

ASYMPTOTICALLY MULTIPLICATIVE QUANTUM INVARIANTS

STAVROS GAROUFALIDIS AND SEOKBEOM YOON

ABSTRACT. The Euler characteristic and the volume are two best-known multiplicative invariants of manifolds under finite covers. On the other hand, quantum invariants of 3-manifolds are not multiplicative. We show that a perturbative power series, introduced by Dimofte and the first author and shown to be a topological invariant of cusped hyperbolic 3-manifolds by Storzer–Wheeler and the first author, and conjectured to agree with the asymptotics of the Kashaev invariant to all orders in perturbation theory, is asymptotically multiplicative under cyclic covers. Moreover, its coefficients are determined by polynomials constructed out of twisted Neumann–Zagier data. This gives a new t -deformation of the perturbative quantum invariants, different than the x -deformation obtained by deforming the geometric representation. We illustrate our results with several hyperbolic knots.

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1. INTRODUCTION

1.1. Multiplicative invariants and quantum invariants. A topological invariant φ of manifolds is said to be *multiplicative* if $\varphi(M') = n\varphi(M)$ for every n -sheeted cover $M' \rightarrow M$. Undoubtedly, the best known example of a multiplicative invariant is the Euler characteristic and, in dimension 3, the volume of a (finite volume) complete hyperbolic 3-manifold, which is a topological invariant as follows from Mostow-rigidity [Thu77].

On the other hand, quantum invariants of 3-manifolds constructed by a topological quantum field theory, such as the Witten–Reshetikhin–Turaev invariant [Wit89, RT90, Tur94], the Turaev–Viro invariant [TV92], and the Kashaev invariant [Kas95], are far from being multiplicative, even under cyclic covers [Gil99]. However, it turns out that quantum invariants are often *asymptotically multiplicative*, that is, they satisfy an equation of the form $\varphi(M') = n\varphi(M) + O(1)$ for suitable n -sheeted covers $M' \rightarrow M$, reminiscent to invariants of coarse geometry [Cal09].

Our goal is to show that some natural perturbative quantum invariants, namely the ones defined in [DG13, DG18] are asymptotically multiplicative under cyclic covers and even more, have a polynomial which determines their values at all cyclic covers.

To avoid technicalities, we will focus on the set \mathcal{M} of knot complements in rational homology 3-spheres which is closed under cyclic coverings of order coprime to a natural number that depends on the manifold in question. A 3-manifold M is in \mathcal{M} if and only if it has betti number $b_1(M) = 1$ and torus boundary; its n -fold cyclic cover will be denoted by $M^{(n)}$.

In [DG13] Dimofte and the first author introduced a power series

$$\Phi_{\mathcal{T}}(h) = \frac{1}{\sqrt{\delta_{\mathcal{T}}}} \left(1 + \sum_{\ell=2}^{\infty} \Phi_{\mathcal{T},\ell} h^{\ell-1} \right) = \frac{1}{\sqrt{\delta_{\mathcal{T}}}} \exp \left(\sum_{\ell=2}^{\infty} \Phi_{\mathcal{T},\ell}^c h^{\ell-1} \right) \quad (1)$$

associated to an ideal triangulation \mathcal{T} (and more precisely, to the Neumann–Zagier data of \mathcal{T}) of a hyperbolic 3-manifold $M \in \mathcal{M}$. These power series are constructed by formal Gaussian integration of a multivariate function which is a product of the infinite Pochhammer symbol (one for each tetrahedron) times the exponential of a quadratic form. Hence the coefficients $\Phi_{\mathcal{T},\ell}$ (resp., $\Phi_{\mathcal{T},\ell}^c$), the so-called ℓ -loop invariants (resp., connected ones) are given by a finite sum over Feynman diagrams of the contraction of tensors and take values in the invariant trace field F of the underlying hyperbolic manifold M .

