Percolation in $\mathbb{Z}^2$

Recall that the critical probability of bond percolation on the d-dimensional cartesian lattice is defined by

$$p_c(\mathbb{Z}^d) = \inf\{p \ : \ P_p[0 \leftrightarrow \infty] > 0\},$$

where $P_p[0 \leftrightarrow \infty]$ is the probability that the origin is in an infinite open cluster. We have seen that

$$0 < p_c(\mathbb{Z}^d) < 1$$

for any $d \geq 2$. For $d = 2$ we have $p_c(\mathbb{Z}^2) = \frac{1}{2}$. This means that we do not have percolation in $\mathbb{Z}^2$ for $p < \frac{1}{2}$, and we have percolation for $p > \frac{1}{2}$. In this lecture we study what happens at the critical probability.

One of the key ingredients in the study of bond percolation in $\mathbb{Z}^2$ is self-duality. Let $LR(l, k)$ denote the event that there is a left-to-right path of open bonds in a fixed $l \times k$ region of $\mathbb{Z}^2$. A first application of self-duality is:

Lemma 1 $P_{\frac{1}{2}}[LR(l, l-1)] = \frac{1}{2}$.

Proof : We either have a left-to-right path of open bonds in the box itself, or a top-to-bottom path of closed bonds in the dual box (which is of size $(l-1) \times l$). Since for $p = \frac{1}{2}$ these two complementary events have equal probability, the claim follows. □

For even l, let $B(l)$ denote the $l \times l$ region of $\mathbb{Z}^2$ centered at the origin. Showing that percolation does not happen for $p = \frac{1}{2}$ is equivalent to showing that, with probability 1, there exists a closed circuit around the origin. Our strategy will be to show that, with probability uniformly bounded away from zero, there exists a closed circuit in any annulus of the form $A(l) = B(3l) - B(l)$. More precisely, we will prove:

Theorem 2 *(Russo–Seymour–Welsh) If $\tau = P_p[LR(l, l)]$ then*

$$P_p[A(l) \textit{ contains an open circuit}] \geq \left(\tau(1 - \sqrt{1-\tau})^4\right)^{12}$$

For $p = \frac{1}{2}$ we have $\tau \geq \frac{1}{4}$ (by Lemma 1, $P_{\frac{1}{2}}[LR(l, l-1)] = \frac{1}{2}$, and the bond that extends any left-to-right path in the $l \times (l-1)$ box is open with probability $\frac{1}{2}$). Thus, since a circuit has the same probability of being open or closed when $p = \frac{1}{2}$, the RSW theorem gives:

$$P_{\frac{1}{2}}[A(l) \text{ contains a closed circuit}] \geq \frac{1}{4^{12}} \left(1 - \frac{\sqrt{3}}{2}\right)^{48}.$$

This implies that for $p = \frac{1}{2}$ we have a closed circuit in at least one of the disjoint annulae $A(l)$, $l = 2, 2 \cdot 3, 2 \cdot 3^2, \ldots$, with probability 1. Hence, *we do not have percolation on $\mathbb{Z}^2$ for $p = \frac{1}{2}$.*

The key step in the proof of the RSW theorem is:

Lemma 3 $P_p[LR(\frac{3}{2}l, l)] \geq (1 - \sqrt{1-\tau})^3$

Proof of the RSW theorem : Let

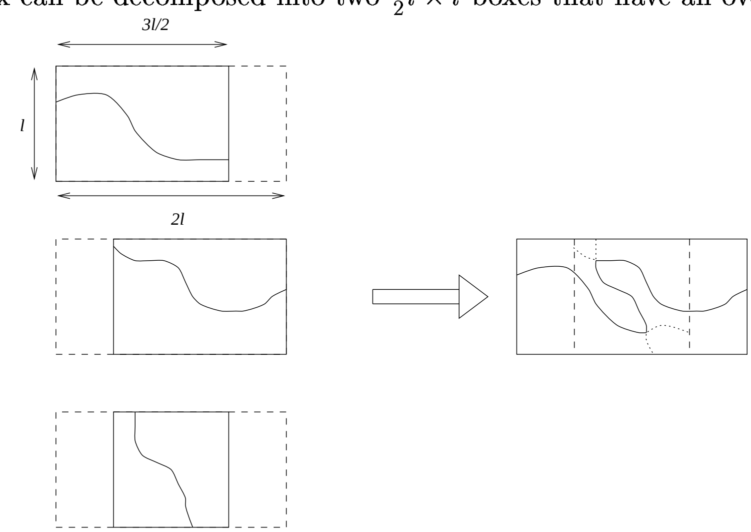
$$\begin{aligned}\tau_1 &= P_p[LR(\tfrac{3}{2}l, l)]\\ \tau_2 &= P_p[LR(2l, l)]\\ \tau_3 &= P_p[LR(3l, l)]\\ \tau_4 &= P_p[A(l) \text{ contains an open circuit}]\end{aligned}$$

We can succesively lower bound τ_i, $i = 2, 3, 4$, in terms of τ and τ_1 as follows.

