



# Algorithms to approximately count and sample conforming colorings of graphs



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## ABSTRACT

Given a multigraph  $G$  and a function  $F$  that assigns a forbidden ordered pair of colors to each edge  $e$ , we say a coloring  $C$  of the vertices is *conforming to  $F$*  if for all  $e = (u, v)$ ,  $(C(u), C(v)) \neq F(e)$ . Conforming colorings generalize many natural graph theoretic concepts, including independent sets, vertex colorings, list colorings,  $H$ -colorings and adapted colorings and consequently there are known complexity barriers to sampling and counting. We introduce natural Markov chains on the set of conforming colorings and provide general conditions for when they can be used to design efficient Monte Carlo algorithms for sampling and approximate counting.

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## 1. Introduction

*Adapted colorings* [13,14] have been studied as a natural generalization of many well-studied discrete models, including independent sets, colorings, list colorings, and  $H$ -colorings [9,12]. Here we introduce “conforming colorings” as a further generalization. Let  $G = (V, E)$  be a (multi)graph and for  $k \in \mathbb{Z}^+$ , let  $[k] = \{1, \dots, k\}$  be a set of colors. We are given a set of edge constraints  $F : E \rightarrow [k] \times [k]$  describing forbidden ordered pairs of colors on the endpoints of each edge, and we are interested in the set of vertex colorings satisfying these constraints. We say that a vertex coloring  $C : V \rightarrow [k]$  is a *conforming coloring* if, for each edge  $e = (u, v)$ , we have  $F(e) \neq (C(u), C(v))$ . Let  $\Omega = \Omega(G, F, k)$  be the set of all conforming colorings of  $G$  with forbidden pairs  $F$  and  $k$  colors. Conforming colorings formalize constraints in many applications including resource allocation, where vertices represent jobs and edge constraints capture incompatible scheduling assignments. We focus on approximately counting and sampling conforming and adapted colorings.

The connection between conforming colorings and many standard graph theoretic objects is straight-forward. For example, when  $k = 2$  and  $F(e) = (1, 1)$  for all edges  $e \in E$ , then in each conforming coloring the vertices colored 1 form an independent set. Likewise, given a graph  $G$ , form a multigraph  $G'$  where each edge is replaced with  $k$  parallel edges, each labeled with distinct  $(i, i)$  for  $1 \leq i \leq k$ , then the conforming colorings of  $G'$  are exactly the proper  $k$ -colorings of  $G$ . We also can formulate weighted versions of these standard models. For example in the hard-core lattice gas model, we are given an activity  $\lambda$  which represents the fugacity of the gas and are interested in sampling independent sets from a weighted distribution where each independent set  $I$  occurs with probability  $\lambda^{|I|}/Z$ , where  $|I|$  is the size of the independent set and  $Z = \sum_I \lambda^{|I|}$  is the normalizing constant known as the partition function. If  $\lambda > 1$ , then dense independent sets are more likely while if  $\lambda < 1$ , sparse independent sets are favored. To formulate this model using conforming colorings, we let  $k > 2$

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and  $F(e) = (1, 1)$  for all edges. Again, in each conforming coloring the vertices colored 1 form an independent set. However, now for each independent set  $I$  there are  $(n - |I|)^{k-1}$  ways to color the remaining vertices. When sampling uniformly from the set of conforming colorings, each independent set  $I$  thus occurs with probability  $\lambda^{|I|}/Z$ , where  $\lambda = 1/(k - 1)$ . Valid configurations are weighted independent sets with low fugacity (favoring sparse independent sets). Likewise, we can construct instances of conforming colorings that correspond to independent sets with high fugacity (favoring dense sets). Given a graph  $G$ , we construct a new graph  $G_r$  with  $r|V|$  vertices and  $r^2|E|$  edges as follows. We replace each vertex with  $r$  copies and replace each edge  $(u, v)$  in  $G$  with the complete bipartite graph  $K_{r,r}$  connecting each copy of  $u$  in  $G_r$  with each copy of  $v$ . We then set  $k = 2$  with colors 1 and 2 and let  $F(e) = (1, 1)$  for all edges  $e$  in  $G_r$ . Notice that if  $(u, v) \in E$  and any of the copies of  $u$  are colored 1 in  $G_r$ , then all of the copies of  $v$  must be colored 2. Thus, the 1 vertices in  $G_r$  correspond to an independent set in  $G$  (where we include a vertex in the independent set if at least one copy in  $G_r$  is colored 1) and each independent set has weight  $\lambda^{|I|}/Z$  where  $\lambda = 2^r - 1$  and  $Z = \sum_I \lambda^{|I|}$  is the normalizing constant.

The same type of construction based on parallel edges allows us to capture the class of  $H$ -colorings, or homomorphisms from a graph  $G$  to  $H$  that preserve adjacency.  $H$ -colorings themselves have been studied as a natural generalization of many discrete models including colorings, independent sets and the Widom–Rowlinson model from statistical physics (see, i.e., [2,7,4,12]). Conforming colorings can model these problems by replacing each edge in  $G$  with parallel edges representing all of the edges of  $H$  that are *not* present, including self-loops. Conforming colorings are more general than the  $H$ -coloring problem because labeling the edges of a graph with forbidden colorings allows non-homogeneity in the coloring restrictions on neighboring vertices. Hell and Nešetřil [12] showed that if  $H$  does not have a loop, then deciding whether there exists an  $H$ -coloring is NP-complete, and it is in P otherwise. Dyer and Greenhill [9] proved that counting  $H$ -colorings exactly is  $\sharp$ -P complete unless each component of  $H$  is trivial (i.e., a complete graph with loops or a complete bipartite graph), in which case the counting problem is also in P.

A special case of conforming colorings that has garnered interest recently is known as *adapted (or adaptable) colorings*. Given an edge coloring  $C : E \rightarrow [k]$ , a vertex coloring  $C' : V \rightarrow [k]$  is adapted to  $C$  if there is no edge  $e = (u, v)$  with  $C(e) = C'(u) = C'(v)$ . Hell and Zhu [14] introduced the *adaptable chromatic number* in 2008. Subsequently there have been a flurry of papers deriving bounds on the adaptable chromatic number in graphs and hypergraphs, the adaptable list chromatic number, and determining when a graph  $G$  is adaptably  $k$ -choosable, where each of these is a natural generalization of the standard graph theoretic notions (see, e.g., [10,13,14,17,19]). Recently, Cygan et al. [5] gave a polynomial time algorithm for finding an adapted 3-coloring given a fixed edge 3-coloring of a complete graph, resolving the so-called “stubborn problem” in the classification of constraint satisfaction problems [3].

In this paper we focus on the problems of approximately counting and randomly sampling conforming and adapted colorings of graphs. Previous research on approximation algorithms in the context of  $H$ -colorings yielded both positive and negative results [4,7], which is not surprising since they include independent sets and colorings as special cases. Our motivation for studying approximation algorithms in the more general class of conforming colorings is similar to that for  $H$ -colorings. Not only does the model capture many fundamental problems that are interesting in their own right, but such a study allows us to examine which approaches to randomized approximate counting can be extended to this more general class of problems.

### 1.1. Previous work

There has been extensive work trying to approximately count various graph structures using Monte Carlo approaches. The main ingredient is designing a Markov chain for sampling configurations that is rapidly mixing. For example, for independent sets we are given a fugacity  $\lambda$  and are interested in sampling independent sets  $I$  from the Gibbs distribution  $\pi(I) = \lambda^{|I|}/Z$ , where  $Z$  is the normalizing constant. Local chains that modify a small number of vertices in each move are known to be efficient on  $\mathbb{Z}^2$  at fugacity  $\lambda < 2.48$  [23] and inefficient when  $\lambda > 5.3646$  [1]. Similarly, local chains on the space of  $k$ -colorings of graphs are efficient if there are enough colors compared to the maximum degree of the graph [11], whereas even finding a single  $k$ -coloring is NP-complete for small degree.

Dyer and Greenhill [9] proved that counting  $H$ -colorings exactly is  $\sharp$ -P complete unless each component of  $H$  is trivial (i.e., a complete graph with loops or a complete bipartite graph), in which case the counting problem is also in P. Previous research on approximation algorithms in the context of  $H$ -colorings yielded both positive and negative results [4,7], which is not surprising since they include independent sets and colorings as special cases. Cooper, Dyer and Frieze [4] considered sampling algorithms for  $H$ -colorings and showed that for a large class of Markov chains convergence will be slow, particularly on graphs with high degree, although they give an efficient algorithm for sampling  $H$ -colorings when  $H$  is a tree with self-loops everywhere. Likewise, Borgs et al. [2] showed that for any finite, connected, non-trivial  $H$ , there are weights on the edges such that all quasi-local ergodic Markov chains will be slowly mixing on finite regions of Allison Martin-Attix  $\mathbb{Z}^d$ . Dyer, Goldberg and Jerrum [7] explored the connection between approximate counting and random sampling in the context of  $H$ -colorings. Sampling and counting are known to be equivalent for problems that are “self-reducible” [16], but  $H$ -colorings do not, in general, have this nice property. Dyer et al. succeed in one direction for  $H$ -colorings, namely showing that an efficient algorithm for random sampling can be used to design an FPRAS for approximate counting. The other direction remains open, as does the reduction for the general class of conforming colorings.

When local algorithms are slow, nonlocal variants can be more effective, but they are typically more challenging to analyze. Examples include the Swendsen Wang algorithm for the Ising and Potts models [22] and the Wang–Swendsen–Kotecký

(WSK) chain for proper colorings that uses moves based on Kempe chains [25]. The WSK chain chooses a vertex  $v$  and a color  $c$  at random and tries to recolor  $v$  with  $c$ . If it cannot be recolored, this is because there is a neighbor colored  $c$ , so we can instead consider the bipartite  $(c, c(v))$  component, where  $c(v)$  is the current color of  $v$ . The WSK chain allows moves that swap the two colors on this whole component. Vigoda studied this chain and showed that a weighted version of the chain is rapidly mixing when  $k \geq 11\Delta/6$ , where  $\Delta$  is the maximum degree [24]. In general variants of the WSK chain have proven difficult to analyze, although we will discuss a variant of this algorithm.

## 1.2. Our results

Let  $\Omega = \Omega(G, F, k)$  be the set of all conforming colorings of  $G$  with forbidden pairs  $F$ . First, we consider the local Markov chain  $\mathcal{M}_l$  on  $\Omega$  that recolors one vertex at a time. Let  $\Delta$  be the max degree in the graph  $G$ , including multi-edges and self-loops. We show that there is a polynomial time algorithm for finding a conforming coloring when the number of colors is at least  $\max(3, \Delta)$ . Moreover, we show that under these conditions, the local Markov chain  $\mathcal{M}_l$  connects the state space of conforming colorings, is rapidly mixing, and there is an FPRAS (fully polynomial randomized approximation scheme) for approximately counting the number of conforming colorings.

For adapted colorings, where each edge is assigned a single color, we prove a stronger result requiring only that  $k \geq \max(\Delta_m, 3)$  where  $\Delta_m$  is related to  $\Delta$  except at most two parallel edges between any two vertices are counted toward the degree of a vertex. Specifically, let  $d_1(v)$  be the number of neighbors of  $v$  with multiplicity one (1 edge between them),  $d_2(v)$  be the number of neighbors with multiplicity two or more,  $d_s(v)$  be the number of self-loops at  $v$  and  $\Delta_m = \max_{v \in V} d_1(v) + 2d_2(v) + d_s(v)$ . For all graphs,  $\Delta_m \leq \Delta$  and if there are no edges with multiplicity greater than 2,  $\Delta_m = \Delta$ .

When we have two colors ( $k = 2$ ), however, the local chain does not always connect the state space  $\Omega$ . Consider, for example, the Cartesian lattice where all of the horizontal edges are  $(1, 1)$  and all of the vertical edges are  $(2, 2)$ ; there are two conforming colorings corresponding to the two proper 2-colorings of the lattice. To handle the case when  $k = 2$ , we introduce a new chain  $\mathcal{M}_c$  that reverses colors on (possibly) large “color-implied” components that are predetermined based on the structure of  $G$  and  $F$ . A *color-implied component* is a connected set of vertices where coloring any vertex in the component implies a unique coloring of the remaining vertices in the component. The chain  $\mathcal{M}_c$  identifies color-implied components and allows moves that change the color of all vertices in a component in a single move. Although some of these moves are Kempe chain moves analogous to those described in [25], in general they are more complicated.

We provide conditions under which we can find conforming 2-colorings efficiently and show  $\mathcal{M}_c$  connects the state space, is rapidly mixing, and there is an FPRAS for approximately counting conforming colorings. We provide conditions under which we can find conforming 2-colorings efficiently and show  $\mathcal{M}_c$  connects the state space, is rapidly mixing, and there is an FPRAS for counting. Finally, we provide an example graph with maximum degree 4 on which both chains require exponential time to mix.

Our mixing results build on ideas used to show fast and slow mixing in the context of colorings and independent sets; however, the proofs required careful fine-tuning to fit the more general setting of conforming colorings and to prove the bounds we achieve here. Moreover, unlike sampling colorings and independent sets where we typically restrict to graphs on which connecting the state space is trivial, in this more general setting establishing ergodicity for the two chains has proven to be considerably more challenging.

## 2. Connectivity and mixing of the local Markov chain $\mathcal{M}_l$

We start by showing how to sample adapted and conforming colorings efficiently when there are sufficient colors. In Section 2.1, we begin exploring how to find a conforming coloring when  $k \geq \max(\Delta, 3)$ , if one exists. Next, in Section 2.2 we define the local Markov chain  $\mathcal{M}_l$  and show that when  $k \geq \max(\Delta, 3)$ ,  $\mathcal{M}_l$  is ergodic. Finally, in Section 2.3 we prove that under this same condition, the chain  $\mathcal{M}_l$  is rapidly mixing and in Section 2.4 we show how to use  $\mathcal{M}_l$  to approximately count conforming colorings.

### 2.1. Finding a conforming coloring

We prove that if  $k \geq \max(\Delta, 3)$  and  $\Omega$  is not “degenerate”, a conforming coloring exists. First, we will define a *degenerate state space*  $\Omega$  that does not have any conforming colorings algorithmically (i.e.,  $\Omega = \emptyset$ ).

