CS 1050 Practice Midterm 2 Solutions

1. Negate the following sentences:

a) For all integers $n \geq 4$, there exists $c \in \mathbb{R}$ such that $n^{100} \geq 2^n$.

Ans: There exists an integer n, $n \geq 4$, such that for all $c \in \mathbb{R}$, $n^{100} < 2^n$.

b) The square of an integer is never odd.

Ans: There exists an integer n such that n^2 is odd.

c) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ [x > y]$ or $[x < y]$.

Ans: $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ [x \leq y]$ and $[x \geq y]$.

d) $\forall x \in \mathbb{Z}^+\ \exists y \in \mathbb{Z}^+\ [x^2 = y^2]$ and $[x \neq y]$.

Ans: $\exists x \in \mathbb{Z}^+\ \forall y \in \mathbb{Z}^+\ [x^2 \neq y^2]$ or $[x = y]$.

2.

Theorem 1. For all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. (by induction)

Base case: $n = 1$: $\sum_{i=1}^{n} i^2 = 1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6}$.

Induction hypothesis: Let $k \geq 1$ and assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Inductive step: We will show the equation holds when $n = k + 1$.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the induction hypothesis)}$$

$$= \frac{k+1}{6}(k(2k+1)+6(k+1)) = \frac{k+1}{6}(2k^2+7k+6) = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.$$

Thus, by induction, for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by

$$f(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

for all reals x_1, x_2.

Prove that f is invertible.

Proof. First we will show that f is onto, i.e., for all $(a, b) \in \mathbb{R} \times \mathbb{R}, \exists (x, y) \in \mathbb{R} \times \mathbb{R}$ such that $f(x, y) = (a, b)$. Let $(a, b) \in \mathbb{R} \times \mathbb{R}$. Let $x = \frac{a+b}{3}$ and let $y = \frac{2a-b}{3}$, where both x and y are reals since a and b are. Then

$$f(x, y) = f(\frac{a+b}{3}, \frac{2a-b}{3}) = (\frac{a+b+2a-b}{3}, \frac{2a+2b-(2a-b)}{3}) = (a, b).$$

Hence f is onto.

Now we will show that f is 1-1. Suppose for $x, y, x', y' \in \mathbb{R}$, $f(x, y) = f(x', y')$. Then $x + y = x' + y'$ and $2x - y = 2x' - y'$. Adding these two equations, we find $x + y + 2x - y = x' + y' + 2x' - y'$ which implies $3x = 3x'$ and hence $x = x'$. Subsituting this into the first equation, we find $x + y = x + y'$ or $y = y'$. Therefore $f(x, y) = f(x', y')$ implies $(x, y) = (x', y')$ and so f is 1-1.

Since we have shown that f is both onto and 1-1, it is invertible.

4. Let $a_1 = 5$, $a_2 = 13$, and, for $n \geq 2$, let $a_{n+1} = 5a_n - 6a_{n-1}$.

Prove the following theorem.

Theorem 2. For all $n \in \mathbb{Z}^+$, $a_n = 3^n + 2^n$.

Proof. (by strong induction)

<u>Base cases:</u> $n = 1 : a_1 = 3^1 + 2^1 = 5$. $n = 2 : a_2 = 3^2 + 2^2 = 13$.

<u>Induction hypothesis:</u> Let $k \geq 2$ be an integer and assume that $\forall n \in \mathbb{Z}, 0 \leq n \leq k, \quad a_n = 3^n + 2^n$.

Inductive step: We want to show that $a_{n+1} = 3^{n+1} + 2^{n+1}$.

$$a_{n+1} = 5a_n - 6a_{n-1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1})$$

$$= (15 - 6)3^{n-1} + (10 - 6)2^{n-1} = 3^{n+1} + 2^{n+1}.$$

Hence, by induction, $a_n = 3^n + 2^n$ for all integers $n \geq 1$.

5. Prove that for all integers $n \geq 1$, $8^n - 2^n$ is a multiple of 6.

Proof. (by induction) Base case: If $n = 1$, then $8^1 - 2^1 = 6$ so it is a multiple of 6. Inductive hypothesis: Let $n \geq 1$ and assume $8^n - 2^n$ is a multiple of 6, i.e., there exists an integer k such that $8^n - 2^n = 6k$. Inductive step: We want to show that $8^{n+1} - 2^{n+1}$ is a multiple of 6 as well.

$$8^{n+1} - 2^{n+1} = 8(8^n - 2^n) + 8 \cdot 2^n - 2 \cdot 2^n$$

$$= 8(6k) + 6(2^n) \quad \text{(by the inductive hypothesis)}$$

$$= 6(8k + 2^n),$$

which is a multiple of 6. Hence, by induction, $8^n - 2^n$ is a multiple of 6 for all integers $n \geq 1$.