It was recently proven in joint work of Storzer–Wheeler and the first author [GSW] that the power series $\Phi_{\mathcal{T}}(h)$ is a topological invariant of cusped hyperbolic 3-manifolds, i.e., that it is invariant under 2–3 Pachner moves as well as other choices made in its definition. This series is conjectured to be the asymptotic expansion to all orders in perturbation theory of two famous quantum invariants, namely the Kashaev invariant $\langle K \rangle_N$ of a knot K [Kas95] and (after complex conjugation) the Andersen–Kashaev state-integral [AK14]. More precisely, the Volume Conjecture of Kashaev [Kas95] asserts that for a hyperbolic knot K , $\log |\langle K \rangle_N|$

is asymptotic to $\text{Vol}(K)/(2\pi)$ as N goes to infinity, and its extension to all orders in $1/N$ asserts that [Gar08, DGLZ09]

$$\langle K \rangle_N \sim N^{\frac{3}{2}} e^{\frac{V(K)}{2\pi i} N} \Phi_{\mathcal{T}}\left(\frac{2\pi i}{N}\right) \quad (2)$$

where $V(K) = i\text{Vol}(K) + \text{CS}(K) \in \mathbb{C}/(4\pi^2\mathbb{Z})$ is the complexified volume of K . This aspect, together with the rich analytic and arithmetic structure of the series $\Phi_{\mathcal{T}}(h)$ is discussed in detail in [GZb, GZa].

1.2. Asymptotic quantum invariants. In our previous paper [GY] we studied how the 1-loop invariant, $\delta_{\mathcal{T}}$ in (1), behaves under finite cyclic covers. To do so, we showed that the Neumann–Zagier data of the n -cyclic cover $\mathcal{T}^{(n)}$ of an ideal triangulation \mathcal{T} is determined by a twisted version of the Neumann–Zagier data of \mathcal{T} . Using this, we introduced a twisted version $\delta_{\mathcal{T}}(t) \in F[t^{\pm 1}]/(\pm t^{\mathbb{Z}})$ of the 1-loop invariant, proved its topological invariance, and showed that it determines the 1-loop invariant of $\mathcal{T}^{(n)}$ for all n by $\delta_{\mathcal{T}^{(n)}} = \prod_{\omega^{n=1}} \delta_{\mathcal{T}}(\omega)$. Note that the twisted 1-loop invariant defined in [GY] has a $t - 1$ factor and is $t - 1$ times the one used here. Finally, we conjectured that $\delta_{\mathcal{T}}(t)$ agrees with the adjoint twisted Alexander polynomial, a palindromic polynomial (see e.g. [DY12, DFJ12]). In the rest of the paper, we will assume that $\delta_{\mathcal{T}}(t)$ is palindromic and denote the set of its roots (with possible repetitions) by $\Lambda = \{\lambda_1^{\pm 1}, \dots, \lambda_r^{\pm 1}\}$. We also denote by E the splitting field of $\delta_{\mathcal{T}}(t)$ over the invariant trace field F of M and by $\|\delta_{\mathcal{T}}\|$ the maximum of the absolute values of the roots of $\delta_{\mathcal{T}}(t)$. The palindromic condition implies that $\|\delta_{\mathcal{T}}\| \geq 1$, and when $\delta_{\mathcal{T}}(t)$ has no roots on the unit circle, it follows that $\|\delta_{\mathcal{T}}\| > 1$.

To simplify the statements of our theorems, we will also assume that (a) $\delta_{\mathcal{T}}(t)$ is non-resonant, i.e. that $\prod_{j=1}^r \lambda_j^{n_j} = 1$ for integers n_j implies that $n_j = 0$ for all j , (b) has no roots on the unit circle, and (c) that the Neumann–Zagier datum is computed with respect to the longitude. This way our theorems have clean statements. However, our proofs apply to the resonant case as well as to Neumann–Zagier data with respect to an arbitrary peripheral curve, and given explicitly as remarks following the proofs of the theorems.

Theorem 1.1. *For every $\ell \geq 2$ there exists $\Psi_{\mathcal{T},\ell}^c \in E$ such that*

$$\Phi_{\mathcal{T}^{(n),\ell}}^c = n\Psi_{\mathcal{T},\ell}^c + O(n^{\ell-1}\|\delta_{\mathcal{T}}\|^{-n}) \quad (3)$$

as n tends to infinity, where $\Psi_{\mathcal{T},\ell}^c$ are

- quantum invariants given as a weighted sum over ℓ -loop Feynman diagrams of multidimensional integrals of rational differential forms over tori.
- multiplicative under cyclic covers, i.e.,

$$\Psi_{\mathcal{T}^{(n),\ell}}^c = n\Psi_{\mathcal{T},\ell}^c \quad (n \geq 1). \quad (4)$$

It is likely that the integral representation of $\Psi_{\mathcal{T},\ell}^c$ can be evaluated using Grothendieck residues [Tsi92, CDS96] thus giving an alternative proof that these invariants take values in the splitting field E .