We begin by defining some notation and terminology that will be used throughout the paper. For an edge  $e = (u, v)$ , let  $F(e) = (F_u(e), F_v(e))$  and  $d(v)$  be the degree of vertex  $v$ . Define a *flower* to be a vertex  $v$  with  $k$  self-loops, each with a distinct color; note there is no conforming coloring of a flower. A vertex is *color-fixed* if it has exactly  $k - 1$  self-loops each with a distinct color or  $k$  self-loops with exactly one color repeated. A color-fixed vertex must have the same color in every conforming coloring. For each color-fixed vertex  $v$ , color  $v$  with its only valid color  $c$ , remove  $v$  from  $G$  and handle each adjacent edge  $e = (u, v)$  as follows. If  $e$  is a self-loop (i.e.,  $u = v$ ) or  $F_v(e) \neq c$ , remove  $e$  as it no longer provides any constraint since  $v$  is fixed to have color  $c$ . Otherwise, if  $F_v(e) = c$ , then  $F_u(e)$  is not a valid color for  $u$  in any conforming coloring, so add a self-loop colored  $F_u(e)$  to  $u$ . Continue this process of removing color-fixed vertices and handling their adjacent edges until either a flower is found, at which point  $G$  does not have any conforming colorings (and is degenerate) or there are no color-fixed vertices (and  $G$  is not degenerate). This procedure produces a subgraph of  $G$  and an edge-coloring of the subgraph

which we will refer to as  $G'$  and  $F'$ . Notice that the vertices in  $G'$  may have self-loops not present in  $G$  however the degree of each vertex has not increased since each self-loop replaced an edge present in  $G$  but not in  $G'$ . By construction,  $G'$  does not have any flowers or color-fixed vertices and there is a bijection between colorings of  $G$  conforming to  $F$  and colorings of  $G'$  conforming to  $F'$ . In the remaining part of this section, we will show how to find, sample and count conforming colorings of  $G$  by finding, sampling and counting conforming colorings of  $G'$ . The following result is proven constructively using graph theoretic techniques by giving an algorithm that iteratively colors vertices.

**Theorem 1.** *Given a graph  $G$  with  $n$  vertices,  $k \geq \max(\Delta, 3)$  and edge  $k$ -constraints  $F$  such that  $\Omega(G, F, k)$  is not degenerate, there exists a conforming  $k$ -coloring of  $G$  and we can find one in time  $O(\Delta n^2)$ .*

**Proof.** Since  $G$  is not degenerate, there exists a subgraph  $G' = (V', E') \subseteq G$  and a coloring  $F'$  as described above. The description above is constructive and we can find  $G'$  and  $F'$  in time  $O(n^2)$ . Recall that  $G'$  does not have any flowers or color-fixed vertices and the conforming colorings of  $G'$  are in bijection with the conforming colorings of  $G$ . Thus, it suffices to show how to find a conforming  $k$ -coloring of  $G'$ . Start by fixing an ordering  $v_1, \dots, v_n$  on the vertices of  $G'$ . Next consider the vertices in order and color each vertex  $v_i$  with the first available color, if possible. Define a *partial conforming coloring* to be a coloring of a subset of the vertices of  $G'$  that does not violate any of the constraints  $F$ . Let  $P_i$  be the partial conforming coloring when we reach vertex  $v_i$ . If we reach a vertex  $v_i$  with no available colors, we give an algorithm for modifying  $P_i$  so that  $P_i$  remains a partial conforming coloring with the same subset of vertices colored, and there is an available color for  $v_i$ . Coloring  $v_i$  with the available color results in a partial conforming coloring  $P_{i+1}$  of the vertices  $v_1, \dots, v_i$ . Repeating this procedure for every vertex  $v_1, \dots, v_n$  in order results in a conforming coloring of  $G$ .

Assume we have reached a vertex  $v \in v_1, \dots, v_n$  with partial coloring  $P$  for which there are no available colors. Since  $G'$  does not have any flowers,  $v$  is not a flower, thus  $v$  is not a component of size one. This implies that  $d(v) = \Delta = k$  and there exists an ordering of the adjacent edges  $e_1, \dots, e_k$  with corresponding adjacent vertices  $u_1, \dots, u_k$  (that need not be distinct from each other or from  $v$ ) such that for  $1 \leq j \leq k$  either  $F_{v_j}(e_j) = j$  and  $F_{u_j}(e_j) = P(u_j)$  or  $e_j$  is a self-loop and  $F(e_j) = (j, j)$ . Consider one such adjacent vertex  $p_1$  ( $p_1 \in u_1, \dots, u_k$ ) that satisfies  $F_v(v, p_1) = c$  and  $F_{p_1}(v, p_1) = P(p_1)$  for some color  $c$ . If there is another valid color for vertex  $p_1$ , then recolor  $p_1$  creating an available color for  $v$ , so we are done. If there is not another valid color in  $P$  for  $p_1$ , consider any maximal path  $p_1, p_2, \dots, p_x$  such that for each  $i > 1$ , there exists an edge  $g_i = (p_{i-1}, p_i)$  with  $F_{p_i}(g_i) = P(p_i)$  where each  $p_i$  is distinct from each other and from  $v$ . Since  $P$  is a valid conforming coloring, for all  $1 < i \leq x$ , we know  $F_{p_{i-1}}(g_i) \neq P(p_{i-1})$ . If there existed a vertex  $p_m$  on the path that has another valid color in  $P$ , let  $m$  be the smallest index such that this is true. We can iteratively recolor  $p_i$  with  $i \leq m$  in descending order, and this would provide an available color for  $v$ . To see this, let  $m$  be the smallest number  $1 < m \leq x$  such that  $p_m$  has another valid color in  $P$ . For each  $1 < i < m$ ,  $p_i$  does not have any additional valid colors and  $F_{p_i}(g_i) = P(p_i)$ . Thus for each color  $c$ , there is an edge  $e_c$  adjacent to  $p_i$  such that  $F_{p_i}(e_c) = c$  or a self loop  $e_c$  with  $F(e_c) = (c, c)$ . Additionally, for each  $p_i$ ,  $d(p_i) \leq k$  so there are no edges  $e_1$  and  $e_2$  both adjacent to  $p_i$  with  $1 < i < m$  such that  $F_{p_i}(e_1) = F_{p_i}(e_2)$ . Thus, if we recolor any vertex adjacent to  $p_i$  (excluding  $p_{i-1}$ ) this will result in an additional valid color for  $p_i$ . We can iteratively recolor each  $p_i$  with  $i \leq m$  in descending order, and this will provide an available color for  $v$ , as desired.

Next we show, by contradiction, that there exists a vertex  $p_m$ ,  $1 < m \leq x$  that has another valid color in  $P$ . Assume to the contrary that we have such a maximal path  $p_1, p_2, \dots, p_x$  and no  $p_i$  has any additional valid colors in  $P$ . Without loss of generality assume  $P(p_x) = k$ . This implies that  $p_x$  must have  $k - 1$  adjacent edges  $f_1, f_2, \dots, f_{k-1}$  with corresponding adjacent vertices  $x_1, \dots, x_{k-1}$  (again note that these vertices might not be distinct from each other or from  $p_x$ ) such that for each  $f_i$ , either  $F_{p_x}(f_i) = i$  and  $F_{x_i}(f_i) = P(x_i)$  or  $f_i$  is a self-loop and  $F(f_i) = (i, i)$ . Since the path is maximal, these vertices can only include  $v$  or vertices already in the path  $(p_1, \dots, p_x)$ . However  $v$  is not colored in  $P_i$  so it cannot be one of these vertices. If there was a vertex  $p_j$  in the path that was adjacent to  $p_x$  then that would imply that  $p_j$  has two adjacent edges  $e, e'$  with  $F_{p_j}(e) = F_{p_j}(e') = P(p_j)$ . Then there exists a color  $c \neq P(p_j)$  that is always valid for  $p_j$ , implying that  $p_j$  has another valid color, a contradiction. Finally, if each  $f_i$  is a self-loop with distinct colors, then this implies that  $p_x$  was color-fixed, a contradiction since  $G'$  does not have any color-fixed vertices.

By repeating this procedure as necessary for each vertex  $v_i$  for which there are no available colors we will obtain a conforming coloring of  $G$ . In the worst case to find a color for every vertex  $v \in G$  we modify the color of the vertices in a path  $p_1, \dots, p_x$  of length at most  $n$ . For every vertex in the path we look at each of its  $\Delta$  adjacent vertices to determine if there is an available color and what the next vertex in the path will be. This results in an  $O(\Delta n^2)$  algorithm.  $\square$

In the special case of adapted colorings, where both colors in each forbidden order pair are the same, we can prove a stronger result. Here, we use the fact that in the adapted setting each neighbor can only prevent a single color from being available to a vertex whereas in the more general setting a single neighbor can prevent multiple colors.

**Theorem 2.** *Given a graph  $G$  with  $n$  vertices,  $k \geq \max(\Delta_m, 3)$  and edge  $k$ -coloring  $F$  such that  $\Omega(G, F, k)$  is not degenerate, there exists an adapted  $k$ -coloring of  $G$  and we can find one in time  $O(\Delta n^2)$ .*

**Proof.** The algorithm is similar to the proof of [Theorem 1](#); we start with the graph  $G'$  that does not have any flowers or color-fixed vertices and iteratively color the vertices with the first available color. Assume we have reached a vertex  $v$  which does not have any available colors. Notice that the vertices  $u_1, \dots, u_k$  adjacent to  $v$  must now be distinct from each other, although they can include multiple self-loops (i.e.,  $u_1, \dots, u_k$  can include  $v$  multiple times). This is because in the adapted coloring

setting each adjacent vertex  $u_i$  (or self-loop  $u_i = v$ ) can only prevent, or block, one color from being a valid color for  $v$  in  $P$  (i.e.,  $u_i$  can only block  $P(u_i)$  from being a valid color for  $v$  and only if there exists an edge  $e = (v, u_i)$  such that  $F(e) = P(u_i)$ ). Notice that the constraint  $k \geq \max(\Delta_m, 3)$  implies that for any vertex  $v$ , the total number of adjacent vertices (there might be more multi-edges) plus self-loops is at most  $k$ . We again consider a maximal path  $p_1, p_2, \dots, p_x$  (with  $p_1 \in u_1, \dots, u_k$ ) but in this case for each  $i > 1$  there exists an edge  $g_i = (p_{i-1}, p_i)$  such that  $F(g_i) = P(p_i)$  and  $P(p_{i-1}) \neq F(g_i)$ . Let  $m$  be the smallest number  $1 < m \leq x$  such that  $p_m$  has another valid color in  $P$ . For each  $1 < i < m$ ,  $p_i$  does not have any additional valid colors and  $F(g_i) = P(p_i)$ . Thus for each color  $c$ , there is either a vertex  $v_c$  with edge  $e_c = (v_c, p_i)$  such that  $F(e_c) = P(v_c) = c$  or a self loop  $e_c$  with  $F(e_c) = c$ . Additionally, for each  $p_i$ , the number of adjacent vertices plus self-loops is at most  $k$  so each adjacent vertex (or self-loop) must have a distinct color in  $P$  (or edge color in the case of self-loops). Thus, if we recolor any vertex adjacent to  $p_i$  (excluding  $p_{i-1}$ ) this will result in an additional valid color for  $p_i$  (i.e., each vertex or self-loop blocks a distinct color so if we recolor that vertex we will have an additional missing color among the neighbors and self-loops of  $p_i$  and thus another valid color). We can iteratively recolor each  $p_i$  with  $i \leq m$  in descending order, and this will provide an available color for  $v$ .

Assume to the contrary that we have such a maximal path  $p_1, p_2, \dots, p_x$  and no  $p_i$  has any additional valid colors in  $P$ . Without loss of generality assume  $P(p_x) = k$ . This implies that  $p_x$  must have  $k - 1$  adjacent edges  $f_1, f_2, \dots, f_{k-1}$  with corresponding adjacent vertices  $x_1, \dots, x_{k-1}$  (these vertices must be distinct or self-loops) such that for each  $f_i$ , either  $F(f_i) = P(x_i) = i$  or  $f_i$  is a self-loop with  $F(f_i) = i$ . Since the path is maximal, these vertices can only include  $v$  or vertices already in the path  $(p_1, \dots, p_x)$ . However  $v$  is not colored in  $P$  so it cannot be one of these vertices. If there was a vertex  $p_j$  in the path that was adjacent to  $p_x$  then that would imply that  $p_j$  has two adjacent edges  $e_x, e_{j-1}$  with distinct (from each other and from  $p_j$ ) adjacent vertices  $p_x$  and  $p_{j-1}$  such that  $F(e_x) = F(e_{j-1}) = P(p_j)$ . We will analyze three cases depending on the multiplicity of  $p_x$  and  $p_{j-1}$ . If they both have multiplicity one then both  $p_x$  and  $p_{j-1}$  do not block any colors from being available for  $p_j$  because  $F(e_x) \neq P(p_x)$  and  $F(e_{j-1}) \neq P(p_x)$ . The constraint  $k \geq \max(\Delta_m, 3)$  implies that  $p_j$  can have at most  $k - 2$  additional distinct neighbors or self-loops (recall for adapted colorings each neighboring vertex can only block one color). Thus there must be at least two valid colors for  $p_j$  in  $P$ , a contradiction. If  $p_x$  and  $p_{j-1}$  both have multiplicity two then the constraint  $k \geq \max(\Delta_m, 3)$  implies that  $p_j$  can have at most  $k - 4$  additional distinct neighbors or self-loops and since  $p_x$  and  $p_{j-1}$  each block at most 1 color there must be at least two colors valid for  $p_j$  in  $P$ , a contradiction. Similarly, if exactly one of  $p_x$  or  $p_{j-1}$  have multiplicity 1, without loss of generality assume  $p_x$ , then  $p_x$  does not block any colors from  $p_j$  because  $F(e_x) \neq P(p_x)$  and  $p_j$  contributes two to  $\Delta_m$  so there can be at most  $k - 3$  additional distinct neighbors or self-loops and again there must be at least two colors valid for  $p_j$  in  $P$ , a contradiction. Finally if every  $f_i$  is a self-loop, each with a distinct color then this would imply that  $p_x$  was color-fixed, a contradiction. The rest of the proof follows exactly as in the proof of [Theorem 1](#).  $\square$

The remaining part of the section is concerned with almost uniformly sampling and approximately counting conforming colorings.

