We begin a discussion of perfect matchings and present Fisher, Kasteleyn, and Temperley's method for counting perfect matchings in a planar graph.

1 Introduction

Definition 1 *In a graph, $G = (V, E)$, a matching $M \subseteq E$ is a set of edges such that no two edges in M share an endpoint (i.e. every vertex has degree ≤ 1).*

Definition 2 *A perfect matching is a matching where every vertex is incident to exactly one edge in the matching.*

Algorithmically, the problem of finding a perfect matching can be solved efficiently. However, we will be interested in counting the number of perfect matching in a particular graph.

Example Let G be a bipartite graph $(G = (V, U, E), E \subseteq V \times U)$ with $|V| = |U| = n$. Here is an example with $n = 5$.

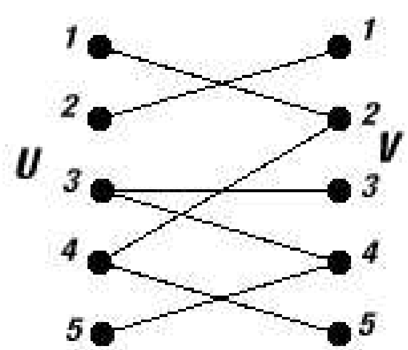

Figure 1: A bipartite graph

Define the adjacency matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

That is for the example graph,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Any permutation with all nonzero entries corresponds to a perfect matching. For example, {(1,2),(2,1),(3,3), (4,5), (5,4)}.

Definition 3 *Let A be an $n \times n$ matrix, and let S(n) be the set of permutations of {1,2,..,n}. The determinant of A, denoted detA, is defined by*

$$detA = \sum_{\pi \in S(n)} sgn(\pi) \prod_{i=1}^{n} a_{i\pi(i)}.$$

Definition 4 *The permanent of A, denoted perA, is defined by*

$$perA = \sum_{\pi \in S(n)} \prod_{i=1}^{n} a_{i\pi(i)}.$$

Claim 1 *If A is the adjacency matrix of a bipartite graph, then perA is the number of perfect matchings.*

This can be seen easily from the definition of A: since all of the a_{ij} are 1 or 0,

$$\prod_{i=1}^{n} a_{i\pi(i)} = \begin{cases} 1 & \text{if } a_{i\pi(i)} = 1 \text{ for all } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, the permutations with all 1's are exactly the perfect matchings. Therefore, $perA$ counts each of the perfect matchings once and all other permutations no times.

The determinant is easy to compute in $O(n^3)$ time, whereas calculating the permanent is as hard as any NP Complete problem, as shown by Valiant in 1979.

Claim 2 *If there is no perfect matching, then $det(A) = 0$.*

Since there are no perfect matchings,

$$\prod_{i=1}^{n} a_{i\pi(i)} = 0, \forall \pi \in S(n).$$

Therefore, $detA = 0$ by definition.

Remark The converse is false. For example, for $K_{n,n}$, $detA = 0$, yet there are obviously several perfect matchings possible.

We will investigate a method for calculating the permanent for planar graphs.

2 Motivation

Combinatorics: In the Cartesian lattice, $\mathcal{Z}^2$, each perfect matching corresponds to a domino tiling in the dual (Cartesian) lattice, as seen in Figure 2.

Statistical Mechanics: The Dimer Model assigns uniform weight to each perfect matching and asks how many tilings there are?

Let M be the set of matchings, and let $m \in M$. The Monomer-Dimer model assigns a weight, $wt(m) = \lambda^{|m|}$, to each matching m. The Gibbs measure assigns probability

$$\Pi(m) = \frac{wt(m)}{\sum_{m \in M} wt(m)},$$

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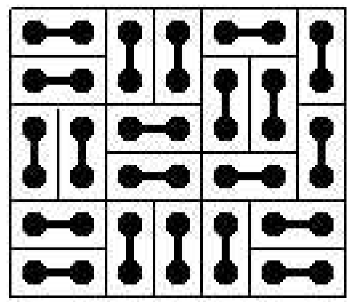

Figure 2: A perfect matching and the corresponding domino tiling

to each $m \in M$. The denominator in this fraction is defined as the partition function, $\mathcal{Z}$.

