\alpha-3},$$

where the Sobolev embedding theorem has been used, we get, from (3.19), that

$$\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|\big(\psi_\tau - \dot{s}\psi_y\big)(\tau)\|^2$$
$$+ \int_0^\tau \|\big(\psi_y, \psi_\tau - \dot{s}\psi_y\big)(s)\|^2\,ds + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\,dy\,ds$$
$$\le O(1)\,\varepsilon\int_0^\tau \Big(\|\psi(s)\|_1^2 + \|\big(\psi_\tau - \dot{s}\psi_y\big)(s)\|^2\Big)\,ds$$
$$+ O(1)N(\tau)\int_0^\tau \|\phi_y(s)\|^2\,ds + O(1)\,\varepsilon^{6\alpha-3}. \tag{3.20}$$

To estimate $\|\phi_y(\tau)\|^2$, we investigate the following relation:

$$\big(E\phi_y - \big(\psi_\tau - \dot{s}\psi_y\big)\big)\partial_y L_1 - \phi_y L_2$$
$$= \Big[\tfrac{1}{2}E\phi_y^2 - \big(\psi_\tau - \dot{s}\psi_y\big)\phi_y - \tfrac{1}{2}\psi_y^2\Big]_\tau$$
$$- \phi_y\big(\psi_\tau - \dot{s}\psi_y\big) + D\phi_y^2 + \{\cdots\}_y. \tag{3.21}$$

Integrating (3.22) over $[0, \tau] \times (-\infty, +\infty)$, and using Young's inequality, one can easily obtain

$$\begin{aligned} &\|\phi_y(\tau)\|^2 + \int_0^\tau \|\phi_y(s)\|^2\, ds \\ &\quad \le O(1)\Big(\|(\psi_y, \psi_\tau - \dot{s}\psi_y)(\tau)\|^2 + \int_0^\tau \|(\psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2\, ds\Big) \\ &\qquad + O(1)(N(\tau) + \varepsilon)\int_0^\tau \|\phi_y(s)\|\, ds + O(1)\,\varepsilon^{6\alpha-3}. \end{aligned} \tag{3.22}$$

Combining (3.20) and (3.22), for δ_0, ε_0, and η_0 suitably small, we arrive at

$$\begin{aligned} &\|(\phi, \psi)(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2 \\ &\quad + \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2\, ds + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\, dy\, ds \\ &\qquad \le O(1)\,\varepsilon^{6\alpha-3} + K\varepsilon \int_0^\tau \|\psi(s)\|^2\, ds, \end{aligned} \tag{3.23}$$

for a positive constant K. Equation (3.23) implies that

$$\|\psi(\tau)\|^2 \le O(1)\,\varepsilon^{6\alpha-3} + K\varepsilon \int_0^\tau \|\psi(s)\|^2\, ds.$$

Thus, Gronwall's inequality gives that

$$\int_0^\tau \|\psi(s)\|^2\, ds \le O(1)\,\varepsilon^{6\alpha-3} \int_0^\tau \exp\{K\varepsilon(\tau - s)\}\, ds. \tag{3.24}$$

Inserting (3.24) into (3.22), we have

$$\begin{aligned} &\|(\phi, \psi)(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2 + \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2\, ds \\ &\quad + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\, dy\, ds \\ &\qquad \le O(1)\,\varepsilon^{6\alpha-3}\Big(1 + K\varepsilon \int_0^\tau \exp\{K\varepsilon(\tau - s)\}\, ds\Big) \\ &\qquad \le O(1)\,\varepsilon^{6\alpha-3}. \end{aligned}$$

This proves Lemma 3.3.

For higher-order estimates, we have

LEMMA 3.4. *Suppose the conditions in Theorem* 3.1 *are satisfied*; *then*, *for all* $\tau \in [0, \tau_0]$, *we have*

$$\|(\phi_{yy}, \psi_{yy})(\tau)\|^2 + \|(\psi_\tau - \dot{s}\psi_y)_y(\tau)\|^2 + \int_0^\tau \|(\phi_{yy}, \psi_{yy}, (\psi_\tau - \dot{s}\psi_y)_y)(s)\|^2\, ds \le O(1)\varepsilon^{6\alpha-3}.$$

Proof. Instead of (3.12) and (3.21), we study

$$(\phi_y + \mu\psi_{yy})\partial_y L_1 + D^{-1}\big(\mu(\psi_\tau - \dot{s}\psi_y)_y + \psi_y\big)\partial_x L_2 = D^{-1}\big(\psi_y + \mu(\psi_\tau - \dot{s}\psi_y)_y\big)F_y, \tag{3.25}$$

and

$$\big(E\phi_{yy} - (\psi_\tau - \dot{s}\psi_y)_y\big)\partial_{yy} L_1 - \phi_{yy}\partial_y L_2 = -\phi_{yy}F_y. \tag{3.26}$$

Repeating the procedure in the proof of Lemma 3.3, it is not difficult to show

$$\Big\|\big(\phi_{yy}, (\psi_\tau - \dot{s}\psi_y)_y, \psi_{yy}\big)(\tau)\Big\|^2 + \int_0^t \|(\phi_{yy}, (\psi_\tau - \dot{s}\psi_y)_y, \psi_{yy})(s)\|^2\, d\tau \le O(1)\varepsilon^{6\alpha-3} + \int_0^\tau\int_{-\infty}^{+\infty}\Big(|D^{-1}\big(\psi_y + \mu(\psi_\tau - \dot{s}\psi_y)_y\big)F_y| + |\phi_{yy}F_y|\Big)\, dy\, ds. \tag{3.27}$$