## 2.2. Ergodicity of the Markov chain $\mathcal{M}_L$

We begin by defining a local Markov chain  $\mathcal{M}_L$  that at each step selects a vertex  $v$  and a color  $c$  uniformly at random and colors vertex  $v$  with color  $c$  if this results in a valid conforming coloring. This chain is known as Glauber dynamics and has been widely studied in the context of colorings and independent sets, for example.

### The local Markov chain $\mathcal{M}_L$

Starting at any initial conforming coloring, iterate the following:

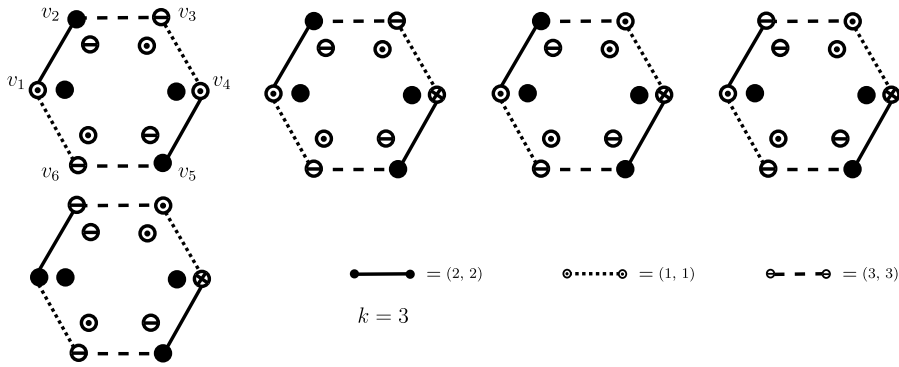
- Pick a vertex  $v$  and a color  $q$  uniformly at random (u.a.r.).
- With probability  $1/2$ , color vertex  $v$  with color  $q$  if this results in a conforming coloring.
- Otherwise, do nothing.

First, we show that when  $k \geq \max(\Delta, 3)$ , the Markov chain  $\mathcal{M}_L$  is ergodic (i.e., irreducible and aperiodic; see [\[21\]](#)). This implies that the chain will converge to a unique stationary distribution that is uniform over the set  $\Omega$  of all conforming colorings. Specifically, we prove that for each  $\sigma, \alpha \in \Omega$  there is a path from  $\sigma$  to  $\alpha$  using only transitions of  $\mathcal{M}_L$ , thus implying the irreducibility of  $\mathcal{M}_L$ . The primary challenge is that there might not be a path between  $\sigma$  and  $\alpha$  that only modifies vertices in the symmetric difference. For example, let  $k = 3$  and consider an even cycle whose edges alternate between  $(1, 1)$  and  $(2, 2)$  and every vertex on the cycle has a  $(3, 3)$  edge to a single vertex. Now consider the coloring  $\alpha$  where the vertices around the cycle alternate between 1 and 2 and the vertices adjacent to the cycle are all colored 3. Let  $\sigma$  be the same coloring except flip the colors around the cycle so 2 vertices are now 1 and vice versa. Notice that any path between  $\sigma$  and  $\alpha$  must include a transition that modifies the color of one of the vertices colored 3. We prove the following theorem.

**Theorem 3.** For any graph  $G$ ,  $k \geq \max(\Delta, 3)$  and edge  $k$ -constraints  $F$ ,  $\mathcal{M}_L$  connects  $\Omega(G, F, k)$ .

**Proof.** If  $G$  is degenerate then it does not have any conforming colorings and  $\mathcal{M}_L$  trivially connects  $\Omega$ . We will assume  $G$  is not degenerate, therefore there exists a subgraph  $G' = (V', E') \subseteq G$  and a coloring  $F'$  as described in the beginning of Section 2 where degenerate is defined. Recall that  $G'$  does not have any flowers or color-fixed vertices and the conforming colorings of  $G'$  are in bijection with the conforming colorings of  $G$ . It suffices to show  $\mathcal{M}_L$  connects  $\Omega(G', F', k)$ . For the





**Fig. 1.** An example cycle  $(v_1, v_2, v_3, v_4, v_5, v_6)$  where  $v_4$  can be recolored and partial path. The outer vertices are the  $\alpha$ -coloring and the inner vertices are the  $\beta$ -coloring of the same vertices.

following proof we will assume that we are working with the graph  $G'$ . For any  $\beta, \alpha \in \Omega$ , let  $\phi(\beta, \alpha)$  be the number of vertices  $v \in V'$  such that  $\beta(v) \neq \alpha(v)$ . It is sufficient to show that for any  $\beta, \alpha \in \Omega$  such that  $\phi(\beta, \alpha) > 0$  there exist paths  $\beta = \beta_0, \beta_1, \dots, \beta_r = \beta'$  and  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_s = \alpha'$ , using only moves of  $\mathcal{M}_L$ , such that  $\phi(\beta', \alpha') < \phi(\beta, \alpha)$ . It is possible that  $\beta = \beta'$  or  $\alpha = \alpha'$ . For any vertex  $v$  such that  $\beta(v) \neq \alpha(v)$  and there exists a color  $c$  such that for all edges  $e = (u, v)$  adjacent to  $v$ ,  $F_v(e) \neq c$  (i.e.,  $c$  is always a valid color for  $v$ ),  $v$  can be recolored  $c$  in both  $\beta$  and  $\alpha$ . This implies that there exists appropriate paths  $(\beta, \beta')$  and  $(\alpha, \alpha')$ , where  $\beta'$  and  $\alpha'$  have  $v$  colored  $c$ . Therefore we can assume for all  $v \in V'$  such that  $\beta(v) \neq \alpha(v)$ ,  $v$  does not have any colors valid in both  $\alpha$  and  $\beta$ ,  $d(v) = \Delta = k$ , and each edge  $e$  incident to  $v$  has a unique color  $F_v(e)$  (there are no incident edges  $e, e'$  such that  $F_v(e) = F_v(e')$ ).

Let  $v \in V'$  satisfy  $\beta(v) \neq \alpha(v)$  and let  $\beta(v) = b$  and  $\alpha(v) = a$ . Since  $a$  is not a valid color for  $v$  in  $\beta$  there exists an edge  $e = (u, v)$  such that  $F_v(e) = a$ ,  $F_u(e) = \beta(u)$  and  $\alpha(u) \neq \beta(u)$ . This implies a cycle  $(v_1, v_2, v_3, \dots, v_t)$  and edges  $e_1 = (v_1, v_2), e_2 = (v_2, v_3), \dots, e_t = (v_t, v_1)$  such that for each edge  $e_i = (x, w)$  either  $F_x(e_i) = \beta(x)$  and  $F_w(e_i) = \alpha(w)$  or  $F_x(e_i) = \alpha(x)$  and  $F_w(e_i) = \beta(w)$  (see Fig. 1).

Now consider any vertex  $v_i$  on the cycle. If it were possible to recolor  $v_i$  differently in either  $\alpha$  or  $\beta$ , without loss of generality assume  $\alpha$ , then we could recolor either  $v_{i+1}$  or  $v_{i-1}$ , again assume  $v_{i+1}$  in  $\alpha$  so that  $\alpha(v_{i+1}) = \beta(v_{i+1})$  and we could continue around the cycle until  $\alpha$  and  $\beta$  agree for every vertex (see Fig. 1). This would provide an appropriate path  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_t = \alpha'$ .

We now consider the case that there is not a vertex on the cycle that can be recolored in either  $\alpha$  or  $\beta$ . This implies that for both  $\alpha$  and  $\beta$ , every color except the current color is invalid for  $v_1$ . Since  $k \geq \max(\Delta, 3)$ , we may assume that  $v_1$  has neighbors outside the cycle therefore there exists a vertex  $u_1$ , adjacent to  $v_1$  with edge  $f_1 = (v_1, u_1)$  and outside the cycle, such that  $F_{u_1}(f_1) = \alpha(u_1) = \beta(u_1)$ . Otherwise, we could recolor  $v_1$  in either  $\alpha$  or  $\beta$ . If it were possible to recolor  $u_1$  in either  $\alpha$  or  $\beta$  then we could recolor  $u_1$ , update the cycle, as in the previous case, and finally return  $u_1$  to its original coloring which satisfies  $\alpha(u_1) = \beta(u_1)$ . Again, this would provide an appropriate path. Consider any maximal path  $u_1, u_2, u_3, \dots, u_x$  such that for each  $i > 1$ ,  $F_{u_i}(u_{i-1}, u_i) = \alpha(u_i) = \beta(u_i)$ , each  $u_i$  is distinct from each other and from every  $v_i$ . Notice that since both  $\alpha$  and  $\beta$  are valid conforming colorings, this implies that for all  $x \geq i > 1$ ,  $F_{u_{i-1}}(u_{i-1}, u_i) \neq \alpha(u_{i-1})$  and  $F_{u_{i-1}}(u_{i-1}, u_i) \neq \beta(u_{i-1})$ . As before, if there existed a  $u_j$  that was recolored in  $\alpha$  or  $\beta$ , assume  $\alpha$ , we can iteratively recolor each  $u_i$  in  $\alpha$  with  $i \leq j$  in descending order, recolor  $v_1$ , recolor the entire cycle so it agrees with  $\beta$  and finally proceed to return each  $u_i$  to the original color in  $\alpha$  (see Fig. 2), this would give an appropriate path from  $\alpha$  to  $\alpha'$ . We show, by contradiction, that such a  $u_i$  must exist.

Without loss of generality assume  $\alpha(u_x) = \beta(u_x) = k$ . As it is not possible to recolor  $u_x$  in either  $\beta$  or  $\alpha$ ,  $u_x$  must have  $k - 1$  adjacent vertices  $x_1, \dots, x_{k-1}$  (not necessarily unique) such that for each  $x_i$ , there exists an edge  $(u_x, x_i)$  such that  $F_{u_x}(u_x, x_i) = i$  and  $F_{x_i}(u_x, x_i) = \alpha(x_i) = \beta(x_i)$ . Also, since the path is maximal, these vertices must either be in the cycle  $(v_1, \dots, v_t)$  or the path  $(u_1, \dots, u_x)$  (or  $u_x$  itself). However every vertex in the cycle is colored differently in  $\alpha$  than  $\beta$  so they cannot be included in the vertices adjacent to  $u_x$ . If there was a vertex  $u_j$  in the path that was adjacent to  $u_x$  then that implies that  $u_j$  has two adjacent edges  $e, e'$  with  $F_{u_j}(e) = F_{u_j}(e') = \alpha(u_j)$ . Thus that there exists a color  $c \neq \alpha(u_j)$  that is always valid for  $u_j$ , implying that  $u_j$  can be recolored, a contradiction to the assumption that no  $u_i$  can be recolored in  $\alpha$  or  $\beta$ . Finally consider the case where all of the vertices  $x_1, \dots, x_{k-1}$  are  $u_x$  itself (i.e.,  $u_x$  has  $k - 1$  self-loops). In this case there must be a color missing from the self-loops otherwise  $u_x$  would be color-fixed; a contradiction since  $G'$  does not have any color-fixed vertices.  $\square$

Next, we will prove a stronger result in the context of adapted colorings with the constraint  $k \geq \max(\Delta_m, 3)$ . Again, we exploit the fact that in the adapted setting each neighbor can only block a single color whereas in the more general setting a single neighbor can block multiple colors.

**Theorem 4.** For any graph  $G$ ,  $k \geq \max(\Delta_m, 3)$  and edge  $k$ -coloring  $F$ ,  $\mathcal{M}_L$  connects  $\Omega(G, F, k)$ .

**Proof.** If  $G$  is degenerate then it does not have any conforming colorings and  $\mathcal{M}_L$  trivially connects  $\Omega$ . We will assume  $G$  is not degenerate, therefore there exists a subgraph  $G' = (V', E') \subseteq G$  and a coloring  $F'$  as described in the beginning of

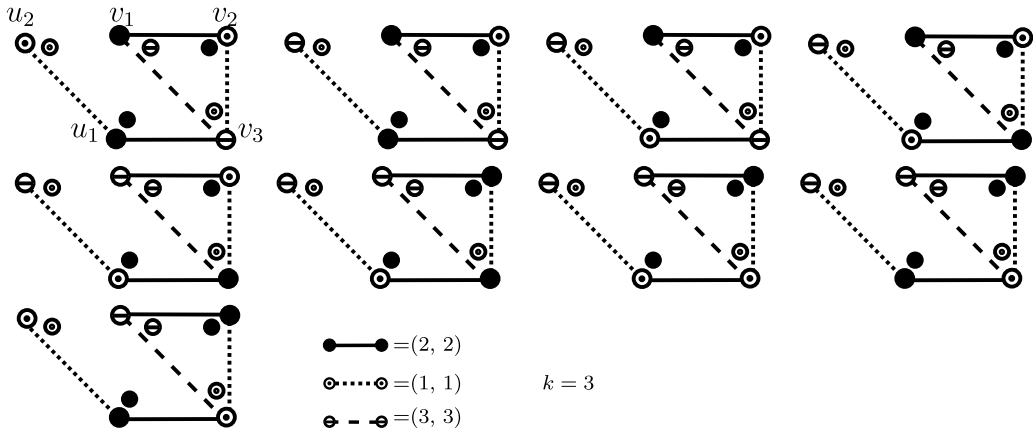


Fig. 2. An example cycle  $(v_1, v_2, v_3)$  and adjacent path  $(u_1, u_2)$ . The figures demonstrate the process of correcting the path and cycle if there is an extra color available at  $u_2$ .

Section 2 where degenerate is defined. Recall that  $G'$  does not have any flowers or color-fixed vertices and the conforming colorings of  $G'$  are in bijection with the conforming colorings of  $G$ . It suffices to show  $\mathcal{M}_L$  connects  $\Omega(G', F', k)$ . For the following proof we will assume that we are working with the graph  $G'$ .