First,

$$\tau_2 \geq \tau_1^2 \tau. \tag{1}$$

Indeed, a $2l \times l$ box can be decomposed into two $\frac{3}{2}l \times l$ boxes that have an overlap of size $l \times l$:

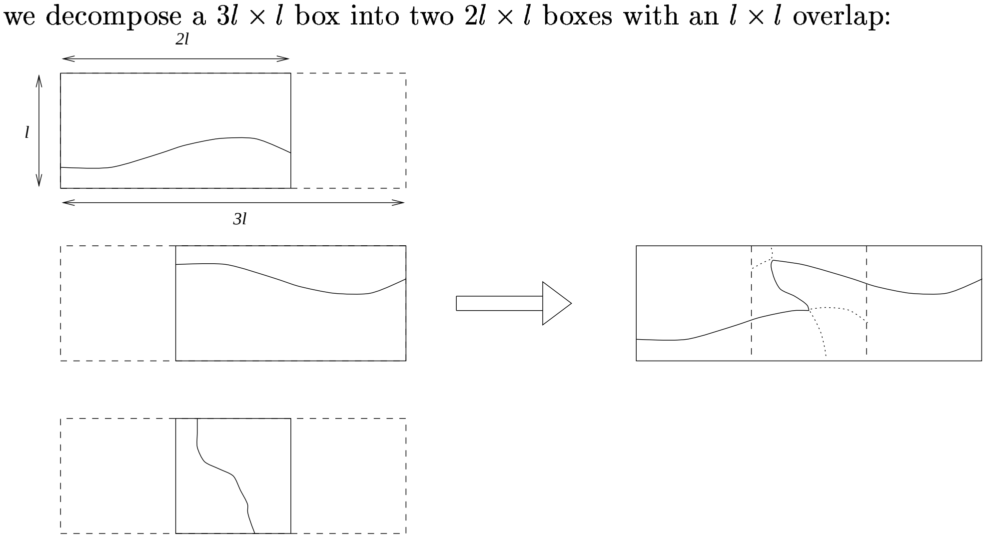


The $2l \times l$ box certainly has a left-to-right path of open bonds if both $\frac{3}{2}l \times l$ boxes have left-to-right open paths and their overlap has a top-to-bottom open path. Thus, inequality (1) follows from the FKG inequality for increasing events.

In a similar way

$$\tau_3 \geq \tau_2^2 \tau; \tag{2}$$

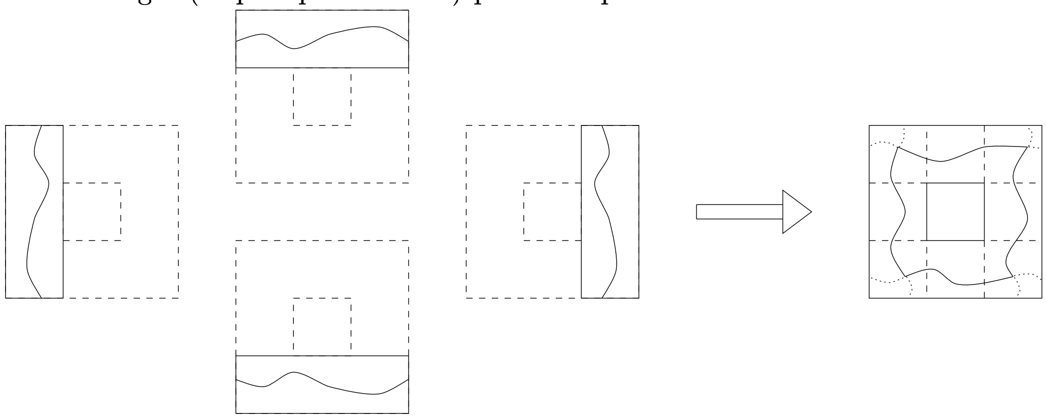
this time we decompose a $3l \times l$ box into two $2l \times l$ boxes with an $l \times l$ overlap:



Finally,

$$\tau_4 \geq \tau_3^4, \tag{3}$$

since $A(l)$ will certainly contain an open circuit if the two $3l \times l$ and the two $l \times 3l$ boxes that cover it contain left-to-right (resp. top-to-bottom) paths of open bonds:



Combining (1–3) we get

$$\tau_4 \geq \tau_3^4 \geq (\tau\tau_2^2)^4 \geq (\tau(\tau_1^2\tau)^2)^4 = (\tau^3\tau_1^4)^4$$

and the theorem follows from Lemma 3. □

To prove Lemma 3 we will need:

Lemma 4 *(The square root trick.) If A_1 and A_2 are increasing events with equal probability then*

$$P_p[A_1] \geq 1 - \sqrt{1 - P_p[A_1 \cup A_2]}.$$

Proof : $1 - P_p[A_1 \cup A_2] = P_p[\overline{A_1} \cap \overline{A_2}] \overset{\text{by FKG}}{\geq} P_p[\overline{A_1}] \cdot P_p[\overline{A_2}] = (1 - P_p[A_1])^2.$ □

Proof of Lemma 3 : Without loss of generality, we may assume that the $\frac{3}{2}l \times l$ box is positioned such that its upper-right corner is at $(l, \frac{l}{2})$. Let B be the box $[-\frac{l}{2}, \frac{l}{2}] \times [-\frac{l}{2}, \frac{l}{2}]$, and B' the box $[0, l] \times [-\frac{l}{2}, \frac{l}{2}]$.

We need to introduce some notations. Let T be the set of left-to-right paths in B. If $\pi \in T$, then

y_π is the point where π crosses for the *last* time the y axis;

π_r is the part of π from y_π to the right border of B;

π'_r is the reflection of π_r around this border (i.e., around the line $x = \frac{l}{2}$);

T^- (T^+) is the set of paths $\pi \in T$ that have $y_\pi \leq 0$ (resp. $y_\pi \geq 0$);

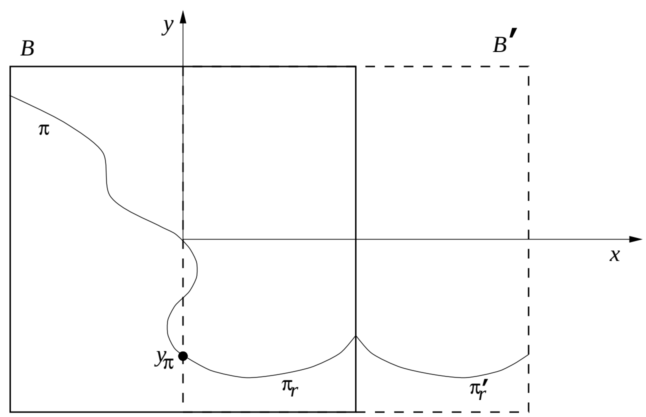
L^- (L^+) is the event that a path in T^- (resp. in T^+) is open;

A_π is the event that path π is open;

L_π is the event that path π is the "lowest" open path of T;

M_π^- (M_π^+) is the event that there is an open path from the top of B' to π_r (resp. π_r');

N^+ (N^-) is the event that there exists a left-to-right open path in B' starting above (resp. below) the x-axis.



Since a configuration in

$$N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)$$

guarantees the existence of a left-to-right path of open bonds in $B \cup B'$, we have

$$P_p\left[LR(\frac{3}{2}l, l)\right] \geq P_p\left[N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)\right].$$

As both N^+ and the union event are increasing, by the FKG inequality we get

$$P_p\left[N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)\right] \geq P_p[N^+] \cdot P_p\left[\bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)\right]$$

Note that $P_p[N^-] = P_p[N^+]$, so, by the square root trick,

$$P_p[N^+] \geq 1 - \sqrt{1 - P_p[N^- \cup N^+]} = 1 - \sqrt{1-\tau}.$$

Hence, to complete the proof it suffices to show that

$$P_p\left[\bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)\right] \geq (1 - \sqrt{1-\tau})^2.$$

Now,

$$P_p\left[\bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)\right] \geq P_p\left[\bigcup_{\pi \in T^-} (M_\pi^- \cap L_\pi)\right] = \sum_{\pi \in T^-} P_p[M_\pi^- | L_\pi] \cdot P_p[L_\pi]$$

Claim : For every $\pi \in T^-$, $P_p[M_\pi^- | L_\pi] \geq P_p[M_\pi^-]$.