1.3. The shape of the quantum invariants of cyclic covers. In this section we describe the shape of the ℓ -loop invariants of n -cyclic covers in terms of the evaluation of polynomials (in a finite dimensional vector space for each fixed ℓ) at $1/(1 - \lambda_j^n)$ for $j = 1, \dots, r$. We abbreviate the polynomial ring $E[x_1, \dots, x_r]$ by $E[x]$ and let $\mathcal{F}_s E[x]$ denote its subspace spanned by elements of degree at most s .

Theorem 1.2. *For every $\ell \geq 2$ there exists a polynomial*

$$p_{\mathcal{T}, \ell}(x_1, \dots, x_r, y) \in \mathcal{F}_{2\ell-2} E[x][y] \quad (5)$$

such that for all but finitely many n , we have

$$\Phi_{\mathcal{T}^{(n)}, \ell} = p_{\mathcal{T}, \ell} \left(\frac{1}{1 - \lambda_1^n}, \dots, \frac{1}{1 - \lambda_r^n}, n \right). \quad (6)$$

The y -degree of $p_{\mathcal{T}, \ell}$ is at most $\ell - 1$.

For example, the 2-loop invariant of n -cyclic covers is given by

$$\Phi_{\mathcal{T}^{(n)}, 2} = n \left(\sum_{1 \leq i < j \leq r} \frac{c_{ij}}{(1 - \lambda_i^n)(1 - \lambda_j^n)} + \sum_{1 \leq i \leq r} \frac{c_i}{1 - \lambda_i^n} + c_0 \right) \quad (7)$$

where c_{ij} , c_i and c_0 are $(r+1)(r+2)/2$ constants in E .

Let us make some comments to complement the above theorem.

1. The ℓ -loop invariants are given by a finite sum over the set of ℓ -loop Feynman diagrams. The proof of the above theorem is local, i.e., valid for the contribution of each Feynman diagram, hence $p_{\mathcal{T}, \ell}$ is a sum of polynomials that depend on Feynman diagrams.
2. The coefficients of $p_{\mathcal{T}, \ell}$ have an integrality property discussed in Remark 5.6 after the proof of the theorem.
3. The theorem is stated for each fixed ℓ and all but finitely many n . On the other hand, we have good reasons to think that the result holds for all n , and present evidence of this in Section 6.

In the special case when $\delta_{\mathcal{T}}(t)$ is quadratic (as is the case for all twist knots), we have an alternative form of the loop invariants of cyclic covers.

Theorem 1.3. *If $\delta_{\mathcal{T}}(t)$ is quadratic, then for each $\ell \geq 2$ there exists a polynomial $q_{\mathcal{T}, \ell}(x, y) \in \mathcal{F}_{2\ell-2} F[x][y]$ such that for all but finitely many n , we have*

$$\Phi_{\mathcal{T}^{(n)}, \ell} = \sum_{t^n=1} q_{\mathcal{T}, \ell} \left(\frac{1}{\delta_{\mathcal{T}}(t)}, \frac{1}{n} \right). \quad (8)$$

1.4. Rationality and determination. Theorem 1.2 determines the ℓ -loop invariants of n -cyclic covers in terms of evaluations of a polynomial $p_{\mathcal{T}, \ell}$. In this section we address the opposite problem of determining the polynomial $p_{\mathcal{T}, \ell}$ from its evaluations.

A corollary of Theorem 1.2 is that the sequence of ℓ -loop invariants of cyclic covers, after multiplied by a suitable power of the n -th cyclic resultant $N_n(\delta_{\mathcal{T}}) = \prod_{\omega^n=1} \delta_{\mathcal{T}}(\omega)$ of $\delta_{\mathcal{T}}(t)$ has a rational generating series. In particular, the sequence of renormalized ℓ -loop invariants of cyclic covers is a generalized power sum (in the sense of [vdP89, EvdPSW03] and briefly reviewed in Section 5.4 below) uniquely determined by a rational function, which we may think of as a twisted version of the loop invariant.