The proof technique is similar to the proof of Theorem 3; we start with two colorings  $\beta$  and  $\alpha$  and show that we can always find a path to reduce the number of vertices whose colors differ between  $\beta$  and  $\alpha$  (i.e., a path from  $\beta$  to  $\beta'$  such that  $\phi(\beta', \alpha) < \phi(\beta, \alpha)$  where  $\phi$  is defined as in Theorem 3). Let  $d_m(v) = d_1(v) + 2d_2(v) + d_s(v)$  where  $d_1, d_2$  and  $d_s$  are defined as in Section 1.2. We assume for all  $v \in V$  such that  $\beta(v) \neq \alpha(v)$ ,  $v$  does not have any colors valid in both  $\alpha$  and  $\beta$  otherwise it is straightforward to find an appropriate path which reduces the distance  $\phi$ . Vertex  $v$  having no colors valid in both  $\alpha$  and  $\beta$  implies that there must be an edge blocking every color. Thus,  $d_m(v) = \Delta_m = k$  and there exists a set of edges  $e_1, e_2, \dots, e_k$  adjacent to  $v$  such that each edge has a unique color and no more than two edges in the set have the same endpoints unless they are self-loops. This is because in the adapted coloring setting each adjacent vertex  $u$  can only prevent or block two colors ( $\alpha(u)$  and  $\beta(u)$ ) from being valid colors for  $v$  in  $\beta$  or  $\alpha$  and only if there exist two edges  $e_i, e_j$  such that  $F(e_i) = \alpha(u)$  and  $F(e_j) = \beta(u)$ .

Let  $v \in V'$  satisfy  $\beta(v) \neq \alpha(v)$  and let  $\beta(v) = b$  and  $\alpha(v) = a$ . Since  $a$  is not a valid color for  $v$  in  $\beta$  there exists an edge  $e = (u, v)$  such that  $F(e) = \beta(u) = a$  and  $\alpha(u) \neq a$ . This implies a cycle  $(v_1, v_2, v_3, \dots, v_t)$  such that for each pair of vertices in the cycle  $(v_i, v_{i+1})$  (similarly for  $(v_t, v_1)$ ) there exists an edge  $(v_i, v_{i+1})$  such that either  $F(v_i, v_{i+1}) = \beta(v_i) = \alpha(v_{i+1})$  or  $F(v_i, v_{i+1}) = \alpha(v_i) = \beta(v_{i+1})$  (see Fig. 1).

Now consider any vertex  $v_i$  on the cycle. If it were possible to recolor  $v_i$  differently in either  $\alpha$  or  $\beta$ , without loss of generality assume  $\alpha$ , then we could recolor either  $v_{i+1}$  or  $v_{i-1}$ , again assume  $v_{i+1}$  in  $\alpha$  so that  $\alpha(v_{i+1}) = \beta(v_{i+1})$  and we could continue around the cycle until  $\alpha$  and  $\beta$  agree for every vertex (see Fig. 1). This would provide an appropriate path  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_t = \alpha'$ .

We now consider the case that there is not a vertex on the cycle that can be recolored in either  $\alpha$  or  $\beta$ . This implies that for both  $\alpha$  and  $\beta$ , every color except the current color is invalid for  $v_1$ . Since  $k \geq \max(\Delta, 3)$ , we may assume that  $v_1$  has neighbors outside the cycle therefore there exists a vertex  $u_1$ , outside the cycle, and edge  $f_1 = (v_1, u_1)$  (i.e.,  $u_1$  is adjacent to  $v_1$ ) such that  $F(f_1) = \alpha(u_1) = \beta(u_1)$ . Otherwise, we could recolor  $v_1$  in either  $\alpha$  or  $\beta$ . If it were possible to recolor  $u_1$  in either  $\alpha$  or  $\beta$  then we could recolor  $u_1$ , update the cycle, as in the previous case, and finally return  $u_1$  to its original coloring which satisfies  $\alpha(u_1) = \beta(u_1)$ . Again, this would provide an appropriate path.

Consider any maximal path  $u_1, u_2, \dots, u_x$  such that for each  $i > 1$ ,

$$F(u_{i-1}, u_i) = \alpha(u_i) = \beta(u_i),$$

each  $u_i$  is distinct from each other and from every  $v_i$ . Notice that since both  $\alpha$  and  $\beta$  are valid conforming colorings, this implies that for all  $x \geq i > 1$ ,  $F(u_{i-1}, u_i) \neq \alpha(u_{i-1})$  and  $F(u_{i-1}, u_i) \neq \beta(u_{i-1})$ . Let  $m$  be the smallest number  $1 < m \leq x$  such that  $u_m$  has another valid color in  $\alpha$  or  $\beta$ . For each  $1 < i < m$ ,  $u_i$  does not have any additional valid colors and  $F(u_{i-1}, u_i) = \alpha(u_i) = \beta(u_i)$ . First consider the coloring  $\beta$ , for each color  $c$ , there is either a vertex  $v_c$  with edge  $e_c = (v_c, u_i)$  such that  $F(e_c) = \beta(v_c) = c$  or a self loop  $e_c$  with  $F(e_c) = c$ . Similarly for  $\alpha$ , for each color  $c$ , there is either a vertex  $v_c$  with edge  $e_c = (v_c, u_i)$  such that  $F(e_c) = \alpha(v_c) = c$  or a self loop  $e_c$  with  $F(e_c) = c$ . Additionally, for each  $u_i$ , the number of adjacent vertices plus self-loops is at most  $k$  so each adjacent vertex (or self-loop) must have a distinct color in  $\alpha$  and in  $\beta$  (or edge color in the case of self-loops). Thus if we recolor any vertex adjacent to  $u_i$  (excluding  $u_{i-1}$ ) this will result in an additional valid color for  $u_i$  (i.e., each vertex or self-loop blocks a distinct color in (in  $\alpha$  and in  $\beta$ ) so if we recolor that vertex we will have an additional missing color among the neighbors and self-loops of  $u_i$  and thus another valid color). We can iteratively recolor every  $u_i$  with  $i \leq m$  in descending order in  $\alpha$ , recolor  $v_1$ , recolor the entire cycle so it agrees with  $\beta$  and finally return

each  $u_i$  to the original color in  $\alpha$  (see Fig. 2), this would give an appropriate path from  $\alpha$  to  $\alpha'$ . We show, by contradiction, that such a  $u_i$  must exist.

Without loss of generality assume  $\alpha(u_x) = \beta(u_x) = k$  and it is not possible to recolor any vertices along the path  $u_1, \dots, u_x$  in either  $\alpha$  or  $\beta$ . Notice that since this implies that there are no multi-edges. If there were any multi-edges then because of the degree constraint we would have an extra color in  $\alpha$  and  $\beta$  since a vertex can only block a single color in the adapted setting yet we add two to the  $d_m(v)$  whenever we have more than one edge to the same neighbor. As it is not possible to recolor  $u_x$  in either  $\beta$  or  $\alpha$ ,  $u_x$  must have  $k - 1$  adjacent vertices  $x_1, \dots, x_{k-1}$  (not necessarily unique) such that for each  $x_i$ , there exists an edge  $(u_x, x_i)$  such that  $F(u_x, x_i) = \alpha(x_i) = \beta(x_i) = i$ . Also, since the path is maximal, these vertices must either be in the cycle  $(v_1, \dots, v_t)$  or the path  $(u_1, \dots, u_x)$  (or  $u_x$  itself). However every vertex in the cycle is colored differently in  $\alpha$  than  $\beta$  so they cannot be included in the vertices adjacent to  $u_x$ . If there was a vertex  $u_j$  in the path that was adjacent to  $u_x$  then that would imply that either  $u_j$  has two adjacent vertices  $x, x'$  with  $F(u_j, x) = F(u_j, x') = \alpha(u_j)$  or there are two edges between  $u_j$  and  $u_x$ . Either way this implies that there exists a color  $c \neq \alpha(u_j)$  that is always valid for  $u_j$ , implying that  $u_j$  can be recolored, a contradiction to the assumption that no  $u_i$  can be recolored in  $\alpha$  or  $\beta$ . Finally consider the case where all of the vertices  $x_1, \dots, x_{k-1}$  are  $u_x$  itself (i.e.,  $u_x$  has  $k - 1$  self-loops). In this case there must be a color missing from the self-loops otherwise  $u_x$  would be color-fixed; a contradiction since  $G'$  does not have any color-fixed vertices.  $\square$

### 2.3. Rapid mixing of $\mathcal{M}_L$

We have established that we can find a conforming color efficiently and the chain  $\mathcal{M}_L$  connects the state space  $\Omega$ . This means we can start at a conforming coloring, perform a random walk with transitions of  $\mathcal{M}_L$  and the limiting distribution will be uniform over all conforming colorings. For this to be useful in practice, we require the chain to converge quickly so that after a few (polynomial) number of steps we can reach a conforming coloring that is close to uniform. We begin with some relevant background on Markov chain mixing. Let  $\mathcal{M}$  be a Markov chain on the state space  $\Omega$  with transition matrix  $\mathcal{P}$  and stationary distribution  $\pi$ . For all  $\epsilon > 0$ , the mixing time  $\tau(\epsilon)$  of  $\mathcal{M}$  is defined as

$$\tau(\epsilon) = \min \left\{ t : \max_{x \in \Omega} \frac{1}{2} \sum_{y \in \Omega} |\mathcal{P}^t(x, y) - \pi(y)| \leq \epsilon, \forall t' \geq t \right\},$$

where  $\mathcal{P}^t(x, y)$  is the  $t$ -step transition probability. We say that a Markov chain is *rapidly mixing* if the mixing time is bounded above by a polynomial in  $n$  and  $\log(\epsilon^{-1})$  and *slowly mixing* if the mixing time is bounded below by an exponential in  $n$ , where  $n$  is the size of each configuration in  $\Omega$ . In the case of conforming colorings,  $n$  is the number of vertices in the underlying graph. Our rapid mixing proof uses a detailed application of the path coupling technique due to Dyer and Greenhill [8]. To prove a stronger result in the adapted coloring setting we use a more sophisticated coupling introduced by Jerrum [15].

First, we state the path coupling theorem due to Dyer and Greenhill [8] which we will use to prove rapid mixing for both  $\mathcal{M}_L$  and  $\mathcal{M}_C$ .

**Theorem 5.** *Let  $\phi$  be an integer valued metric defined on  $\Omega \times \Omega$  which takes values in  $\{0, \dots, B\}$ . Let  $U$  be a subset of  $\Omega \times \Omega$  such that for all  $(x_t, y_t) \in \Omega \times \Omega$  there exists a path  $x_t = z_0, z_1, \dots, z_r = y_t$  between  $x_t$  and  $y_t$  such that  $(z_i, z_{i+1}) \in U$  for  $0 \leq i < r$  and  $\sum_{i=0}^{r-1} \phi(z_i, z_{i+1}) = \phi(x_t, y_t)$ . Let  $\mathcal{M}$  be a Markov chain on  $\Omega$  with transition matrix  $\mathcal{P}$ . Consider any random function  $f : \Omega \rightarrow \Omega$  such that  $\Pr[f(x) = y] = \mathcal{P}(x, y)$  for all  $x, y \in \Omega$ , and define a coupling of the Markov chain by  $(x_t, y_t) \rightarrow (x_{t+1}, y_{t+1}) = (f(x_t), f(y_t))$ . If  $E[\phi(x_{t+1}, y_{t+1})] \leq \phi(x_t, y_t)$ , for all  $(x_t, y_t) \in U$ , let  $\alpha > 0$  satisfy  $\Pr[\phi(x_{t+1}, y_{t+1}) \neq \phi(x_t, y_t)] \geq \alpha$  for all  $t$  such that  $x_t \neq y_t$ . The mixing time of  $\mathcal{M}$  then satisfies*

$$\tau(\epsilon) \leq \left\lceil \frac{eB^2}{\alpha} \right\rceil \lceil \ln \epsilon^{-1} \rceil.$$

We are now ready to prove the following theorem bounding the mixing time of  $\mathcal{M}_L$ .

**Theorem 6.** *Given a graph  $G$ ,  $k \geq \max(\Delta, 3)$  and edge  $k$ -constraints  $F$  the mixing time of  $\mathcal{M}_L$  on  $\Omega(G, F, k)$  satisfies  $\tau(\epsilon) \leq \lceil ekn^5 \rceil \lceil \ln \epsilon^{-1} \rceil$ .*

**Proof.** We use a path coupling argument with the natural coupling. Notice that a move of  $\mathcal{M}_L$  consists of selecting a vertex  $v$  and a color  $q$ . The coupling simply uses the same vertex and color to generate both  $x_{t+1}$  and  $y_{t+1}$ . We let  $U$  be the set of configurations  $(x_t, y_t)$  that differ by the coloring of one vertex and define  $\phi(x_t, y_t)$  to be the length of the shortest path  $x_t = z_0, z_1, \dots, z_r = y_t$  between  $x_t$  and  $y_t$  such that  $(z_i, z_{i+1}) \in U$ . Note that  $\phi$  is not always the same as the Hamming distance. Without loss of generality assume that vertex  $x_t(v) = 1, y_t(v) = 2$  and  $(x_t, y_t) \in U$ . Moves which increase the distance occur if we select a vertex  $u$  adjacent to  $v$  for which edge  $e = (u, v)$  (or any edge  $(u, v)$  if there are multiple edges) satisfies  $F_v(e) = 2$  or  $F_v(e) = 1$  and select color  $F_u(e)$ . Let  $i$  be the number of such vertices. The distance is decreased if we select  $v$  and a color that succeeds in both  $x_t$  and  $y_t$ . Notice that a self-loop only blocks one good move (that decreases



distance), the color associated with the self-loop. Since  $x_t(v) = 2$  and  $y_t(v) = 1$  we know that these colors always succeed in both. All other moves are neutral. The expected change in distance therefore satisfies

$$E[\Delta\phi(x_t, y_t)] \leq \frac{1}{2|V|k}(i - (k - (\Delta - i))) \leq \frac{1}{2|V|k}(\Delta - k) \leq 0.$$

Combining this with Theorem 3 and applying Theorem 5 with  $B = |V|^2 = n^2$  (an upper bound on the maximum length of a path between any two vertices) and  $\alpha = 1/(|V|k) = 1/(kn)$  implies

$$\tau(\epsilon) \leq \lceil ekn^5 \rceil \lceil \ln \epsilon^{-1} \rceil. \quad \square$$

Next, we will prove a stronger result in the context of adapted colorings with the constraint  $k \geq \max(\Delta_m, 3)$ . Here, we use a more sophisticated coupling introduced by Jerrum [15].

