Math 8223A – Combinatorial Methods for Statistical Physics Models
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We continue by proving the Stronger Form of the FKG Inequality.

The FKG Inequality (cont.)

Recall the FKG inequality, and the stronger version.

Theorem 1 (FKG) *If A and B are increasing events, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B).$$

Theorem 2 (FKG, Stronger Version) *Let μ be a probability measure on Ω that satisfies the FKG condition, and let f and g be increasing functions on Ω. Then*

$$\sum_{a\in\Omega} f(a)g(a)\mu(a) \geq \sum_{a\in\Omega} f(a)\mu(a) \sum_{b\in\Omega} g(b)\mu(b).$$

We've already seen how theorem 2 implies theorem 1 by taking $f = 1_A$ and $g = 1_B$, the indicator functions of the sets A and B, and observing that P_p satisfies the FKG condition. Therefore we will focus on proving theorem 2. To do this, we prove a preliminary result about a certain product measure. In this lemma, we consider $\Omega = \mathcal{P}(X)$ (the power set of X), partially ordered by inclusion.

Lemma 1 *Let μ_1 and μ_2 be probability measures satisfying*

$$\mu_1(a \cup b)\mu_2(a \cap b) \geq \mu_1(a)\mu_2(b) \tag{1}$$

for all $a, b \in \Omega$.

Then there exists a measure ν on $\Omega \times \Omega$ such that

1. $\sum_{a\in\Omega} \nu(a,b) = \mu_2(b)$ *for all $b \in \Omega$,*
2. $\sum_{b\in\Omega} \nu(a,b) = \mu_1(a)$ *for all $a \in \Omega$,*
3. $\nu(a,b) = 0$ *unless $a \geq b$.*

<u>**Markov Chain**</u>

Let $n = |X|$.
We define a Markov chain $\mathcal{MC}(\mu)$ on Ω governed by the following transitions (where $a, b \in \Omega$)

$$T_\mu(a,b) = \begin{cases} \frac{1}{2n}\min(1, \frac{\mu(b)}{\mu(a)}) & \text{if } |b \odot a| = 1 \\ 0 & \text{if } |b \odot a| > 1 \\ 1 - \sum_{b\neq a} T_\mu(a,b) & \text{if } b = a \end{cases}$$

where $a \odot b$ is the bitwise symmetric difference of the vectors a and b.
(Such a Markov chain is commonly referred to as the Metropolis chain.)
We implement this Markov chain by starting at $a \in \Omega$ and we repeat

$$\left[\begin{array}{l} \text{Pick } x \in_u X, \text{ and } r \in_u [0,1]. \\ \text{If } r \leq \frac{1}{2n}\min(1, \frac{\mu(b)}{\mu(a)}) \text{ and } x \in a, \text{ remove } x. \\ \text{If } r > 1 - \frac{1}{2n}\min(1, \frac{\mu(b)}{\mu(a)}) \text{ add } x \text{ to } a. \\ \text{Otherwise, do nothing.} \end{array}\right.$$

Note that $\mathcal{MC}(\mu)$ is ergodic and reversible, so the unique stationary distribution, π, satisfies detailed balance.

Coupled Markov Chain

Define a coupled chain, $\hat{T}_{\mu_1,\mu_2}$ on $\Omega \times \Omega$ in the following manner:
Starting at the initial state $(a_1, a_2) = (X, \phi)$, repeat

$$\left[\begin{array}{l} \text{Pick } x \in_u X, \text{ and } r \in_u [0,1]. \\ \text{Update } a_1 \text{ according to } T_{\mu_1}. \\ \text{Update } a_2 \text{ according to } T_{\mu_2}. \end{array}\right.$$

Because of this coupling we have transitions defined by $(a_1, a_2) \stackrel{\hat{T}_{\mu_1,\mu_2}}{\longrightarrow} (b_1, b_2)$ with probability

$$\begin{cases} \min(T_{\mu_1}, T_{\mu_2}) & \text{if } \exists x \in X \ni b_1 = a_1 \cup \{x\},\ b_2 = a_2 \cup \{x\}, \text{ or} \\ & \text{if } \exists x \in X \ni b_1 = a_1 \backslash \{x\},\ b_2 = a_2 \backslash \{x\} \\ T_{\mu_2} - T_{\mu_1} & \text{if } a_1 = b_1,\ |b_2 \odot a_2| = 1 \\ T_{\mu_1} - T_{\mu_2} & \text{if } |a_1 \odot b_1| = 1,\ b_2 = a_2 \\ 0 & \text{if } |a_1 \odot b_1| > 1 \text{ or } |a_2 \odot b_2| > 1 \\ 1 - \sum_{(b_1,b_2) \neq (a_1,a_2)} \hat{T}_{\mu_1,\mu_2}((a_1,a_2),(b_1,b_2)) & \text{if } (a_1,a_2) = (b_1,b_2) \end{cases}$$

Then, $\hat{T}_{\mu_1,\mu_2}$ has a unique stationary distribution, call it ν. It is easy to see that conditions 1 and 2 of the theorem are satisfied merely from the definition of the coupled chain.