Proposition 1.4. (a) For every $\ell \geq 2$, there exists a rational function $\Phi_{\mathcal{T},\ell}^{\text{rat}}(t) \in E(t)$ regular at $t = 0$ such that

$$\Phi_{\mathcal{T},\ell}^{\text{rat}}(t) = \sum_{n=0}^{\infty} N_n(\delta_{\mathcal{T}})^{\ell-1} \Phi_{\mathcal{T}^{(n)},\ell} t^n. \quad (9)$$

(b) The polynomial $p_{\mathcal{T},\ell}(x, y)$ of (5) is determined by Λ and $(\ell-1) \binom{r+2\ell-2}{r}$ consecutive values of $\Phi_{\mathcal{T}^{(n)},\ell}$.

The above proposition gives a t -deformation of the perturbative series $\Phi_{\mathcal{T}}(h)$ which is different from the x -deformation of $\Phi_{\mathcal{T}}(h)$ defined in [DG13] and studied in detail [GGMn]; for instance, see Equations (123) and (238) for the 4_1 and the 5_2 knots, respectively. The x -deformed series $\Phi_{\mathcal{T}}(x, h)$ is reciprocal, i.e., satisfies $\Phi_{\mathcal{T}}(x^{-1}, h) = \Phi_{\mathcal{T}}(x, h)$, as follows from Weyl duality, or from the fact that x denotes one of the two eigenvalues x and x^{-1} of the holonomy of the meridian. On the other hand, the rational function of (9) is not invariant under $t \mapsto t^{-1}$ as the case of $\ell = 2$ and the 5_2 knot illustrates.

The above corollary determines the polynomial $p_{\mathcal{T},\ell}(x, y)$ from finitely many values of the ℓ -loop invariants of cyclic covers, together with the set Λ . The next proposition removes the assumption that Λ is known, at the cost of using infinitely many values of the ℓ -loop invariants of cyclic covers. This is a generalization of some results of Fried and Hillar who recover a palindromic polynomial with no cyclotomic factors from its cyclic resultants [Fri88, Hil05]. Its proof uses asymptotics, much in the spirit of recovering the Poincaré map from the asymptotics of the wave-trace functions [Gui96, ISZ02].

Proposition 1.5. (a) Let $R(x, y) \in \mathbb{C}(x)[y]$ be a rational function, regular at $x = 0$, where $x = (x_1, \dots, x_r)$ and $\Lambda_+ = \{\lambda_1, \dots, \lambda_r\}$ be a multiplicatively independent set of nonzero complex numbers with absolute values less than 1. Then the rational function $R(x, y)$ and the set Λ_+ are uniquely determined by infinitely many values of $R(\lambda_1^n, \dots, \lambda_r^n, n)$.

(b) The polynomial $p_{\mathcal{T},\ell}$ is determined by infinitely many values of $\Phi_{\mathcal{T}^{(n)},\ell}$ for each fixed ℓ .

Finally, we mention that the structure of Equation (5) of perturbative quantum invariants of cyclic covers is a very general (even if unnoticed) statement of perturbative quantum field theory. In particular, one can define asymptotically multiplicative quantum invariants using perturbation theory of the trivial connection, as described in the power series expansion of the colored Jones polynomial and the Le–Murakami–Ohtsuki invariant [LMO98], as well as perturbation theory at abelian $SU(2)$ -connections, as described by the rational form of the Kontsevich integral of a knot by Kricker [GK04]. We will discuss this subject, of a slightly different flavor, in a later investigation.

2. A REVIEW OF THE LOOP INVARIANTS

Let M be an oriented 1-cusped 3-manifold and \mathcal{T} be an ideal triangulation of M with N tetrahedra $\Delta_1, \dots, \Delta_N$ and with N edges e_1, \dots, e_N . The shape of Δ_j is described by one complex variable $z_j \in \mathbb{C} \setminus \{0, 1\}$ and each edge of Δ_j is assigned with one of the following parameters with opposite edges having same parameters:

$$z_j, \quad z'_j := \frac{1}{1 - z_j}, \quad z''_j := 1 - \frac{1}{z_j}.$$

The shapes $z = (z_1, \dots, z_N)$ satisfy a system of equations, one per every edge of \mathcal{T} , hence N equations. Precisely, the equation for an edge e_i is given by