For $\lambda \gg 1$, the Monomer-Dimer model favors perfect matchings since it favors large $|m|$. Therefore, we need to know the number of perfect matchings to calculate $\mathcal{Z}$.

Chemistry: The Kekulé structure is a representation of chemical compounds (usually with the hydrogens removed). Chemists have found that how difficult a compound is to synthesize seems to be directly related to the number of perfect matchings that can be made using the bonds in the Kekulé structures. For example, benzene, anthracene, and phenanthrene have 2,4, and 5 perfect matchings, respectively. Accordingly, benzene is the most difficult of the the three compounds to artificially reproduce, and phenanthrene is the easiest.

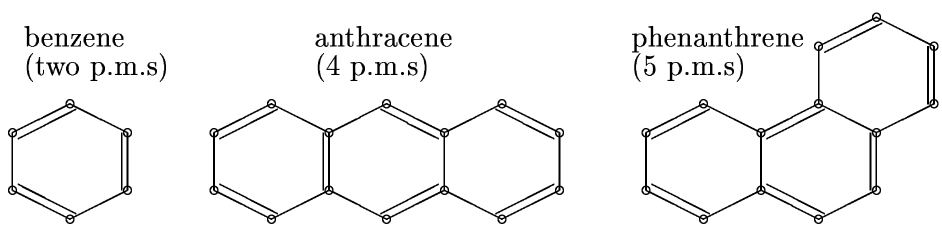

Figure 3: Kekulé structures

A second application to chemistry relates bond strength in certain carbon compounds to the probability that the bond is contained in a perfect matching.

3 Kasteleyn's Method

We will now see Kasteleyn's algorithm for counting perfect matching in planar graphs. Recall that Valiant showed that counting perfect matchings in general is likely to be intractable, so though the following method works for planar graphs, it is not likely to be generalized for counting perfect matching in general graphs. However, there are generalizations of Kasteleyn's method to graphs of fixed genus.

Let $G = (V, E)$ with $|V| = 2n$. Orient G to get $\vec{G}$. Let $B = (b_{ij})$ be the skew symmetric adjacency matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \vec{E} \\ -1 & \text{if } (j,i) \in \vec{E} \\ 0 & \text{otherwise.} \end{cases}$$

We need a few definitions before we can continue.

Definition 5 *Let B be a $2n \times 2n$ skew symmetric adjacency matrix (i.e. $B^T = -B$), and let*

$$\mathcal{P} = \{\{i_1, j_1\}, \{i_2, j_2\}, ..., \{i_n, j_n\}\}$$

be a partition over pairs of the 2n vertices. Let

$$b_{\mathcal{P}} = sgn \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_n \end{pmatrix} b_{i_1,j_1} b_{i_2,j_2} \cdots b_{i_n,j_n}.$$

The Pfaffian of B, denoted Pf(B), is defined by

$$Pf(B) = \sum_{\mathcal{P}} b_{\mathcal{P}}.$$

Remarks $b_{\mathcal{P}}$ is well defined since

1. changing the order within a pair, $\{i_k, j_k\}$, does not change $b_{\mathcal{P}}$ since both the sign of the permutation and the sign of b_{i_k,j_k} are changed.

2. switching the order of two pairs, say $\{i_k, j_k\}$ and $\{i_m, j_m\}$, requires an even number of transpositions, so the sign of the permutation does not change; therefore, $b_{\mathcal{P}}$ remains fixed.

Definition 6 *An even cycle is a cycle with an even number of edges.*

Definition 7 *An evenly oriented cycle is an even cycle with an even number of edges going in the direction of a cycle in either direction. Similarly, and oddly oriented cycle is an even cycle with an odd number of edges going in the direction of the cycle in either direction.*

Note that the union of two perfect matchings is a collection of alternating even cycles (including some trivial cycles, if both perfect matchings contain the same edge).