Let J_π denote the set of bonds of B' "trapped" between the left border of B' and π. Note that M_π^- does not depend on whether or not the edges in J_π are open. Moreover, note that M_π and J_π are increasing events (this is not true about L_π). So,

$$P_p[M_\pi^-|L_\pi] = P_p[M_\pi^-|J_\pi] = \frac{P_p[M_\pi^- \cap J_\pi]}{P_p[J_\pi]} \overset{\text{by FKG}}{\geq} \frac{P_p[M_\pi^-]P_p[J_\pi]}{P_p[J_\pi]} = P_p[M_\pi^-],$$

proving the claim.

For every $\pi \in T^-$, $P_p[M_\pi^-] = P_p[M_\pi^+]$ because π_r' is the reflection of π_r. So, using the square root trick and the fact that $P_p[M_\pi^- \cup M_\pi^+] \geq \tau$, we get

$$P_p[M_\pi^-] \geq 1 - \sqrt{1 - P_p[M_\pi^- \cup M_\pi^+]} \geq 1 - \sqrt{1-\tau}.$$

Therefore,

$$\sum_{\pi \in T^-} P_p[M_\pi^-|L_\pi] \cdot P_p[L_\pi] \geq (1 - \sqrt{1-\tau}) \cdot \left(\sum_{\pi \in T^-} P_p[L_\pi] \right) \geq (1 - \sqrt{1-\tau}) \cdot P_p[T^-].$$

The proof is completed by applying the square root trick once again for $P_p[T^-] = P_p[T_+]$. □

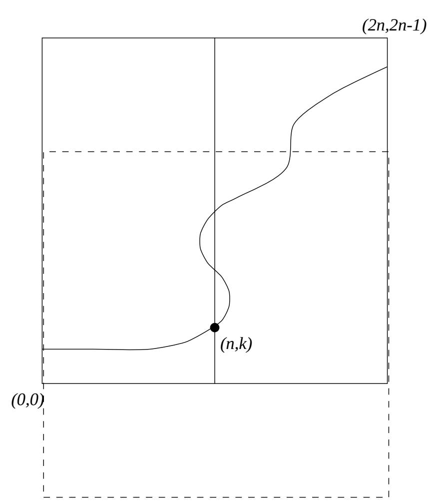
Continuing the study of bond percolation at the critical probability, we next establish lower- and upper-bounds on the probability that the open cluster containing the origin extends past $B(n)$.

Theorem 5 *(Power law inequalities.) There exist constants A and α such that*

$$\tfrac{1}{2} n^{-\frac{1}{2}} \leq P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)] \leq A n^{-\alpha},$$

where $\partial B(n)$ denotes the border of $B(n)$.

Proof : For any $k \in [0, 2n-1]$ let $A(k)$ denote the event that there is an open path from (n,k) to the border of the $(2n+1) \times (2n+1)$ box centered at (n,k) (which is nothing but a translation of $B(n)$).



Note that there are two *disjoint* paths of this kind when the box $[0, 2n] \times [0, 2n-1]$ has a left-to-right path of open bonds passing through (n, k). Since a left-to-right open path in the box $[0, 2n] \times [0, 2n-1]$ has to cross the line $x = n$ at least once, we get that

$$P_{\frac{1}{2}}[LR(2n+1, 2n] \leq \sum_{k=0}^{2n-1} P_{\frac{1}{2}}[A(k) \circ A(k)].$$

But $P_{\frac{1}{2}}[LR(2n+1, 2n] = \frac{1}{2}$ by Lemma 1, and $P_{\frac{1}{2}}[A(k) \circ A(k)] \leq P_{\frac{1}{2}}[A(k)]^2$ by the BK inequality. Since $P_{\frac{1}{2}}[A(k)] = P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)]$, it follows that

$$\tfrac{1}{2} \leq 2n P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)]^2,$$

and this gives the left-hand side inequality.

To prove the right-hand side inequality we will rely again on duality. Note that if there is an open path from the origin to the boundary of $B(n)$, then there is no closed circuit in each of the $(\frac{1}{2}, \frac{1}{2})$ centered annulae of external radiae $3, 3^2, \ldots, 3^{\lfloor \log_3 n \rfloor - 1}$ *of the dual lattice*. But we know from the RSW theorem that each such annulus contains a closed circuit with some probability $\xi > 0$, so

$$P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)] \leq (1-\xi)^{\lfloor \log_3 n \rfloor - 1} \leq (1-\xi)^{\log_3 n - 2} = \frac{n^{\log_3(1-\xi)}}{(1-\xi)^2}.$$

□