$$\prod_{j=1}^N z_j^{\mathbf{G}_{ij}} \prod_{j=1}^N z'_j{}^{\mathbf{G}'_{ij}} \prod_{j=1}^N z''_j{}^{\mathbf{G}''_{ij}} = 1 \quad (10)$$

where \mathbf{G}_{ij} (resp., \mathbf{G}'_{ij} , \mathbf{G}''_{ij}) is the number of edges of Δ_j which are incident to e_i in \mathcal{T} and have shape parameter z_j (resp., z'_j , z''_j). The exponents of (10) form three $N \times N$ integral matrices \mathbf{G} , \mathbf{G}' and \mathbf{G}'' which are known to be singular. To remove such singularity, we choose a peripheral curve γ of M and replace the last row of \mathbf{G} , \mathbf{G}' and \mathbf{G}'' (one edge equation) by its completeness equation. We denote by \mathbf{G}_γ , \mathbf{G}'_γ and \mathbf{G}''_γ the resulting three $N \times N$ integral matrices accordingly. Following [NZ85], we set

$$\begin{aligned} \mathbf{A} &= \mathbf{G} - \mathbf{G}', & \mathbf{B} &= \mathbf{G}'' - \mathbf{G}', \\ \mathbf{A}_\gamma &= \mathbf{G}_\gamma - \mathbf{G}'_\gamma, & \mathbf{B}_\gamma &= \mathbf{G}''_\gamma - \mathbf{G}'_\gamma \end{aligned}$$

and refer to \mathbf{A} and \mathbf{B} (resp., \mathbf{A}_γ and \mathbf{B}_γ) as Neumann-Zagier matrices of \mathcal{T} (with respect to γ). Finally, the propagator matrix of \mathcal{T} with respect to γ is defined by ([DG13])

$$\Pi_\gamma = \hbar \left(-\mathbf{B}_\gamma^{-1} \mathbf{A}_\gamma + \Delta_{z'} \right)^{-1}$$

where \hbar is a formal variable and $\Delta_{z'}$ is a diagonal matrix with diagonal $z' = (z'_1, \dots, z'_N)$.

In what follows, we fix a peripheral curve γ of M . Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The Feynman rule defines a weight $W_{\mathcal{T}}(G; \iota)$ for a vertex-labeling $\iota : V(G) \rightarrow \{1, \dots, N\}$

$$W_{\mathcal{T}}(G; \iota) = \prod_{v \in V(G)} \Gamma_{\iota(v)}^{(d(v))} \prod_{(v, v') \in E(G)} (\Pi_\gamma)_{\iota(v), \iota(v')} \quad (11)$$

where $\Gamma_{\iota(v)}^{(d(v))}$ is a rational function in the shape parameter $z_{\iota(v)}$ depending on the degree $d(v)$ of v (its precise definition will not be needed in this paper) and $(v, v') \in E(G)$ is an edge joining v and v' . Note that the edge orientation does not matter here, as Π_γ is a symmetric matrix. We refer to [DG13, Section 1.6] for details. Also, a weight $W_{\mathcal{T}}(G)$ associated to G is defined by

$$W_{\mathcal{T}}(G) = \frac{1}{\sigma(G)} \sum_{\iota} W_{\mathcal{T}}(G; \iota) \quad (12)$$

where the sum is over all vertex-labelings $\iota : V(G) \rightarrow \{1, \dots, N\}$ and $\sigma(G)$ is the symmetry factor of G .

Definition 2.1 ([DG13]). The ℓ -loop invariant of \mathcal{T} (with respect to γ) is defined by

$$\Phi_{\mathcal{T}, \ell} = \text{coeff} \left[\sum_{G \in \mathcal{G}_\ell} W_{\mathcal{T}}(G), \hbar^{\ell-1} \right] + \Gamma^{(0)} \quad (13)$$

where $\Gamma^{(0)}$ is the vacuum contribution (see [DG13, Eqn 1.18]) and

$$\mathcal{G}_\ell = \{G : \#(1\text{-vertices}) + \#(2\text{-vertices}) + \#(\text{loops}) \leq \ell\}.$$

Here $\text{coeff}[f(\hbar), \hbar^m]$ denotes the coefficient of \hbar^m in a power series $f(\hbar)$.

