CS 7535

Homework 3

Assigned Thursday, Nov 13

Due Tuesday, Dec 2

1. Let $G = (V, E)$ be a $\sqrt{n} \times \sqrt{n}$ region on the Cartesian lattice. We are interested in sampling each independent set of size k with weight proportional to λ^k, for some given parameter $0 < \lambda < 1$.

   We considered the following "vertex-move" Markov chain $\mathbf{M}_V$ (this is the heat-bath version). Starting with an independent set $I \subseteq V$, we transition as follows:

   - Uniformly pick vertex $v \in V$.
   - Let $I' = I \cup \{v\}$ with probability $\frac{\lambda}{\lambda+1}$, and $I' = I \setminus \{v\}$ with probability $\frac{1}{\lambda+1}$.
   - If I' is a valid independent set, set $I = I'$, otherwise do nothing.

   Recall that we showed that the Markov chain above samples according to the desired distribution, and we gave a range of λ for which the chain converges in polynomial time.

   Now, consider a different "edge-move" Markov chain $\mathbf{M}_E$ (introduced by Luby and Vigoda) where, given an independent set I, we do the following in each step:

   - Uniformly pick edge $e = (v, w) \in E$.
   - Include or remove either v or w to the independent set with appropriate probabilities.
   - If this results in an invalid independent set, reject the move.

   (a) Find the appropriate probabilities so that this new edge-move Markov chain $\mathbf{M}_E$ has the same stationary distribution as the earlier vertex-move Markov chain $\mathbf{M}_V$ in problem 1. Next, provide a path coupling argument on $\mathbf{M}_E$. For what range of λ can you show the chain converges in polynomial time? (It should be a larger range)

   Hint: When bounding the number of bad moves, consider grouping the potential bad moves by the vertex you are attempting to add to the set. You have two cases for bounding the number in each grouping: when this vertex has no neighbors in I, and where it has at least one neighbor in I.

   (b) Use the Comparison Theorem from class to show that $\mathbf{M}_V$ also mixes in polynomial time for this larger range of λ.

2. Consider an Ising model where we are interested in sampling spin configurations on the complete graph $G = (V, E)$. As a reminder, our state space is over $\Omega = \{+, -\}^V$. For our notation, in a configuration $\sigma \in \Omega$, let $A(\sigma) \subset E = \{(v, w) \mid \sigma(v) = \sigma(w)\}$ be the set of edges where the spins on the endpoints agree. Then we seek to sample $\sigma \in \Omega$ proportional to $\lambda^{|\{A(\sigma)\}|}$ for some given λ.

   We consider the Glauber Dynamics Markov chain $\mathbf{M}_G$ that, starting at configuration σ,

   - Choose $v \in V$ and $b \in \{+, -\}$ uniformly at random.
   - Let $\sigma' = \sigma$ except with $\sigma'(v) = b$
   - Transition to σ' with probability $\min(1, \frac{\pi(\sigma')}{\pi(\sigma)})$.

   Consider a partition Ω into subsets $\Omega_0, \Omega_1, \ldots, \Omega_n$, where each Ω_i is the set of configurations that contain exactly i + spins. In other words, $\Omega_i = \{\sigma \in \Omega\ s.t.\ |\{v \in V\ s.t.\ \sigma(v) = +\}| = i\}$.

   Notice that since the Markov chain only changes one vertex at a time, if we start at a configuration in Ω_i we can only move to an element in Ω_{i-1}, or Ω_{i+1} or stay where we were.

   (a) Show that this Markov chain is ergodic and has the correct stationary distribution. Then show that $\Omega_{n/2}$ is an exponentially small bottleneck in Ω when $\lambda > 1$.

   Since we have this bottleneck, we consider what happens if we restrict to $\Omega' = \bigcup_{i \geq n/2} \Omega_i$. If our Markov chain above attempts to leave Ω', we reject the move. We will show that this Markov chain is rapidly mixing.

   First, since the above Markov chain cannot move between pairs of distinct states in Ω_i, we define a new Markov chain $\mathbf{M}_S$ that can by including swaps:

   - With probability 1/2, make moves according to the original Markov chain above.
   - With probability 1/2, choose two vertices in $v, w \in V$ uniformly at random, and swap their spins (i.e., $\sigma'(w) = \sigma(v)$, and $\sigma'(v) = \sigma(w)$).

   (b) Use the decomposition theorem to show that the Markov chain $\mathbf{M}_S$ is rapidly mixing. The following strategy will be helpful:

      i. Show that the restriction of the above Markov chain $\mathbf{M}_S$ to any Ω_i is rapidly mixing.
      ii. Show that the projection of our Markov chain $\mathbf{M}_S$ onto the number line, where $\sigma \to |v \in V\ s.t.\ \sigma(v) = +|$, is a unimodal distribution over $\{0, 1, \ldots, n\}$.
      iii. Conclude that the projection Markov chain is also rapidly mixing.
      iv. Use the Decomposition theorem from class to show that the Markov chain $\mathbf{M}_S$ is rapidly mixing.

   (c) Use the Comparison theorem from class to argue that the original Glauber dynamics $\mathbf{M}_G$ is also rapidly mixing.