Mid Term 1 Solutions

1. We need to show that the sum of any three consecutive odd numbers is divisible by 3. Let the smallest odd number in a series of three consecutive odd numbers be $2n+1$. The next two odd numbers are $2n+1+2 = 2n+3$ and $2n+1+2+2 = 2n+5$. Their sum is $(2n+1)+(2n+3)+(2n+5) = 6n+9 = 3(2n+3)$, which is divisible by 3. Hence, proved.

2. $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $f(x,y,z) = (x+y+z, x)$. To prove that f is onto, we select an arbitrary $(u,v) \in \mathbb{R}\times\mathbb{R}$. We need to show that there exists atleast one $(x,y,z) \in \mathbb{R}$, such that $f(x,y,z) = (u,v)$. For this, let $x = v,\ y = 0,\ z = u - v$. Since $u, v \in \mathbb{R},\ x, y, z \in \mathbb{R}$. We get $f(x,y,z) = (v+0+u-v, v) = (u,v)$, which is what we needed. Hence, proved.

3. We are given that $A \subseteq B \cap C^c$. Therefore, $A \subseteq B \cap C^c \Rightarrow \forall x \in A, x \in B \cap C^c \Rightarrow \forall x \in A, x \in B\ and\ x \in C^c \Rightarrow \forall x \in A, x \in C^c$ (Statement 1). Consider any $y \in C$. Suppose $y \in A$. Then $y \in A\ and\ y \in C\ \Leftrightarrow y \in A\ and\ y \notin C^c$, which is a contradiction to the above Statement 1. Therefore $y \notin A$, that is, $y \in A^c$. This means that $\forall y \in C,\ y \in A^c$, which means that $C \subseteq A^c$.

4. To prove that $\forall x \in \mathbb{Z}^+\ \exists y \in \mathbb{R},\quad y < x < 2y$. Let $x \in \mathbb{Z}^+$. We will set $y = x - \frac{1}{3}$. Now, clearly $y \in \mathbb{R}$. Also since $x - \frac{1}{3} < x$, we have $y < x$. Also, since $x \in \mathbb{Z}^+$, we have that $x \geq 1$. This implies that $2x \geq x+1$ (adding x on both sides). So, $2x - \frac{2}{3} \geq x + 1 - \frac{2}{3}$. So, $2(x - \frac{1}{3}) \geq x + \frac{1}{3} > x$. Therefore, $2y > x$. So, $y < x < 2y$. Hence, proved.

5.a We have to write out the negation of $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{R},\quad y < x < 2y$. Now $\neg(\forall x \in \mathbb{Z}\ \exists y \in \mathbb{R},\quad y < x < 2y) = \exists x \in \mathbb{Z}, \neg(\exists y \in \mathbb{R},\quad y < x < 2y)$. Also, $\exists x \in \mathbb{Z} \neg(\exists y \in \mathbb{R},\quad y < x < 2y) = \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}, \neg(y < x < 2y) = \exists x \in \mathbb{Z}\ \forall y \in \mathbb{R}, either\quad x \leq y\quad or\quad x \geq 2y$, which is the negation of the original statement.

5.b We need to prove $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}, x \leq y\ or\ x \geq 2y$ to be true in order to prove the conjecture to be false. For this we select $x = 0$. For any $y \in \mathbb{R}$, either $y \geq 0$, which means $y \geq x$, or $y < 0$ which implies that $2y < 0$ that is, $2y \leq x$. Therefore, with $x = 0$, for all $y \in \mathbb{R}$, either $x \leq y$ or $x \geq 2y$. Therefore $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}, x \leq y\ or\ x \geq 2y$. Hence, proved.