3. FLOWS AND LOOP INVARIANTS OF CYCLIC COVERS

In this section we explain how to express the loop invariants of n -cyclic covers as a sum over Feynmann diagrams of graphs with $\mathbb{Z}/n\mathbb{Z}$ -flows; see Theorem 3.7 below. This is possible since the NZ matrices of cyclic covers are expressed in terms of the twisted NZ matrices using circulant matrices, as was found in [GY]. We explain this first.

3.1. Twisted NZ matrices. Let M be an oriented 1-cusped 3-manifold with an ideal triangulation \mathcal{T} and a surjective morphism $\alpha : \pi_1(M) \rightarrow \mathbb{Z}$. We denote by $M^{(\infty)}$ the infinite cyclic cover of M corresponding to the kernel of α and $\mathcal{T}^{(\infty)}$ the ideal triangulation of $M^{(\infty)}$ induced from \mathcal{T} . Similarly, we denote by $M^{(n)}$ the finite n -cyclic cover corresponding to the subgroup $\alpha^{-1}(n\mathbb{Z})$ and $\mathcal{T}^{(n)}$ the ideal triangulation of $M^{(n)}$ induced from \mathcal{T} .

We choose a fundamental domain of M in $M^{(\infty)}$ and denote by \tilde{e}_i (resp., $\tilde{\Delta}_j$) the lift of an edge e_i (resp., a tetrahedron Δ_j) of \mathcal{T} to the fundamental domain. We also choose a generator t of the deck transformation group \mathbb{Z} of $M^{(\infty)}$ so that every tetrahedron of $\mathcal{T}^{(\infty)}$ is given by $t^k \cdot \tilde{\Delta}_j$ for $k \in \mathbb{Z}$ and $1 \leq j \leq N$. We then define an $N \times N$ integral matrix \mathbf{G}_k (resp., $\mathbf{G}'_k, \mathbf{G}''_k$) for $k \in \mathbb{Z}$ by letting its (i, j) -entry be the number of edges of $t^k \cdot \tilde{\Delta}_j$ which are incident to \tilde{e}_i in $\mathcal{T}^{(\infty)}$ and have shape parameter z_j (resp., z'_j, z''_j). Clearly, $\mathbf{G}_k, \mathbf{G}'_k$ and \mathbf{G}''_k are zero matrices for all but finitely many $k \in \mathbb{Z}$. It follows that (below we view t as a formal variable)

$$\mathbf{A}(t) := \sum_{k \in \mathbb{Z}} (\mathbf{G}_k - \mathbf{G}'_k) t^k \quad \text{and} \quad \mathbf{B}(t) := \sum_{k \in \mathbb{Z}} (\mathbf{G}''_k - \mathbf{G}'_k) t^k \quad (14)$$

are $N \times N$ matrices with entries in $\mathbb{Z}[t^{\pm 1}]$. We call $\mathbf{A}(t)$ and $\mathbf{B}(t)$ twisted Neumann-Zagier matrices of \mathcal{T} . See [GY] for some basic properties of twisted Neumann-Zagier matrices.

The twisted Neumann-Zagier matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ determine the Neumann-Zagier matrices $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ of $\mathcal{T}^{(n)}$ for all $n \geq 1$. Precisely, for $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}\}$

$$\mathbf{X}^{(n)} = \begin{pmatrix} \sum_{k \equiv 0} \mathbf{X}_k & \sum_{k \equiv 1} \mathbf{X}_k & \cdots & \sum_{k \equiv n-1} \mathbf{X}_k \\ \sum_{k \equiv n-1} \mathbf{X}_k & \sum_{k \equiv 0} \mathbf{X}_k & \cdots & \sum_{k \equiv n-2} \mathbf{X}_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k \equiv 1} \mathbf{X}_k & \sum_{k \equiv 2} \mathbf{X}_k & \cdots & \sum_{k \equiv 0} \mathbf{X}_k \end{pmatrix}, \quad \mathbf{X}(t) = \sum_{k \in \mathbb{Z}} \mathbf{X}_k t^k$$

where \equiv means the equality of integers in modulo n . In particular, $\mathbf{X}^{(1)} = \mathbf{X}(1)$ and $\mathbf{X}^{(n)}$ is a block circulant matrix for $n \geq 2$. In what follows, we omit the superscript (n) for $n = 1$.