**Theorem 7.** *Given a graph  $G$ ,  $k \geq \max(\Delta_m, 3)$  and edge  $k$ -coloring  $F$  the mixing time of  $\mathcal{M}_L$  on  $\Omega(G, F, k)$  satisfies  $\tau(\epsilon) \leq \lceil ekn^5 \rceil \lceil \ln \epsilon^{-1} \rceil$ .*

**Proof.** As in the proof of Theorem 6, we use a path coupling argument with  $U$  and  $\phi$  defined as in the proof of Theorem 6. Here, however, we use a more complex coupling similar to one introduced by Jerrum [15]. First, select a vertex  $u$  and a color  $c$  uniformly at random. If  $(x_t, y_t) \in U$ , without loss of generality assume that  $x_t(v) = 1$  and  $y_t(v) = 2$ . Thus,  $x_t$  and  $y_t$  differ only on vertex  $v$ . Then if we select a vertex  $u$  connected to  $v$  by both an edge colored 2 and an edge colored 1 and select color  $c = x_t(v)$  in  $x_t$  we will attempt to color vertex  $u$  color  $x_t(v)$  while in  $y_t$  we will attempt to color vertex  $u$  color  $y_t(v)$ . Similarly, if we select a vertex  $u$  connected to  $v$  by both an edge colored 2 and an edge colored 1 and select color  $c = y_t(v)$  in  $x_t$  we will attempt to color  $u$   $y_t(v)$  while in  $y_t$  we will attempt to color  $u$   $x_t(v)$ . Otherwise, we will attempt to color  $u$  color  $c$  in both  $x_t$  and  $y_t$ . Moves which increase the distance occur if we select a vertex  $u$  connected to  $v$  by an edge colored 2 and no edge colored 1 and select color 2 or if we select a vertex  $u$  connected to  $v$  by an edge colored 1 and no edge colored 2 and select color 1. Additionally if  $u$  is connected to  $v$  by both an edge colored 1 and one colored 2 then there is a bad move when we select color 2 (in this case our coupling causes  $u$  to be colored 2 in  $x_t$  and 1 in  $y_t$ , if possible) which increases the distance by one. Notice that if we select color 1 in this case, because of our careful coupling, both moves are rejected and this is a neutral move. The distance is decreased if we select  $v$  and a color that succeeds in both  $x_t$  and  $y_t$ . Notice that a self-loop only blocks one good move (i.e., disallows a color at  $v$  in  $x_t$  and  $y_t$ ), the color associated with the self-loop. Unlike the conforming coloring case, each adjacent vertex  $u$ , regardless of multiplicity, can only block one good move,  $x_t(u) = y_t(u)$ . All other moves are neutral. The expected change in distance therefore satisfies  $E[\Delta\phi(x_t, y_t)] \leq \frac{1}{2|V|k}(\Delta_m - k) \leq 0$ . Combining this with Theorem 3 and applying Theorem 5 with  $B = |V|^2 = n^2$  (an upper bound on the maximum length of a path between any two vertices) and  $\alpha = 1/(|V|k) = 1/(kn)$  implies

$$\tau(\epsilon) \leq \lceil ekn^5 \rceil \lceil \ln \epsilon^{-1} \rceil. \quad \square$$

#### 2.4. Approximate counting using $\mathcal{M}_L$

In this section, we use  $\mathcal{M}_L$  to approximately count conforming colorings. To do this we design an FPRAS which, in our context, is a randomized algorithm that given a graph  $G$  with  $n$  vertices, edge coloring  $F$  and error parameter  $0 < \epsilon \leq 1$ , produces a number  $N$  such that  $\mathcal{P}[(1 - \epsilon)N \leq A(G, F) \leq (1 + \epsilon)N] \geq \frac{3}{4}$ , where  $A(G, F)$  is the number of colorings of  $G$  conforming to  $F$  and runs in time polynomial in  $n$  and  $\epsilon^{-1}$ . Next, we will show constructively that such an algorithm exists. Our proof uses similar techniques to [16].

**Theorem 8.** *Given a graph  $G$ ,  $k \geq \max(\Delta, 3)$  and edge  $k$ -constraints  $F$ , there exists an FPRAS for counting the number of vertex  $k$ -colorings conforming to  $F$ .*

**Proof.** Theorem 6 tells us that subject to the given degree restrictions, we can approximately uniformly sample conforming colorings efficiently. We will generate samples and use them to approximately count the number of conforming colorings.

Select any vertex  $v$  in  $G$  and sample conforming colorings to approximate the probability  $p$  that for a random sample from  $\Omega$ ,  $v$  is colored  $c$ , where  $c$  is the most likely of the colors. Next, we will give a procedure for creating a graph  $G_v$  and edge constraints  $F_v$  which no longer contain  $v$  such that  $|\Omega(G_v, F_v, k)|$  is the number of conforming colorings in  $\Omega(G, F, k)$  with  $v$  colored  $c$ . Delete any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) \neq c$  as we cannot have a violation to the conforming coloring condition on such edges. For any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) = c$ , vertex  $u$  cannot be colored  $F_u(u, v)$  so add a self-loop at  $v$  with color  $(F_u(u, v), F_u(u, v))$  and remove edge  $(u, v)$ . Notice that adding this self-loop does not change the degree of  $u$  since we have removed the edge  $(u, v)$ . It is important that the degree does not increase because it ensures that the new graph we create still satisfies our degree restrictions so we can continue to sample. We can now remove vertex  $v$  and any adjacent edges. Thus  $p$  is the ratio  $|\Omega(G_v, F_v, k)|/|\Omega(G, F, k)|$  and we can estimate  $|\Omega(G_v, F_v, k)|$  recursively, thus we can estimate  $|\Omega(G, F, k)|$  as  $|\Omega(G_v, F_v, k)|/p$ . It follows from [16] and Theorem 6 that this procedure gives us an FPRAS.  $\square$

In the special case of adapted colorings with  $k \geq \max(\Delta_m, 3)$ , we prove a stronger result.

**Theorem 9.** Given a graph  $G$ ,  $k \geq \max(\Delta_m, 3)$  and edge  $k$ -coloring  $F$ , there exists an FPRAS for counting the number of vertex  $k$ -colorings adapted to  $F$ .

**Proof.** The argument closely follows the proof of Theorem 8. The only difference is we apply Theorem 7 instead of Theorem 6. We obtain a stronger result because Theorem 7, which applies only to adapted colorings, gives the same sampling result for the stronger constraint  $k \geq \max(\Delta_m, 3)$ .  $\square$

### 3. The chain $\mathcal{M}_C$ and conforming 2-colorings

For the remaining part of the paper we will look at the special case of 2-colorings and develop algorithms to approximately sample and count under certain conditions. This is an important special case because it generalizes the problem of independent sets and thus insights for sampling conforming 2-colorings can tell us how to sample independent sets. Surprisingly, in the case of conforming 2-colorings the widely-used Glauber dynamics do not connect the state space. Consider, for example, the Cartesian lattice where all horizontal edges are colored (1, 1) and all vertical edges are colored (2, 2); there are two conforming colorings corresponding to the two proper 2-colorings. We develop a new algorithm for which even proving connectivity becomes much more difficult. More specifically, to handle the case  $k = 2$ , we introduce a non-local chain  $\mathcal{M}_C$  based on “color-implied” components which we show is ergodic. Some moves of  $\mathcal{M}_C$  are Kempe chain moves analogous to those in [25], but in general they are more complicated. Additionally, under conditions based on the degrees of the color-implied components we can find a conforming 2-coloring,  $\mathcal{M}_C$  is rapidly mixing and we have an FPRAS. On the other hand, we show that there are settings when  $\mathcal{M}_C$  requires exponential time with  $\Delta = 4$ .

#### 3.1. Color-implied components

A color-implied component is a connected set of vertices where coloring any vertex in the component implies a unique coloring of the remaining vertices in the component; thus, each component has at most two valid conforming colorings. We begin with some terminology that will be helpful to formally define color-implied components. For  $b \in \{1, 2\}$ , we define a path  $P = v_1, v_2, \dots, v_x$  to be a  $b$ -alternating path from  $v_1$  to  $v_x$  if the following two conditions hold:  $F_{v_1}(v_1, v_2) = b$  and for all  $1 \leq i \leq x-2$ ,  $F_{v_{i+1}}(v_i, v_{i+1}) \neq F_{v_{i+1}}(v_{i+1}, v_{i+2})$ . We say that a path  $P_1 = v_1, v_2, \dots, v_x$  ends in color  $c$  if  $F_{v_x}(v_{x-1}, v_x) = c$ . Define two vertices  $u$  and  $v$  to be *color-implied* if there is a 1-alternating path and a 2-alternating path from  $u$  to  $v$  that end in different colors or  $u = v$ . We show that color-implied is an equivalence relation, thus determining a partition of the vertices of  $G$  into connected components  $C_1, C_2, \dots, C_s$  (e.g. Fig. 3). Each component has at most two conforming colorings. To see this, color any vertex  $v$  in the component and this uniquely determines the color of all the remaining vertices because of the 1 and 2-alternating paths between them. There are two colors choices for  $v$ , so there are two conforming colorings of the vertices in the component that contains  $v$ . Using a modified version of depth first search (DFS) we can find this partition in polynomial time.

We will reuse some terminology but in the context of conforming 2-colorings, it is necessary to modify and expand the definition of a color-fixed vertex. Here, we define a *color-fixed* vertex to be a vertex that has the same color in every conforming coloring. If we determine a graph has color-fixed vertex we will remove it from the graph as follows. Assume there is a color-fixed vertex  $v$  such that  $v$  has color  $c$  in every conforming 2-coloring. Delete vertex  $v$ . Delete any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) \neq c$ ; these no longer constrain the coloring. For any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) = c$ , vertex  $u$  cannot be colored  $F_u(u, v)$  thus vertex  $u$  is color-fixed with color  $F_u(u, v)$  where  $F_u(u, v)$  satisfies  $F_u(u, v) \neq F_u(u, v)$ . Remove vertex  $u$  and repeat this edge procedure recursively for  $u$  and any subsequent vertices which are determined to be color-fixed. It is possible that this procedure will determine a vertex is color-fixed with two different colors which implies that there is no conforming coloring of  $G$ . We now have a new graph  $G' = (V', E') \subseteq G$  and coloring  $C$  of the vertices in  $G$  that are not in  $V'$  such that the set of conforming colorings of  $G$  is the same as the set of conforming colorings of  $G'$  combined with the coloring  $C$ .

We are now ready to show how to find the color-implied components in polynomial time. Starting with any vertex  $v$ , we will use a slightly modified DFS to find all of the vertices connected to  $v$  by a 1-alternating path and for each vertex record which color is implied. We modify a standard DFS as follows. Follow all edges  $e$  leaving  $v$  such that  $F_v(e) = 1$ . The first time a vertex  $u$  is encountered in the search record whether the vertex was reached by a path ending in color 1 or 2 (this is the opposite of the implied or forced color of  $u$  when  $v$  is colored 1). If at any point you determine that a vertex  $u$  was reached by both a path ending in color 1 and a path ending in color 2 (i.e., you encounter  $u$  on a path that ends in a different color than the color already recorded) then this implies that  $v$  cannot be colored 1 in any conforming color and is thus color-fixed. In this case, remove  $v$  and all adjacent edges as described in the paragraph above. When processing a vertex  $u \neq v$  with recorded color  $c$ , only follow edges  $e$  such that  $F_u(e) = \bar{c}$ . Repeat this procedure for 2-alternating paths. The intersection of these sets (the vertices connected to  $v$  by both a 1-alternating and 2-alternating path), which we call  $I$ , is a potential color-implied component. Taking the opposite of the stored colors gives two colorings of the vertices in  $I$ . If any vertices have the same color in both colorings, then these vertices are color-fixed with that color; remove them from the graph as described above. The remaining vertices form a color-implied component. Analyze these two colorings of  $I$  to determine if they cause any violations of the constraints on the edges between vertices in  $I$ . If exactly one of these colorings causes a violation then all of the vertices in  $I$  are color-fixed with the other color; remove all vertices in  $I$  as described above.

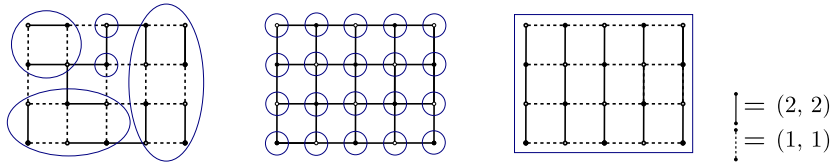


Fig. 3. The color-implied components associated with each edge coloring are circled.

If both colorings cause a violation, then no valid conforming coloring exists. Otherwise, if both colorings are valid then  $I$  is a color-implied component. Select a new vertex not already in a color-implied component and not determined to be color-fixed and repeat until all vertices are determined to be either in a color-implied component or color-fixed. It takes  $O(n^2)$  time to find each component, thus the overall time to find the color-implied components is  $O(n^3)$ .

Now, we give a detailed proof that the color-implied relation is an equivalence relation. The equivalence classes defined by the color-implied relation, which we refer to as color-implied components, will be used by the Markov chain  $\mathcal{M}_C$ . Later, we show that all edges between two color-implied components have the same color (here color refers to which of the two possible colorings of the adjacent components the edge is labeled with) and thus we can define a coloring  $F^{(C)}$  of the component graph  $G^{(C)}$ , whose vertices are the color-implied components, which we will define formally later.

**Lemma 1.** *The color-implied relation is an equivalence relation.*

**Proof.** To prove that color-implied is an equivalence relation we need to show that it is reflexive, symmetric and transitive. By definition color-implied is reflexive. To show symmetry, we prove that if  $u$  color-implies  $v$  then  $v$  color-implies  $u$ . Assume  $u$  color-implies  $v$ , then there are 1 and 2-alternating paths from  $u$  to  $v$  then end in different colors. The path that ends in color 1 is a 1-alternating path from  $v$  to  $u$  and the path that ends in color 2 is a 2-alternating path from  $v$  to  $u$ . Thus,  $v$  color-implies  $u$  and the relation color-implies is symmetric. To show transitivity, assume vertex  $u$  color-implies vertices  $v$  and vertex  $v$  color-implies vertex  $x$ . Since  $u$  color-implies  $v$ , there are 1-alternating and 2-alternating paths from  $u$  to  $v$  that end in different colors. Let  $P_1$  be the alternating path that ends in color 1 and  $P_2$  be the alternating path that ends in color 2. Let  $P_3$  be the 2-alternating path from  $v$  to  $x$  and  $P_4$  be the 1-alternating path from  $v$  to  $x$  (these exist because  $v$  color-implies  $x$ ). Let path  $P_{13}$  be path  $P_1$  concatenated with path  $P_3$  and similarly let  $P_{24}$  be path  $P_2$  concatenated with  $P_4$ . The two paths  $P_{13}$  and  $P_{24}$  are 1 and 2-alternating paths between  $u$  and  $x$  (if  $P_1$  is  $c$ -alternating then so is  $P_{13}$  and similarly for paths  $P_2$  and  $P_{24}$ ). Colored-implies is therefore reflective, symmetric and transitive and thus an equivalence relation.  $\square$

Let the component graph,  $G^{(C)}$  be the graph whose vertices are the components  $C_1, C_2, \dots, C_s$  and there is an edge  $(C_i, C_j) \in G^{(C)}$  if there exists  $(v_i, v_j) \in G : v_i \in C_i, v_j \in C_j$ . The component graph does not have any multi-edges or self-loops. For each component  $C_i$ , let  $\rho(C_i)$  and  $\bar{\rho}(C_i)$  be the two conforming colorings of  $C_i$ . For any vertex  $v$  in component  $C$ , let  $\rho(v)$  be the color of vertex  $v$  in the coloring  $\rho(C)$  and analogously let  $\bar{\rho}(v)$  be the color of vertex  $v$  in  $\bar{\rho}(C)$ . We will show that all edges between two color-implied components have the same color (here color refers to which of the two possible colorings of the adjacent components the edge is labeled with) and thus we can define a coloring  $F^{(C)}$  of the component graph  $G^{(C)}$  as follows.