Lemma 1 Let $\vec{G}$ be an orientation of G. For any two perfect matchings, $\mathcal{P}_1$ and $\mathcal{P}_2$, let $EC(\mathcal{P}_1, \mathcal{P}_2)$ be the number of evenly oriented cycles in $\mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$b_{\mathcal{P}_1} \cdot b_{\mathcal{P}_2} = (-1)^{EC(\mathcal{P}_1, \mathcal{P}_2)}.$$

Proof of Lemma 1 We start with two observations which simplify the proof of the Lemma.
Observation 1: The claim in the Lemma is independent of the orientation of G. Suppose we switch the orientation of an edge, $\mathbf{e} \in G$. If $\mathbf{e} \notin \mathcal{P}_1 \cup \mathcal{P}_2$, changing the orientation of $\mathbf{e}$ has no effect. Also, if $\mathbf{e} \in \mathcal{P}_1 \cap \mathcal{P}_2$, changing the orientation of $\mathbf{e}$ changes the sign of both $b_{\mathcal{P}_1}$ and $b_{\mathcal{P}_2}$.

Finally, **e** could be in one of the perfect matchings but not the other. Without loss of generality, let $\mathbf{e} \in \mathcal{P}_1 \backslash \mathcal{P}_2$. Then, since **e** is part of an even cycle, reversing **e** changes the cycle's orientation, $b_{\mathcal{P}_1}$ changes sign and the value of k changes by ± 1. Therefore, if the lemma holds for any orientation of G, it must hold for all of them.

Observation 2: The numbering of the vertices also has no effect on the claim. Reordering the vertices simply permutes rows and columns of B, so $b_{\mathcal{P}}$ changes sign for all partitions $\mathcal{P}$ or they all stay fixed, so the product of any two of them remains unchanged. So if the lemma holds for one numbering of the vertices, it must also hold for all possible numberings.

Now consider the union of two perfect matchings, $\mathcal{P}_1 \cup \mathcal{P}_2$, in an arbitrary graph with $|V| = 2n$. Order the cycles in the union. Label the vertices with the numbers $\{1, 2, 3, 4, ..., 2n\}$ and orient the edges in the following manner. Take the lowest numbered cycle whose vertices have not been numbered. Choose a vertex on that cycle, and number it with the lowest remaining number from the set of labels, say k. Then label k+1 so that (k,k+1) is an edge in $\mathcal{P}_1$. If the cycle is trivial, then just orient the edge from k to k+1. Otherwise, we must label the rest of vertices in the cycle. Do this by going around the cycle in the direction of k to k+1, labeling the vertices in the order we get to them until all of the vertices in the cycle have a label. Orient the edges in the cycle so that they go around the cycle in the same direction as k to k+1. Move on to the next cycle and continue until all of the vertices have been labeled and all of the edges have been oriented. (See Figure 3 for an example of the labeling and orientation.)

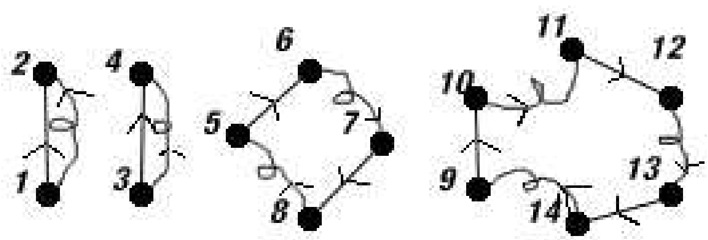

Figure 4: The union of two perfect matchings in $\mathcal{P}_1$ (straight edges) and $\mathcal{P}_2$ (curved edges)

Using this labeling and orientation, we see that $b_{\mathcal{P}_1}$ is trivially equal to 1 from the definition. Now consider $b_{\mathcal{P}_2}$. From the orientation that we chose, the contribution of the $b_{i,j}$ terms is 1. So the only contributions of -1 come from the permutations of the cycles. The permutations do not change for the trivial cycles since we have not changed the order of the vertices. For the nontrivial cycles the lowest label moves from the beginning of the cycle to the end, which, since there is an even number of vertices in the cycle, this change requires an odd number of transpositions. So for each non-trivial cycle a factor of -1 appears in the permutation in $b_{\mathcal{P}_2}$. Since, here, all of the even cycles are evenly oriented, this proves the lemma.