We observe the following facts,

$$\int_0^\tau\int_{-\infty}^{+\infty}\Big(|D^{-1}\big(\psi_y + \mu(\psi_\tau - \dot{s}\psi_y)_y\big)| + |\phi_{yy}|\Big)\varepsilon|R_y|\, dy\, ds \le \alpha_1\int_0^\tau\Big\|\big(\phi_{yy}, \psi_y, (\psi_\tau - \dot{s}\psi_y)_y\big)(s)\Big\|^2 ds + C(\alpha_1)\varepsilon^2\int_0^\tau\int_{-\infty}^{+\infty} R_y^2\, dy\, ds, \tag{3.28}$$

$$\varepsilon^2\int_0^\tau\int_{-\infty}^{+\infty} R_y^2\, dy\, ds = \varepsilon\int_0^{\varepsilon\tau}\int_{-\infty}^{+\infty} R_x^2(x,\eta)\, dx\, d\eta \le O(1)\varepsilon^{4\alpha-1} \le O(1)\varepsilon^{6\alpha-3}, \tag{3.29}$$

$$|(F - \varepsilon R)_y| \le O(1)\big(\eta_0 + |\phi_{yy}|\big)\phi_y^2 + O(1)|\phi_y|\,|\phi_{yy}|, \tag{3.30}$$

and

$$\int_0^\tau \int_{-\infty}^{+\infty} \Big(|D^{-1}\big(\psi_y + \mu(\psi_\tau - \dot{s}\psi_y)_y\big)| + |\phi_{yy}| \Big) |(F - \varepsilon R)_y| \, dy \, ds$$
$$\leq O(1) N(\tau) \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|_1^2 \, ds, \tag{3.31}$$

for any positive number α_1. Equations (3.27)–(3.31) with α_1 suitably small imply Lemma 3.4.

Now with the help of Lemmas 3.3 and 3.4, Theorem 3.1 follows.

Turn to the initial value problem (3.7) now. The local (in time) existence and uniqueness of the solution in the space X is standard. In view of the local result, the a priori estimates, and Theorem 3.1, it follows from a standard continuity argument for hyperbolic systems that

THEOREM 3.5. *Suppose* (H_1)–(H_3) *are satisfied. Let* ε_0, η_0, δ_0, *and* K_0 *be the suitable constants as in Theorem* 3.1, *such that* $|v_0^r - v_0^l| + |u_0^r - u_0^l| + |p_0^r - p_0^l| \leq \eta_0$. *Then for each* $\varepsilon \in (0, \varepsilon_0]$, *there exists a unique solution* (ϕ, ψ) *to* (3.7) *in* $X(0, \frac{T}{\varepsilon})$ *satisfying*

$$\sup_{0 \leq \tau \leq T/\varepsilon} \Big(\|(\phi, \psi)(\tau)\|_2^2 + \|\psi_\tau(\tau)\|_1^2 \Big) + \int_0^{T/\varepsilon} \Big(\|(\phi_y, \psi_y, \psi_\tau)(\tau)\|_1^2 \Big) \, d\tau$$
$$\leq K_0 \varepsilon^{6\alpha - 3}, \tag{3.32}$$

for $(\phi, \psi) \in X(0, \frac{T}{\varepsilon})$.

It follows from Theorem 3.5 and the structure of S^ε that, for each $\varepsilon \in (0, \varepsilon_0]$, there exists a smooth solution $\bar{S}^\varepsilon$ to (1.1) on $[0, T] \times \mathbf{R}^1$ such that (1.8) is satisfied.

Next we study the desired asymptotic behavior of $\bar{S}^\varepsilon(x, t)$. From (3.1), (3.3), (3.5), (3.6), and (3.32), we get that

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(\bar{S}^\varepsilon - S^\varepsilon)(\cdot, t)\|^2 &= \sup_{0 \leq t \leq T} \|\big(\tilde{\phi}_x, \tilde{\psi}_x, \tilde{w}\big)(\cdot, t)\|^2 \\ &= \varepsilon^2 \sup_{0 \leq t \leq T} \|(\phi_x, \psi_x, \psi_t)\|^2 \\ &= \sup_{0 \leq \tau \leq T/\varepsilon} \|(\phi_y, \psi_y, \psi_\tau)\|^2 \\ &\leq K_0 \varepsilon^{6\alpha - 3}. \end{aligned} \tag{3.33}$$

On the other hand, by the construction of the approximation solutions (see Lemma 3.2), we have

$$\sup_{0 \le t \le T} \|S^{\varepsilon} - \chi_0\|^2 \le O(1)\, \varepsilon^{\alpha}.$$

Hence,

$$\sup_{0 \le t \le T} \|\bar{S}^{\varepsilon} - \chi_0\|^2 \le O(1)\, \varepsilon^{\alpha},$$

which is (1.9).

To prove (1.10), we use Sobolev's inequality and (3.32) to obtain

$$\begin{aligned}
\sup_{x \in \mathbf{R}^1} |(\bar{S}^{\varepsilon} - S^{\varepsilon})| &= \sup_{y \in \mathbf{R}^1} |(\phi_y, \psi_y, \psi_\tau)| \\
&\le O(1) \|(\phi_y, \psi_y, \psi_\tau)\|^{1/2} \|(\phi_{yy}, \psi_{yy}, \psi_{\tau y})\|^{1/2} \\
&\le O(1)\, \varepsilon^{(6\alpha - 3)/2}.
\end{aligned}$$

This and Lemma 3.2(i) yield (1.10), if we choose $\alpha > \frac{5}{6}$. The proof of Theorem 1 is completed then.

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