**Definition 1.** For any edge  $e = (C_i, C_j)$  in the component graph  $G^{(C)}$ :

- if for all edges  $(u, v) : u \in C_i, v \in C_j, F_u(u, v) = \rho(u)$  then  $F^{(C)}_e = \rho(C_i)$ ,
- otherwise,  $F^{(C)}_e = \bar{\rho}(C_i)$ .

Next, we give an explicit bijection between colorings of the component graph conforming to  $F^{(C)}$  and colorings of the original graph conforming to  $F$ . Using this map, we can then sample conforming colorings of  $G$  by sampling conforming colorings of  $G^{(C)}$ .

**Lemma 2.** *Let the component graph  $G^{(C)}$  and associated edge coloring  $F^{(C)}$  be defined as above. There is a bijection between the colorings of  $G$  conforming to  $F$  and colorings of  $G^{(C)}$  conforming to  $F^{(C)}$ .*

**Proof.** Given any two adjacent components  $C_i$  and  $C_j$ , let  $E_{ij}$  be the set of edges  $(u, v) : u \in C_i, v \in C_j$ . First, we show that it suffices to prove that either for all edges  $(u, v) \in E_{ij}, F_u(u, v) = \rho(u)$  or for all edges  $(u, v) \in E_{ij} : F_u(u, v) = \bar{\rho}(u)$ . Consider a conforming coloring  $\alpha$  of  $G^{(C)}$ , the coloring of each of the components in  $\alpha$  gives a coloring of the vertices in the component. This coloring does not violate any constraints within the components (by the definition of the component colorings) and cannot violate any constraints between components because these constraints are satisfied as long as the edge constraint between the components are satisfied since the edges between two components are all colored the same. Given a conforming coloring  $\beta$  of the original graph this induces a coloring on the component graph; look at each component  $C$  and determine whether it is colored  $\rho(C)$  or  $\bar{\rho}(C)$  (the only two choices). If the coloring  $\beta$  violated a constraint between two components in  $G^{(C)}$  then this constraint corresponds to at least one edge between a vertex in each of the components and this constraint would be violated by  $\beta$ .

Next, we prove that for every edge  $(C_i, C_j)$  in  $G^{(C)}$ , either for all edges  $(u, v) \in E_{ij} : u \in C_i, F_u(u, v) = \rho(u)$  or for all edges  $(u, v) \in E_{ij} : u \in C_i, F_u(u, v) = \bar{\rho}(u)$ . We assume by way of contradiction there exist two edges  $e_1 = (x, y), e_2 = (z, w)$  with  $x, z \in C_i$  and  $y, w \in C_j$  such that  $F_x(e_1) = \rho(x)$  and  $F_z(e_2) = \bar{\rho}(z)$ . We show this implies that  $z$  color-implies  $y$  contradicting  $z$  and  $y$  being in different color-implied components ( $z \in C_i$  and  $y \in C_j$ ). Since  $\rho(C_i)$  is a valid conforming coloring of  $C_i$  there exists a  $\rho(z)$ -alternating path  $P_1$  from  $z$  to  $x$  that ends in an edge  $e$  with  $F_x(e) = \bar{\rho}(x)$ . If we take path  $P_1$  and append edge  $e_1$  we have a  $\rho(z)$ -alternating path from  $z$  to  $y$ . For any color  $c$  let  $\bar{c}$  be the other color (i.e.,  $\bar{1} = 2$ ). Since  $w$  and  $y$  are in the same component there exists a  $\bar{F}_w(e_2)$ -alternating path  $P_2$  from  $w$  to  $y$ . Taking edge  $e_2$  and appending path  $P_2$  gives a  $\bar{\rho}(z)$ -alternating path from  $z$  to  $y$ . Thus we have shown that  $z$  and  $y$  are color-implied, a contradiction.  $\square$

### 3.2. The Markov chain $\mathcal{M}_C$

We now define the non-local Markov chain  $\mathcal{M}_C$ , which is based on color-implied components and connects the state space  $\Omega(G, F, 2)$ .

#### The Color-Implied Component Chain $\mathcal{M}_C$

Starting at any initial conforming coloring, iterate the following:

- Pick an integer  $i \in 1, 2, \dots, s$  u.a.r.
- With probability  $1/2$ , color component  $C_i$  with color  $\rho(C_i)$  if this results in a conforming coloring.
- With probability  $1/2$ , color component  $C_i$  with color  $\bar{\rho}(C_i)$  if this results in a conforming coloring.
- Otherwise, do nothing.

**Theorem 10.** For any graph  $G$  and edge 2-constraints  $F$ , the Markov chain  $\mathcal{M}_C$  connects  $\Omega(G, F, 2)$ .

**Proof.** Define  $\beta, \alpha \in \Omega$ . We will show that there is always a path from  $\beta$  to  $\alpha$  using transitions of  $\mathcal{M}_C$ . Let  $\phi(\beta, \alpha)$  be the number of vertices  $v$  such that  $\beta(v) \neq \alpha(v)$ . We will show that for any  $\beta, \alpha \in \Omega$  such that  $\phi(\beta, \alpha) > 0$  there exists a transition of  $\mathcal{M}_C, (\beta, \beta')$  such that  $\phi(\beta', \alpha) < \phi(\beta, \alpha)$ . Let  $v$  be a vertex such that  $\beta(v) \neq \alpha(v)$ . Let  $C_v$  be the component containing  $v$  (note that  $v$  might be the only vertex in  $C_v$ ). Let  $G^{(C)}(v)$  be the component graph  $G^{(C)}$  restricted to  $C_v$  and those components  $C_i$  with  $\beta(C_i) \neq \alpha(C_i)$  and that are connected to  $C_v$  through a path which only includes components  $C_i$  for which  $\beta(C_i) \neq \alpha(C_i)$ . So  $G^{(C)}(v)$  is the maximal connected set of components on which  $\beta$  and  $\alpha$  disagree that includes  $C_v$ . This implies that for every component  $C_j$  such that  $C_j \notin G^{(C)}(v)$  and  $C_j$  is adjacent to some component in  $G^{(C)}(v)$ ,  $\beta(C_j) = \alpha(C_j)$ . Next, we will define a partial orientation on the edges of  $G^{(C)}(v)$  such that if a vertex has an out-going edge, that edge prevents the vertex from being recolored in  $\beta$ . Consider any two adjacent components  $C_i, C_j \in G^{(C)}(v)$ . If  $\beta(C_i) = F_{C_i}^{(C)}(C_i, C_j)$  then direct the edge toward  $C_i$ , if  $\beta(C_j) = F_{C_j}^{(C)}(C_i, C_j)$  then direct the edge toward  $C_j$ , otherwise leave the edge undirected (since  $\beta$  is a conforming coloring it is not possible for both  $\beta(C_i) = F_{C_i}^{(C)}(C_i, C_j)$  and  $\beta(C_j) = F_{C_j}^{(C)}(C_i, C_j)$  to be true).

Given, the orientation defined above it is sufficient to show that  $G^{(C)}(v)$  contains a sink, which we define as a vertex with no out-going edges, because any such vertex can be recolored in  $\beta$  thus decreasing the distance between  $\beta$  and  $\alpha$ . If  $G^{(C)}(v)$  contains a single component  $C_v$  then it is always possible to recolor  $C_v$  in  $\beta$  since this coloring of  $C_v$  is valid in  $\alpha$ , and  $\beta$  and  $\alpha$  agree on all vertices adjacent to  $G^{(C)}(v)$ . To see that a sink always exists, start at any vertex in  $G^{(C)}(v)$  and arbitrary follow out-going edges as long as possible. This process only stops if a vertex with no out-going edges is reached in which case we have located a sink or if we encounter a vertex twice. Encountering a vertex twice implies that there is a cycle  $Y$  in  $G^{(C)}(v)$ , which we will show is not possible. We use the edges in  $Y$  to construct two alternating paths between vertices in different components (a contradiction), an approach similar to the one used in the proof of Lemma 2.

Without loss of generality, let  $Y = (C_1, C_2, \dots, C_t)$  with edges directed  $(C_1, C_2), (C_2, C_3), \dots, (C_{t-1}, C_t), (C_t, C_1)$ . Since edge  $(C_1, C_2)$  exists in  $G^{(C)}$  this implies that an edge  $e = (v_1, v_2) : v_1 \in C_1, v_2 \in C_2, F_{v_1}(e) = \bar{\beta}(v_1), F_{v_2}(e) = \beta(v_2)$  exists, implying that there is a  $\bar{\beta}(v_1)$  alternating path from  $v_1$  to  $v_2$ . Next, we will show that there is also a  $\beta(v_1)$  alternating path from  $v_1$  to  $v_2$ , a contradiction. To do this we will create the path by following the cycle around from  $C_i$  to  $C_j$  in the opposite direction. The edges in the cycle will imply the existence of an alternating path. Edge  $(C_t, C_1)$  implies that an edge  $e_1 = (u_1, v_t) : u_1 \in C_1, v_t \in C_t, F_{u_1}(e_1) = \beta(u_1), F_{v_t}(e_1) = \bar{\beta}(v_t)$  exists. Since  $u_1$  and  $v_1$  are in the same component there is a  $\beta(v_1)$  alternating path  $P_1$  from  $v_1$  to  $u_1$  that ends in  $\bar{\beta}(u_1)$  if we take  $P_1$  and concatenate it with  $e_1$  we have a  $\beta(v_1)$  alternating path from  $v_1$  to  $v_t$  which ends with  $e_1$ . Next, edge  $(C_{t-1}, C_t)$  implies that an edge  $e_2 = (v_{t-1}, u_t) : v_{t-1} \in C_{t-1}, u_t \in C_t, F_{u_t}(e_2) = \beta(u_t), F_{v_{t-1}}(e_2) = \bar{\beta}(v_{t-1})$  exists. Since  $u_t$  and  $v_t$  are in the same component there is a  $\beta(v_t)$  alternating path  $P_2$  from  $v_t$  to  $u_t$  that ends in  $\bar{\beta}(u_t)$  if we take  $P_2$  and concatenate it with  $e_2$  we have a  $\beta(v_t)$  alternating path from  $v_t$  to  $v_{t-1}$ . If we combine this path with our path from  $v_1$  to  $v_t$  so our total path is  $P_1 \cup e_1 \cup P_2 \cup e_2$  we have a  $\beta(v_1)$  alternating path from  $v_1$  to  $v_{t-1}$ . Similarly we can extend this path to a  $\beta(v_1)$  alternating path from  $v_1$  to  $v_2$ , a contradiction.  $\square$

### 3.3. Finding a conforming 2-coloring

We show that  $\mathcal{M}_C$  is rapidly mixing when every vertex  $v$  of the component graph  $G^{(C)} = G^{(C)}(G, F)$  has  $d(v) \leq 2$  or  $d(v) \leq 4$  and  $v$  is *monochromatic*. We say a vertex  $v \in G^{(C)}$  is *monochromatic* if for any two edges  $e_1, e_2$  adjacent to  $v$ ,  $F_v^{(C)}(e_1) = F_v^{(C)}(e_2)$ . In the adapted setting, if  $C_i$  is a single vertex then this corresponds to having all adjacent edges colored the same. We use path coupling to prove that a related chain  $\mathcal{M}_E$  mixes rapidly and then use the comparison technique (see, [6,20]) to relate the mixing time of  $\mathcal{M}_E$  to the mixing time of  $\mathcal{M}_C$ . The chain  $\mathcal{M}_E$  is a generalization of the edge chain introduced by Luby and Vigoda [18]. Under these same conditions, we give a polynomial time algorithm for finding a conforming coloring and an FPRAS for approximately counting the number of conforming colorings by showing the model is self-reducible and appealing to [16].

**Theorem 11.** *Given a graph  $G$  with  $n$  vertices, edge 2-coloring  $F$ , the color-implied components and the component graph  $G^{(C)} = G^{(C)}(G, F)$  such that for every vertex  $v \in G^{(C)}$  either  $d(v) \leq 2$  or  $d(v) \leq 4$  and  $v$  is monochromatic, then we can find a conforming 2-coloring or determine there is no conforming 2-coloring in time  $O(n)$ .*

**Proof.** We give an algorithm for finding a conforming coloring  $\alpha$  of the vertices of the component graph  $G^{(C)}$ , which implies a conforming coloring of the vertices of  $G$ . If you are not given the component graph  $G^{(C)}$  and associated color-implied components, you can find these in time  $O(n^3)$  as described in Section 3.1. If the component graph  $G^{(C)}$  contains a monochromatic vertex  $C$ , we determine a coloring  $\alpha(C)$  and remove  $C$  to create smaller graph on which we can recurse. Without loss of generality assume that all edges adjacent to  $C$  in  $G^{(C)}$  satisfy  $F_C^{(C)}(e) = \rho(C)$ . Let  $\alpha(C) = \bar{\rho}(C)$  and remove  $C$  and all of edges adjacent to  $C$  from  $G^{(C)}$ . Notice that because all edges  $e$  adjacent to  $C$  are colored  $F_C^{(C)}(e) = \rho(C)$  by coloring  $C$  color  $\bar{\rho}(C)$ , component  $C$  no longer puts any coloring constraints on its adjacent vertices so we can simply remove  $C$  and all of its adjacent edges. If a conforming coloring of  $G^{(C)}$  exists, then there must be a conforming coloring of the new graph without component  $C$ . Repeat this procedure until there are no more monochromatic components. From the degree constraint, this implies that we are only left with degree two components so we have a collection of cycles. Each vertex  $v$  in every cycle must have two adjacent edges  $e_1$  and  $e_2$  with one edge (assume  $e_1$ ) satisfying  $F_v^{(C)}(e_1) = \rho(v)$  and the other edge (assume  $e_2$ ) satisfying  $F_v^{(C)}(e_2) = \bar{\rho}(v)$ . Next we will show that such a cycle cannot exist. Specifically we will assume to the contrary that such a cycle exists and show that this implies that two vertices in different components around the cycle are color-implied, a contradiction.