3.2. Block diagonalizations of NZ matrices. Let μ and λ be peripheral curves of M satisfying $\alpha(\mu) = 1$ and $\alpha(\lambda) = 0$. We refer to μ (resp., λ) as the meridian (resp., longitude). Note that μ^n and λ represent peripheral curves of $M^{(n)}$. For notational convenience we will confuse μ^n with μ ; for instance, we simply write $\mathbf{A}_\mu^{(n)}$ and $\mathbf{B}_\mu^{(n)}$ instead of the Neumann-Zagier matrices $\mathbf{A}_{\mu^n}^{(n)}$ and $\mathbf{B}_{\mu^n}^{(n)}$ of $\mathcal{T}^{(n)}$ with respect to μ^n .

Theorem 3.1. $(\mathbf{B}_\gamma^{(n)})^{-1}\mathbf{A}_\gamma^{(n)}$ is a block circulant matrix for $n \geq 2$ and $\gamma \in \{\mu, \lambda\}$. Moreover,

$$V (\mathbf{B}_\gamma^{(n)})^{-1}\mathbf{A}_\gamma^{(n)} V^{-1} = \begin{pmatrix} \mathbf{B}_\gamma^{-1}\mathbf{A}_\gamma & & & \\ & \mathbf{B}(\omega)^{-1}\mathbf{A}(\omega) & & \\ & & \ddots & \\ & & & \mathbf{B}(\omega^{n-1})^{-1}\mathbf{A}(\omega^{n-1}) \end{pmatrix}$$

where $\omega = e^{\frac{2\pi\sqrt{-1}}{n}}$ and V is a block Vandermonde matrix given as in (21).

As a consequence, the propagator matrix of $\mathcal{T}^{(n)}$ with respect to $\gamma \in \{\mu, \lambda\}$

$$\Pi_\gamma^{(n)} = \hbar \left(-(\mathbf{B}_\gamma^{(n)})^{-1}\mathbf{A}_\gamma^{(n)} + \Delta_{z'}^{(n)} \right)^{-1} \quad (15)$$

admits a block diagonalization in terms of

$$\Pi(t) := \hbar \left(-\mathbf{B}(t)^{-1}\mathbf{A}(t) + \Delta_{z'} \right)^{-1}.$$

Here $\Delta_{z'}^{(n)}$ is the diagonal matrix whose diagonal is n times repetitions of z' .

Corollary 3.2. We have

$$V \Pi_\gamma^{(n)} V^{-1} = \begin{pmatrix} \Pi_\gamma & & & \\ & \Pi(\omega) & & \\ & & \ddots & \\ & & & \Pi(\omega^{n-1}) \end{pmatrix} \quad (16)$$

for $\gamma \in \{\mu, \lambda\}$.

Remark 3.3. The proof of [GY, Theorem 1.7] shows that $\mathbf{X}_\lambda = P(t)\mathbf{X}(t)|_{t=1}$ where

$$P(t) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ \frac{t^{p_1+t^{q_1}}}{t-1} & \frac{t^{p_2+t^{q_2}}}{t-1} & \dots & \frac{t^{p_N+t^{q_N}}}{t-1} \end{pmatrix}$$

for some integers p_i and q_i . Hence, $\mathbf{B}_\lambda^{-1}\mathbf{A}_\lambda$ equals to $\mathbf{B}(t)^{-1}\mathbf{A}(t)$ at $t = 1$, and $\Pi_\lambda = \Pi(1)$. This shows that Equation (16) admits full cyclic symmetry for $\gamma = \lambda$.

3.3. Feynman diagrams with flows. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We fix an orientation of each edge of G and regard an element of $E(G)$ as an oriented edge.

Definition 3.4. A $\mathbb{Z}/n\mathbb{Z}$ -flow on G is a map $\varphi : E(G) \rightarrow \mathbb{Z}/n\mathbb{Z}$ such that for all $v \in V(G)$

$$\sum_{\substack{e \in E(G) \\ e \text{ into } v}} \varphi(e) = \sum_{\substack{e \in E(G) \\ e \text{ out of } v}} \varphi(e).$$

The set of $\mathbb{Z}/n\mathbb{Z}$ -flows on G is naturally an abelian group isomorphic to $H_1(G; \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^d$ where d is the first betti number of G .