□

Now we will use this lemma to prove a result in Linear Algebra that is essential to Kasteleyn's method.

Theorem 1 *Let B be a $2n \times 2n$ skew symmetric matrix. Then $det(B) = Pf(B)^2$.*

Proof: Consider permutations that contribute to $det(B)$.

$$\prod_{i=1}^{n} b_{i\pi(i)} = \pm 1.$$

First we show that permutations that contain any odd cycles do not contribute to the determinant. We will show this by paring such permutations by a bijection. Take a permutation,π, with at least one odd cycle and pair it with another permutation, $\pi\prime$, by reversing the "first" odd cycle. Reversing an odd cycle does not change the parity of the permutation, but each of the $b_{ij}s$ in the cycle do change sign. Since there are an odd number of the $b_{ij}s$ in an odd cycle, the net result is that the contributions of π and $\pi\prime$ to $det(B)$ cancel each other. Thus, only the permutations with only even cycles contribute to the determinant.

Define C be the set of all cycle covers of G, and define $C_E \subset C$ such that any $c \in C_E$ has only even cycles. Define EC(c) to be the number of non-trivial even cycles in c, and define EOC(c) to be the number of evenly oriented cycles in c. Then define O(c) be the set of possible orientations $\overrightarrow{c}$ of $c \in C$(i.e the $2^{EC(c)}$ ways of directing each cycle in c. Finally, define π_o to be the permutation determined by the orientation $o \in O(C)$.

$$\begin{aligned}
det(B) &= \sum_{\pi \in S(n)} sgn(\pi) \prod_{i=1}^{n} b_{i\pi(i)} = \sum_{c \in C} \sum_{o \in O(c)} sgn(\pi_o) \prod_{i=1}^{n} b_{i\pi_o(i)} \\
&= \sum_{c \in C_E} \sum_{o \in O(c)} sgn(\pi_o) \prod_{i=1}^{n} b_{i\pi_o(i)} = \sum_{c \in C_E} 2^{EC(c)} (-1)^{EOC(c)} \\
&= \sum_{c \in C_E} 2^{EC(c)} \cdot b_{\mathcal{P}_1} \cdot b_{\mathcal{P}_2} \quad (\text{for } \mathcal{P}_1 \cup \mathcal{P}_2 = c) \\
&= \sum_{c \in C_E} \sum_{\mathcal{P}_1 \cup \mathcal{P}_2 = c} b_{\mathcal{P}_1} \cdot b_{\mathcal{P}_2} \\
&= \sum_{\mathcal{P}_1} b_{\mathcal{P}_1} \cdot \sum_{\mathcal{P}_2} b_{\mathcal{P}_2} = Pf(B)^2.
\end{aligned}$$

□

Kasteleyn observed that any planar graph can be oriented so that every "nice" cycle (meaning a cycle whose removal (i.e. deleting the vertices and edges) leaves a perfect matching) is oddly oriented. From our Lemma above, we see that this means $b_{\mathcal{P}}$ has the same sign, for all perfect matchings $\mathcal{P}$, since the product of any two of them is always 1 regardless of which perfect matchings we choose. Therefore, for this orientation, we find that the Pfaffian counts exactly the number of perfect matchings. Furthermore, since $det(B) = Pf(B)^2$, if we had an easy method of finding such an orientation of G, we would have an easy method for calculating the number of perfect matchings; simply calculate $det(B)$ and take the square root.

Finding an odd orientation:
First observe that it is sufficient to find an odd orientation of each of the faces of G (including odd faces bounded by an odd number of edges, where here "oddly oriented" means an odd number edges oriented in the clockwise direction. Now consider the following Lemma.

Lemma 2 Let $\overrightarrow{G}$ be a planar graph whose faces are all oddly oriented. Then each cycle in $\overrightarrow{G}$ will be oriented with opposite parity as the number of vertices it encloses.

Proof of Lemma 2 Proof by induction.

Consider the base case, i.e. a simple face that encloses no vertices. Since we have oriented each simple face oddly, the lemma obviously holds.