Let  $Y = (C_1, C_2, \dots, C_t)$  with edges  $e_1 = (C_1, C_2), e_2 = (C_2, C_3), \dots, e_{t-1} = (C_{t-1}, C_t), e_t = (C_t, C_1)$  such that  $F_{C_1}^{(C)}(e_t) \neq F_{C_1}^{(C)}(e_1)$  and for  $1 < i \leq t, F_{C_i}^{(C)}(e_{i-1}) \neq F_{C_i}^{(C)}(e_i)$ . Without loss of generality, assume that  $F_{C_1}^{(C)}(e_1) = \bar{\rho}(C_1)$  and  $F_{C_1}^{(C)}(e_t) = \rho(C_1)$ . Since edge  $e_1 = (C_1, C_2)$  exists in  $G^{(C)}$  this implies that an edge  $e = (v_1, v_2) : v_1 \in C_1, v_2 \in C_2, F_{v_1}^{(C)}(e) = \bar{\rho}(v_1)$  exists, implying that there is a  $\bar{\rho}(v_1)$  alternating path from  $v_1$  to  $v_2$ . Next, we will show there is also a  $\rho(v_1)$  alternating path from  $v_1$  to  $v_2$ , a contradiction. To do this we will create the path by following the cycle around from  $C_1$  to  $C_2$  in the opposite direction. The edges in the cycle will imply the existence of an alternating path. Edge  $e_t = (C_t, C_1)$  with  $F_{C_1}^{(C)}(e_t) = \rho(C_1)$  implies that an edge  $f_1 = (u_1, v_t) : u_1 \in C_1, v_t \in C_t, F_{u_1}(f_1) = \rho(u_1)$  exists. Since  $u_1$  and  $v_1$  are both in  $C_1$  and  $\rho$  is one of the two conforming colorings of  $C_1$ , there is a  $\rho(v_1)$  alternating path  $P_1$  from  $v_1$  to  $u_1$  that ends in an edge  $f$  with  $F_{u_1}(f) = \bar{\rho}(u_1)$  if we take  $P_1$  and concatenate it with  $f_1$  we have a  $\rho(v_1)$  alternating path from  $v_1$  to  $v_t$  which ends with  $f_1$ . Next, edge  $(C_{t-1}, C_t)$  implies that an edge  $f_2 = (v_{t-1}, u_t) : v_{t-1} \in C_{t-1}, u_t \in C_t, F_{u_t}(f_2) \neq F_{v_t}(f_1)$  exists because we know that  $F_{C_t}^{(C)}(C_t, C_1) \neq F_{C_{t-1}}^{(C)}(C_t, C_{t-1})$  by assumption. Since  $u_t$  and  $v_t$  are in the same component, there are  $\bar{\rho}(v_t)$  and  $\rho(v_t)$  alternating paths from  $v_t$  to  $u_t$  that end in  $\rho(u_t)$  and  $\bar{\rho}(u_t)$  respectively. Let  $P_2$  be the  $\bar{\rho}(v_t)$  alternating path if  $F_{v_t}(f_1) = \rho(v_t)$  and otherwise be the  $\rho(v_t)$  alternating path. If we take  $P_2$  and concatenate it with  $e_2$  we have an alternating path from  $v_t$  to  $v_{t-1}$ . If we combine this path with our path from  $v_1$  to  $v_t$  so our total path is  $P_1 \cup e_1 \cup P_2 \cup e_2$  we have a  $\rho(v_1)$  alternating path from  $v_1$  to  $v_{t-1}$ . Similarly, we can extend this path to a  $\rho(v_1)$  alternating path from  $v_1$  to  $v_2$ , a contradiction.

Since such a cycle cannot exist, after iteratively coloring all monochromatic components as described above, we are left with a valid conforming coloring of  $G$ . This procedure can be completed in time  $O(n)$ . First go through each component and determine if it is monochromatic, adding these components to a queue. Next, for each component in the queue, color it appropriately as described above and remove its edges. Next look at all of its adjacent components that have not already been colored and add them to the queue if they are now monochromatic. The initial pass takes time  $O(n)$  since there are at most  $n$  components, each with at most 4 neighbors. Each component gets processed once at which point we look at all of the component's neighbors and their edges to determine if they should get added to the queue. This takes time constant for each component (since the maximum degree is 4), giving a running time of  $O(n)$ .  $\square$

### 3.4. Rapid mixing of $\mathcal{M}_C$

To prove that  $\mathcal{M}_C$  is rapidly mixing, we first use path coupling to prove that a related chain  $\mathcal{M}_E$  mixes rapidly, and then use the comparison technique (see, [6,20]) to relate the mixing time of  $\mathcal{M}_E$  to the mixing time of  $\mathcal{M}_C$ . The chain  $\mathcal{M}_E$  is a generalization of the edge chain introduced by Luby and Vigoda in the context of independent sets [18].



### The Colored Implied Component Edge Chain $\mathcal{M}_E$

Starting at any initial conforming coloring, iterate the following:

- Select an edge  $e = (u, v)$  in  $G^{(C)}$  u.a.r.
- Select one of the 3 valid colorings for  $u$  and  $v$  u.a.r. (i.e.,  $\rho(u), \bar{\rho}(v)$ ).
- Color  $u$  and  $v$  with these colors if this results in a conforming coloring.
- Otherwise, do nothing.

Since the moves of  $\mathcal{M}_C$  are a subset of the moves of  $\mathcal{M}_E$ , [Theorem 10](#) also proves that  $\mathcal{M}_E$  connects the state space  $\Omega$ . Let  $\mathcal{P}_E$  be the transition matrix of  $\mathcal{M}_E$ . For all  $\beta_1, \beta_2 \in \Omega$ ,  $\mathcal{P}_E(\beta_1, \beta_2) = \mathcal{P}_E(\beta_2, \beta_1)$  so detailed balance implies that  $\mathcal{M}_E$  converges to the uniform distribution over  $\Omega$  [[21](#)]. The following theorem establishes that  $\mathcal{M}_E$  is rapidly mixing.

**Theorem 12.** *Given a graph  $G$  with  $n$  vertices and edge 2-constraints  $F$  with component graph  $G^{(C)} = G^{(C)}(G, F)$  such that for every vertex  $v$  in  $G^{(C)}$  either  $v$  satisfies  $d(v) \leq 2$  or  $v$  satisfies  $d(v) \leq 4$  and  $v$  is monochromatic, then the mixing time of  $\mathcal{M}_E$  on  $\Omega(G, F, 2)$  satisfies  $\tau(\epsilon) \leq \lceil 6en^3 \rceil \lceil \ln \epsilon^{-1} \rceil$ .*

**Proof.** We use path coupling with a natural coupling and distance function. Notice that a move of  $\mathcal{M}_E$  consists of selecting an edge  $e$  in  $G^{(C)}$  and a new valid coloring of the two components adjacent to  $e$ . Recall that each component  $C$  has exactly two valid colorings which we call  $\rho(C)$  and  $\bar{\rho}(C)$ . The coupling selects the same edge and coloring to generate both  $x_{t+1}$  and  $y_{t+1}$ . Let  $U$  be the set of configurations  $(x_t, y_t)$  that differ by the coloring of one component and define  $\phi(x_t, y_t)$  to be the Hamming distance (the number of components  $C_i$  such that  $x_t(C_i) \neq y_t(C_i)$ ). Let  $(x_t, y_t) \in U$ , without loss of generality we will assume that  $C_i$  is colored  $\rho(C_i)$  in  $x_t$  and  $\bar{\rho}(C_i)$  in  $y_t$  and all other components  $C_j$  are colored  $\rho(C_j)$  in both  $x_t$  and  $y_t$ . A good move, which decreases the distance  $\phi(x_t, y_t)$ , occurs if we select an edge containing  $C_i$  and a valid coloring. A bad move, that increases the distance, occurs only if we select an edge adjacent to an edge containing  $C_i$ . Selecting any other edges will result in the same change in  $x_t$  and  $y_t$  and therefore does not change  $\phi(x_t, y_t)$ . Consider any edge  $e = (C_i, C_j)$ , we will show that if we only consider moves which involve edges containing  $C_j$ ,  $E[\Delta\phi(x_t, y_t)] \leq 0$ . If we can show this for all such  $C_j$  then this implies that for all moves  $E[\Delta\phi(x_t, y_t)] \leq 0$ . There may be an edge between neighbors of  $C_i$  but in this case any bad moves involving this edge are counted twice.

First, consider the case where  $F_{C_j}^{(C)}(e)$  is the same color for every edge  $e$  adjacent to  $C_j$ , and  $C_j$  has degree at most 4. Since  $x_t(C_j) = y_t(C_j)$  and  $x_t(C_i) \neq y_t(C_i)$  we know that  $(\rho(C_j), \rho(C_i))$  and  $(\rho(C_j), \bar{\rho}(C_i))$  are both valid conforming colorings, which implies that  $F_{C_j}^{(C)}(e) = \bar{\rho}(C_j)$  and this is true for every edge adjacent to  $C_j$ . Consider any component  $C_k \neq C_i$  which is adjacent to  $C_j$ . If the edge  $(C_j, C_k)$  satisfies  $F_{C_k}^{(C)}(C_j, C_k) = \rho(C_k)$  then this edge contributes one bad move with distance two,  $(\bar{\rho}(C_j), \bar{\rho}(C_k))$ . However, this implies that no other edges can have any bad moves since a bad move for these edges would involve coloring  $\bar{\rho}(C_j)$  which would not be valid given the constraint on edge  $(C_k, C_j)$ . In the other case, if edge  $(C_j, C_k)$  satisfies  $F_{C_k}^{(C)}(C_j, C_k) = \bar{\rho}(C_k)$  then this edge contributes one bad move with distance one,  $(\bar{\rho}(C_j), \rho(C_k))$  and puts no constraints on the other edges. In all cases there are two good moves corresponding to selecting edge  $(C_i, C_j)$  and coloring it  $(\bar{\rho}(C_i), \rho(C_j))$  or  $(\rho(C_i), \rho(C_j))$ , each of which decreases the distance by one. If there is a move that increase the distance by two this means that one vertex  $C_k$  adjacent to  $C_j$  satisfies  $F_{C_k}^{(C)}(C_j, C_k) = \rho(C_k)$ . However, this prevents any other bad moves. In this case there are two good moves that decrease the distance by one and one bad move that increases the distance by two, so the overall expected change in distance is zero. If there are no bad moves that increase the distance by two, so all components  $C_k$  (except  $i$ ) that are adjacent to  $C_j$  satisfy  $F_{C_k}^{(C)}(C_k, C_j) = (\bar{\rho}(C_k), \bar{\rho}(C_j))$ . This implies that either  $(\bar{\rho}(C_i), \bar{\rho}(C_j))$  or  $(\rho(C_i), \bar{\rho}(C_j))$  is an additional good move. Since  $C_j$  has degree at most four, there are at most three bad moves that increase the distance by one. So in this case there are three good moves and at most three bad moves each of which changes the distance by 1. Thus, we have shown that the expected change in distance is at most zero when considering only moves that involve vertex  $C_j$ .

If  $C_j$  is only adjacent to  $C_i$  then it is not involved in any bad moves. Next consider the case where  $C_j$  has degree two and is adjacent to two vertices  $C_i$  and  $C_k$ . Again we know that there are at least two good moves involving edge  $(C_i, C_j)$ , corresponding to  $(\rho(C_i), \rho(C_j))$  and  $(\bar{\rho}(C_i), \rho(C_j))$  each of which decreases the distance by one. There are 3 possible colorings for edge  $(C_j, C_k)$ . We know that one of them is  $(\rho(C_j), \rho(C_k))$ , the coloring in  $x_t$  and  $y_t$ , and therefore is a neutral move. This implies that there are at most two bad moves involving edge  $(C_j, C_k)$ . If only one of them is a bad move (the other could be neutral,  $(\rho(C_j), \bar{\rho}(C_k))$  or invalid) then since it increases the distance by at most two and there are two good moves this case is neutral. Next consider the case where edge  $(C_j, C_k)$  allows all three colorings and two of them are bad moves. This implies that the other two valid colorings are  $(\bar{\rho}(C_j), \bar{\rho}(C_k))$  and  $(\bar{\rho}(C_j), \rho(C_k))$ . However this implies that either  $(\bar{\rho}(C_j), \rho(C_i))$  or  $(\bar{\rho}(C_j), \bar{\rho}(C_i))$  is also a valid good move. So, there are two bad moves, one of which increases the distance by two and one of which increases the distance by one. This is balanced by three good moves each of which decreases the distance by one. Therefore in all cases we have shown that  $E[\Delta\phi(x_t, y_t)] \leq 0$ . Combining this with [Theorem 10](#) we can apply [Theorem 5](#) with  $B \leq |V| = n$  and  $\alpha \geq 1/(6|V|) = 1/(6n)$  since there are at most  $2n$  edges (each component has max degree 4) and 3 coloring choices for each edge of  $G^{(C)}$ . This implies that the mixing time of  $\mathcal{M}_E$  on  $\Omega(G, F, 2)$  satisfies

$$\tau(\epsilon) \leq \lceil 6en^3 \rceil \lceil \ln \epsilon^{-1} \rceil. \quad \square$$

Next, we will use the comparison method and [Theorem 12](#) to bound the mixing time of  $\mathcal{M}_C$ . Let  $\mathcal{P}'$  and  $\mathcal{P}$  be two reversible Markov chains with mixing times  $\tau(\epsilon)$  and  $\tau'(\epsilon)$  respectively on the same state space  $\Omega$  with the same stationary distribution  $\pi$ . The comparison method (see [\[6\]](#) and [\[20\]](#)) allows us to relate the mixing times of these two chains. Let  $E(\mathcal{P}) = \{(x, y) : \mathcal{P}(x, y) > 0\}$  and  $E(\mathcal{P}') = \{(x, y) : \mathcal{P}'(x, y) > 0\}$  denote the sets of edges of the two graphs, viewed as directed graphs. For each pair of state  $x, y$  in  $E(\mathcal{P}')$ , with  $\mathcal{P}'(x, y) > 0$ , define a path  $\gamma_{xy}$  using a sequence of states  $x = x_0, x_1, \dots, x_k = y$  with  $\mathcal{P}(x_i, x_{i+1}) > 0$  for all  $0 \leq i < k$ , and let  $|\gamma_{xy}|$  denote the length of the path. Let  $\Gamma(z, w) = \{(x, y) \in E(\mathcal{P}') : (z, w) \in \gamma_{xy}\}$  be the set of paths that use the transition  $(z, w)$  of  $\mathcal{P}$ . Let  $\pi_* = \min_{x \in \Omega} \pi(x)$ . Finally, define

$$A = \max_{(z,w) \in E(\mathcal{P})} \left\{ \frac{1}{\pi(z)\mathcal{P}(z,w)} \sum_{\Gamma(z,w)} |\gamma_{xy}| \pi(x) \mathcal{P}'(x,y) \right\}.$$

We use the following formulation of the comparison method [\[20\]](#).