In order to show the third condition, we need to show monotonicity, i.e. that if $\hat{T}_{\mu_1,\mu_2}((a_1,a_2),(b_1,b_2)) > 0$ and $a_1 \geq a_2$, then $b_1 \geq b_2$ (where the partial order means that $a \geq b$ if all of the open bonds in b are also open in a).

Assume that $a_1 \geq a_2$ (i.e. $a_1 \supseteq a_2$). Then we can write $a_1 = a_2 \cup C$, where $C \subseteq X$ is the additional bonds open in a_1, and $a_2 \cap C = \phi$. We may also assume that $a_1 \neq a_2$ (otherwise monotonicity follows trivially) so that $|C| \geq 1$.

Let $x \in X$. One bad case that might happen is that $x \notin a_2$ (so $x \notin a_1$), but $P(\text{add } x \text{ to } a_2) > P(\text{add } x \text{ to } a_1)$. This can't happen. Why? With $a = a_1$, $b = a_2 \cup \{x\}$, and the FKG condition (1) on μ_1 and μ_2, we have that $\mu_1(a_1 \cup \{x\})\mu_2(a_2) \geq \mu_1(a_1)\mu_2(a_2 \cup \{x\})$ or that $\frac{\mu_1(a_1 \cup \{x\})}{\mu_1(a_1)} \geq \frac{\mu_2(a_2 \cup \{x\})}{\mu_2(a_2)}$. This means if we add x to a_2, then we will also add x to a_1.

The other bad case that might happen is that $x \in a_2$ (so $x \in a_1$), but $P(\text{remove } x \text{ from } a_1) > P(\text{remove } x \text{ from } a_2)$. However, similar to the previous case, with $a = a_1 \backslash \{x\}$, $b = a_2$, and condition (1) on the measures, we have $\mu_1(a_1)\mu_2(a_2 \backslash \{x\}) \geq \mu_1(a_1 \backslash \{x\})\mu_2(a_2)$ or that $\frac{\mu_2(a_2 \backslash \{x\})}{\mu_2(a_2)} \geq \frac{\mu_1(a_1 \backslash \{x\})}{\mu_1(a_1)}$. Therefore, removing x from a_1 means that we also remove it from a_2.

This implies monotonicity, and hence, condition 3 of the measure ν. □

There are two corollaries to this lemma.

Corollary 1 *If f is increasing and μ_1, μ_2 satisfy*

$$\mu_1(a \cup b)\mu_2(a \cap b) \geq \mu_1(a)\mu_2(b)$$

for all $a, b \in \Omega$, then

$$\sum_{a \in \Omega} f(a)\mu_1(a) \geq \sum_{a \in \Omega} f(a)\mu_2(a).$$

Proof: Let ν be the measure as in theorem 1. Then

$$\begin{aligned}\sum_{a\in\Omega} f(a)\mu_1(a) &= \sum_{a,b\in\Omega} f(a)\nu(a,b)\\ &= \sum_{a\geq b} f(a)\nu(a,b)\\ &\geq \sum_{a\geq b} f(b)\nu(a,b)\\ &= \sum_{a,b\in\Omega} f(b)\nu(a,b)\\ &= \sum_{b\in\Omega} f(b)\mu_2(b)\end{aligned}$$

□

The second corollary is the FKG inequality (theorem 2).

Corollary 2 (FKG, Stronger Version) *Let μ be a probability measure on Ω that satisfies the FKG condition, and let f and g be increasing functions on Ω. Then*

$$\sum_{a\in\Omega} f(a)g(a)\mu(a) \geq \sum_{a\in\Omega} f(a)\mu(a)\sum_{b\in\Omega} g(b)\mu(b).$$

Proof: Let $\mu_1(a) = \frac{g(a)\mu(a)}{\sum_{a\in\Omega} g(a)\mu(a)}$ and let $\mu_2 = \mu$. Then, we claim that μ_1 and μ_2 satisfy the condition given in corollary 1.

$$\begin{aligned}\mu_1(a\cup b)\mu_2(a\cap b) &= \frac{g(a\cup b)\mu(a\cup b)}{\sum_a g(a)\mu(a)}\mu(a\cap b)\\ &\geq \frac{g(a)\mu(a\cup b)}{\sum_a g(a)\mu(a)}\mu(a\cap b)\\ &\geq \frac{g(a)}{\sum_a g(a)\mu(a)}\mu(a)\mu(b)\\ &= \mu_1(a)\mu_2(b)\end{aligned}$$

Now apply corollary 1 to μ_1, μ_2, and f. □