Now consider an arbitrary cycle, $\mathcal{F}_1$ that contains k vertices and assume that the lemma holds. Consider the cycle formed by taking the symmetric difference of $\mathcal{F}_1$ and a simple face, $\mathcal{F}_2$, that coincides with $\mathcal{F}_1$ at p edges. The new cycle will contain k + p - 1 vertices since the edges where $\mathcal{F}_1$ and $\mathcal{F}_2$ agree have been removed. We now have several cases and subcases.

Consider the case where k is even and p is odd. From the inductive hypothesis, we know that $\mathcal{F}_1$ is oddly oriented. We also know that the symmetric difference contains an even number of vertices. Now we have two subcases. First let an even number of the p edges shared by $\mathcal{F}_1$ and $\mathcal{F}_2$ be oriented in the clockwise direction with respect to $\mathcal{F}_1$. Since $\mathcal{F}_1$ is oddly oriented removing these edges in the clockwise direction leaves an odd number of edges in the clockwise direction that contribute to the symmetric difference. Since p is odd and the number of edges in the clockwise direction with respect to $\mathcal{F}_1$ is even, we have an odd number of the p edges in the clockwise direction with respect to $\mathcal{F}_2$. Therefore, $\mathcal{F}_2$ contributes an even number of edges in the clockwise direction to the symmetric difference, and the total number of edges in the symmetric difference oriented in the clockwise direction is odd. Similarly, if an odd number of the p edges are oriented in the clockwise direction with respect to $\mathcal{F}_1$, the symmetric difference is oddly oriented since $\mathcal{F}_1$ contributes an even number of clockwise edges and $\mathcal{F}_2$ contributes an odd number.

Now consider the case with k even and p even. The analysis follows exactly as in the last case, except here we find that k + p - 1 is odd. So if an even number of the p edges that $\mathcal{F}_1$ and $\mathcal{F}_2$ share are oriented in the clockwise direction with respect to $\mathcal{F}_1$, we find that both $\mathcal{F}_1$ and $\mathcal{F}_2$ contribute an odd number of clockwise edges to the symmetric difference, so the symmetric difference is evenly oriented. Similarly, if an odd number of the p edges are oriented clockwise with respect to $\mathcal{F}_1$, then both cycles contribute an odd number of clockwise edges to the symmetric difference, and the symmetric difference is evenly oriented.

The argument for the final two cases follows exactly from these arguments, so these are left to the reader. With these arguments, the proof is complete.

□

The union of two perfect matchings contains only trivial and nice cycles. All nice cycles enclose an even number of vertices, so if all of the faces of $\overrightarrow{G}$ are oddly oriented, all nice cycles are also oddly oriented from Lemma 2.

Example
Consider the Cartesian Lattice. Orient all vertical edges "upward" and alternate the orientation of the horizontal edges by giving all of the edges on the same row the same orientation and alternating the orientation of the rows. Obviously, every face is oddly oriented since the horizontal edges either both go with or against any cycle, while one of the vertical edges goes with the cycle and the other does not. Therefore, we can use this orientation to calculate the number of perfect matchings in a graph on the Cartesian lattice.

We now describe an algorithm which will determine an orientation of G that will have only oddly oriented faces. First choose a vertex and find a spanning tree from that vertex, orienting the edges arbitrarily. For example, we could orient all the edges in the tree away from the root. This tree determines a spanning tree in the dual graph, (i.e. the graph where each of the faces of G becomes a vertex, and each vertex in G becomes a face in the dual. Then do a depth first search of the spanning tree in the dual orienting each edge in G so that it forms an oddly oriented cycle as we cross it.

So now we have an algorithm for counting the number of perfect matchings in a planar graph.

Algorithm (F,K,T)

1. Orient G so that all faces (except possibly the outer one) are oddly oriented.
2. Let B be the skew symmetric adjacency matrix of G.
3. The number of perfect matchings is the square root of $det(B)$.

Conclusion This algorithm can count the number of perfect matchings for planar graphs in matrix multiplication time. This method cannot be generalized for general graphs, but in future lectures we will see other methods for counting perfect matchings on lattice graphs that are even more efficient.