**Theorem 13.** *With the above notation, for  $0 < \epsilon < 1$ , we have*

$$\tau(\epsilon) \leq \frac{4 \log(1/(\epsilon \pi_*))}{\log(1/2\epsilon)} A \tau'(\epsilon).$$

Now we are ready to combine [Theorem 12](#), which bounds the mixing time of  $\mathcal{M}_E$ , with the comparison theorem ([Theorem 13](#)) to prove [Theorem 14](#) bounding the mixing time of  $\mathcal{M}_C$ .

**Theorem 14.** *Given a graph  $G$  with  $n$  vertices and edge 2-constraints  $F$  with component graph  $G^{(C)} = G^{(C)}(G, F)$  such that for every vertex  $v \in G^{(C)}$  either  $d(v) \leq 2$  or  $d(v) \leq 4$  and  $v$  is monochromatic, then the mixing time of  $\mathcal{M}_C$  on  $\Omega(G, F, 2)$  with the given constraints satisfies  $\tau(\epsilon) = O(n^4 \ln \epsilon^{-1})$ .*

**Proof.** Let  $\mathcal{P}$  be the transition matrix of  $\mathcal{M}_C$  and  $\mathcal{P}'$  be the transition matrix of  $\mathcal{M}_E$ . In order to use [Theorem 13](#) we first need to for each  $(x, y) \in \Omega$  such that  $\mathcal{P}'(x, y) > 0$  define a path  $\gamma_{xy}$  between  $x$  and  $y$  using only valid moves of  $\mathcal{M}_C$ . Without loss of generality we will consider two types of moves  $\mathcal{M}_E$  can make. First consider the case where the move only affects one component  $u$ , and assume  $\mathcal{M}_E$  changes the coloring of  $u$  from  $\rho(u)$  to  $\bar{\rho}(u)$ . This move is also a move in  $\mathcal{M}_C$  so we will use this single edge as our path. Next, it is possible that  $\mathcal{M}_E$  changes the coloring of two components. Without loss of generality assume  $u$  and  $v$  are adjacent and the move changes the coloring of  $u$  and  $v$  from  $\rho(u), \rho(v)$  to  $\bar{\rho}(u), \bar{\rho}(v)$ . Since  $G^{(C)}$  is not a multigraph, exactly one of the two colorings  $\rho(u), \bar{\rho}(v)$  and  $\bar{\rho}(u), \rho(v)$  must be a valid coloring of  $u$  and  $v$ . Assume  $\bar{\rho}(u), \rho(v)$  is valid then we will define the path  $\gamma_{xy}$  to have length two and will first recolor  $u$  from  $\rho(u)$  to  $\bar{\rho}(u)$  and next recolor  $v$  from  $\rho(v)$  to  $\bar{\rho}(v)$ . Given an edge  $(u, v)$  of  $\mathcal{M}_C$  there are at most five paths  $\gamma_{xy}$  that the edge could be on corresponding to the path  $(u, v)$  and the paths associated with each edge adjacent to the component whose coloring is changed between  $u$  and  $v$ . Notice that if component  $C_i$  is changed from  $\rho(C_i)$  to  $\bar{\rho}(C_i)$  then for each neighbor  $C_j$ , the length of the path must correspond to the move of  $\mathcal{M}_E$  that changes the coloring from  $\bar{\rho}(C_i)\bar{\rho}(C_j)$  to  $\rho(C_i)\rho(C_j)$ . Using these paths we can now determine an upper bound on  $A$  as defined in [Theorem 13](#) as follows :

$$\begin{aligned} A &= \max_{(z,w) \in E(\mathcal{P})} \left\{ \frac{1}{\pi(z)\mathcal{P}(z,w)} \sum_{\Gamma(z,w)} |\gamma_{xy}| \pi(x) \mathcal{P}'(x,y) \right\} \\ &\leq \max_{(z,w) \in E(\mathcal{P})} 10 \left\{ \frac{\mathcal{P}'(x,y)}{\mathcal{P}(z,w)} \right\}. \end{aligned}$$

Let  $S$  be the number of vertices in the component graph  $G^{(C)}$  (the number of color-implied components). We know that  $\mathcal{P}(z, w) = 1/2|S|$  and if we let  $E$  be the number of edges in  $G^{(C)}$  then  $\mathcal{P}'(x, y) = 1/3E$ . For all  $z, w, x, y \in \Omega$  with  $(z, w) \in E(\mathcal{P})$  and  $(x, y) \in E(\mathcal{P}')$ ,  $\frac{\mathcal{P}'(x,y)}{\mathcal{P}(z,w)} = \frac{3E}{2|S|} \leq 3$  because the vertices in  $G^{(C)}$  have maximum degree 4 implying  $E/|S| \leq 2$ . This implies that  $A \leq 30$ . Combining [Theorems 12](#) and [13](#) prove that the mixing time of  $\mathcal{M}_C$  on  $\Omega(G, F, 2)$  with the given constraints satisfies

$$\tau(\epsilon) \leq 12 \left( \frac{n - \log \epsilon}{\log(1/2\epsilon)} \right) \lceil 6en^3 \rceil \lceil \ln \epsilon^{-1} \rceil. \quad \square$$

### 3.5. Approximately counting using $\mathcal{M}_C$

Finally, we show that under the same conditions for which we have proved rapid mixing for  $\mathcal{M}_C$ , we have an FPRAS for approximately counting conforming colorings.

**Theorem 15.** *Given a graph  $G$  with  $n$  vertices and edge 2-constraints  $F$  with component graph  $G^{(C)} = G^{(C)}(G, F)$  such that for every vertex  $v \in G^{(C)}$  either  $d(v) \leq 2$  or  $d(v) \leq 4$  and  $v$  is monochromatic, there exists an FPRAS for counting the number of vertex 2-colorings conforming to  $F$ .*

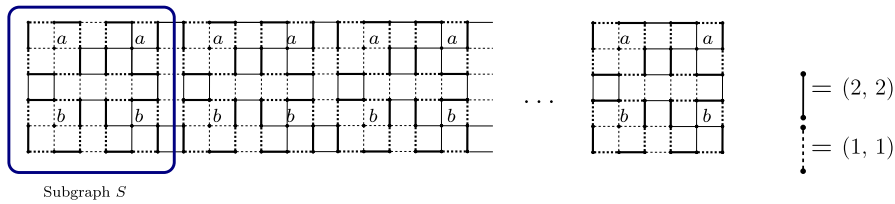


Fig. 4. An edge coloring for which the Markov chain  $\mathcal{M}_C$  mixes slowly.

**Proof.** Theorem 14 tells us that given the restrictions on components we can efficiently approximately uniformly sample conforming colorings. Combining this with a well-known result of Jerrum, Valiant and Vazirani [16], it is sufficient to prove that the model is self-reducible. In fact this proof will be a generalization of self-reducibility of independent sets. Select any vertex  $v$  in  $G$  and sample conforming colorings to approximate the probability  $p$  that for a random sample from  $\Omega$ ,  $v$  is colored 1 or 2, whichever is most likely. Next, with probability  $p$  color vertex  $v$  with color 2 and with probability  $1 - p$  color vertex  $v$  with color 1. Without loss of generality, assume  $v$  is assigned color 2. Delete any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) = 1$ , they no longer provide any constraint since  $v$  is colored 2. For any edges  $(u, v)$  incident to  $v$  for which  $F_v(u, v) = 2$ , vertex  $u$  cannot be colored  $F_u(u, v)$  so color vertex  $u$  color  $\bar{F}_v(u, v)$ . Delete  $v$  and repeat this edge procedure for any vertices which receive a fixed color. Afterwards, at least one vertex will have been deleted. Continue recursively by choosing a new vertex  $v$ . It follows from [16] and Theorem 14 that this procedure results in an FPRAS.  $\square$

3.6. An example when  $\mathcal{M}_C$  mixes slowly

While it is encouraging that  $\mathcal{M}_C$  can be shown to be rapidly mixing for some special cases, we show that it can also be slow. Specifically, we prove that there is a graph on which  $\mathcal{M}_C$  requires exponential time by demonstrating that the state space contains a bottleneck that requires exponential expected time to cross (see Fig. 4).

We start by defining conductance which we will use in the proof. The *conductance* of an ergodic Markov chain  $\mathcal{M}$  with stationary distribution  $\pi$  is

$$\Phi_{\mathcal{M}} = \min_{\substack{S \subseteq \Omega \\ \pi(S) \leq 1/2}} \frac{1}{\pi(S)} \sum_{s_1 \in S, s_2 \in \bar{S}} \pi(s_1) \mathcal{P}(s_1, s_2).$$

The following theorem relates conductance and mixing time (see, e.g., [21]).

**Theorem 16.** For any Markov chain with conductance  $\Phi_{\mathcal{M}}$ ,  $\forall \epsilon > 0$  we have

$$\tau(\epsilon) \geq \left( \frac{1}{4\Phi_{\mathcal{M}}} - \frac{1}{2} \right) \log \left( \frac{1}{2\epsilon} \right).$$

Next, we will use Theorem 16 to prove the following theorem.

**Theorem 17.** There exists a family of graphs  $G$  with  $36n$  vertices and  $\Delta = 4$  and an edge 2-coloring  $F$  for which the mixing time of  $\mathcal{M}_C$  on  $\Omega(G, F, 2)$  satisfies

$$\tau(\epsilon) = \Omega(4^n \log(1/2\epsilon)).$$

**Proof.** We will show that for the graph and edge 2-coloring given in Fig. 4,  $\mathcal{M}_C$  takes exponential time to converge. The component graph  $G^{(C)}$  for Fig. 4, has a large component  $C$  containing all vertices except for those labeled  $a$  and  $b$ , each of these forms a single vertex component. For each of the two colorings of  $C$  either the  $a$  vertices or the  $b$  vertices are free to change colors giving an exponential number of configurations. However there is only a single coloring of the  $a$  and  $b$  vertices that allows the color of  $C$  to change, creating a bottleneck. We will now formalize this argument.

To create the graph  $G$  and coloring  $F$ , connect  $n$  copies of the subgraph  $S$  and associated edge 2-coloring as shown in Fig. 4. Notice that all the vertices except for those labeled  $a$  and  $b$  form a single color-implied component  $C$ . This component has two conforming colorings corresponding to the even and odd colorings (the two proper colorings) which we will call  $\rho$  and  $\bar{\rho}$ . More precisely, a vertex  $v$  located at coordinates  $(v_x, v_y)$  is *even* if  $v_x + v_y \equiv 0 \pmod{2}$ ; otherwise,  $v$  is *odd* (the origin is the lower left vertex). In  $\rho(C)$  even vertices are colored 1 and odd vertices are colored 2, while in  $\bar{\rho}(C)$  even vertices are colored 2 and odd vertices are colored 1. Let  $A$  be the set of vertices labeled  $a$  in Fig. 4 and  $B$  be the set of vertices color  $b$ . The component graph  $G^{(C)}$ , has a large component  $C$  containing all vertices not in  $A$  and  $B$  and  $4n$  single vertex components corresponding to the vertices in  $A$  and  $B$ . More specifically,  $G^{(C)}$  is the star graph with center vertex, component  $C$ . We will show that while there are exponential configurations where component  $C$  is colored  $\rho(C)$  or  $\bar{\rho}(C)$ , there is only a single coloring of the vertices in  $A$  and  $B$  that allows the color of  $C$  to change, creating a bottleneck in the state space. Let  $S_\rho$  be the set of conforming colorings of  $G$  where  $C$  is colored  $\rho(C)$  and  $S_{\bar{\rho}}$  be the remaining conforming colorings where  $C$  is colored

$\bar{\rho}(C)$ . If  $C$  is colored  $\rho(C)$ , all vertices in  $B$  can independently be colored 1 or 2 so  $|S_\rho| = 2^{2n}$ . Similarly, if  $C$  is colored  $\bar{\rho}(C)$  all vertices in  $A$  can be independently be colored 1 or 2 so  $|S_{\bar{\rho}}| = 2^{2n} = 4^n$ . However in order to switch between the two colorings of  $C$ , every vertex in  $A$  and  $B$  must be colored differently than their adjacent edges. Let  $\beta_c$  be this configuration with  $C$  colored  $\rho(C)$ . Since  $\pi(S_\rho) = 1/2$ , combining the observations above gives the following bound on  $\Phi_{\mathcal{M}_C}$ :

$$\Phi_{\mathcal{M}_C} \leq \frac{1}{\pi(S_\rho)} \sum_{\beta_1 \in S_\rho, \beta_2 \in S_{\bar{\rho}}} \pi(\beta_1) \mathcal{P}(\beta_1, \beta_2) \leq \frac{\pi(\beta_c)}{\pi(S_\rho)} = 4^{-n}.$$

Combining with [Theorem 16](#) proves that the mixing time of  $\mathcal{M}_C$  satisfies

$$\tau(\epsilon) \geq \left(4^{n-1} - \frac{1}{2}\right) \log\left(\frac{1}{2\epsilon}\right). \quad \square$$

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