In what follows, we fix a peripheral curve $\gamma \in \{\mu, \lambda\}$. As a generalizations of (11), for a vertex-labeling $\iota : V(G) \rightarrow \{1, \dots, N\}$, we define a weight

$$W_{\mathcal{T}}^{(n)}(G; \iota) := \frac{1}{n^{d-1}} \sum_{\varphi} \left(\prod_{v \in V(G)} \Gamma_{\iota(v)}^{(d(v))} \prod_{(v, v') \in E(G)} (\Pi_{\varphi(v, v')})_{\iota(v), \iota(v')} \right) \quad (17)$$

where the sum is over all $\mathbb{Z}/n\mathbb{Z}$ -flows on G , (v, v') is an oriented edge of G running from v to v' , and

$$\Pi_{\varphi(e)} = \begin{cases} \Pi_{\gamma} & \text{if } \varphi(e) = 0 \\ \Pi(\omega^{\varphi(e)}) & \text{otherwise} \end{cases}.$$

Also, as in (12), we set

$$W_{\mathcal{T}}^{(n)}(G) := \frac{1}{\sigma(G)} \sum_{\iota} W_{\mathcal{T}}^{(n)}(G; \iota) \quad (18)$$

where the sum is over all vertex-labelings $\iota : V(G) \rightarrow \{1, \dots, N\}$. We remark that for $n = 1$ there is only one flow, the trivial one (assigning 0 to all edges), hence (17) and (18) reduce to (11) and (12), respectively.

Lemma 3.5. $W_{\mathcal{T}}^{(n)}(G)$ does not depend on the choice of edge orientation of G .

Example 3.6. Let us consider a Feynman diagram G with edge orientation as in Figure 1. For notational simplicity we let a vertex-labeling ι assign the vertices to $(i, j) \in \{1, \dots, N\}^2$ and a map φ assign the oriented edges to $(a, b, c) \in (\mathbb{Z}/n\mathbb{Z})^3$ as in Figure 1.

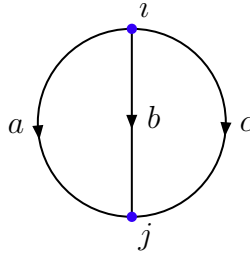


FIGURE 1. A 2-loop Feynman diagram.

Since φ is a flow if and only if $a + b + c \equiv 0$ in modulo n , we have

$$W_{\mathcal{T}}^{(n)}(G) = \frac{1}{8n} \sum_{1 \leq i, j \leq N} \sum_{a, b \in \mathbb{Z}/n\mathbb{Z}} \Gamma_i^{(3)} \Gamma_j^{(3)} (\Pi_a)_{ij} (\Pi_b)_{ij} (\Pi_{-a-b})_{ij}.$$

Note that $\sigma(G) = 8$. In particular, if the peripheral curve γ is chosen to be the longitude, we have (see Remark 3.3)

$$W_{\mathcal{T}}^{(n)}(G) = \frac{1}{8n} \sum_{1 \leq i, j \leq N} \sum_{a, b \in \mathbb{Z}/n\mathbb{Z}} \Gamma_i^{(3)} \Gamma_j^{(3)} \Pi(\omega^a)_{ij} \Pi(\omega^b)_{ij} \Pi(\omega^{-a-b})_{ij}. \quad (19)$$

Theorem 3.7. *The ℓ -loop invariant of $\mathcal{T}^{(n)}$ satisfies*

$$\Phi_{\mathcal{T}^{(n)}, \ell} = \text{coeff} \left[\sum_{G \in \mathcal{G}_{\ell}} W_{\mathcal{T}}^{(n)}(G), \hbar^{\ell-1} \right] + \Gamma^{(0)} \quad (20)$$

for all $n \geq 1$.

4. PROOFS OF THE FLOW STATEMENTS

We devote this section to prove theorems in Section 3.

4.1. Some facts on block circulant matrices. We here list some elementary facts on block circulant matrices. We refer to [Dav79] for details. Let C be a block circulant matrix of the form

$$C = \begin{pmatrix} C_0 & C_1 & \cdots & C_{n-1} \\ C_{n-1} & C_0 & \cdots & C_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ C_1 & C_2 & \cdots & C_0 \end{pmatrix}.$$

It is known that C is block diagonalizable by conjugating a block Vandermonde matrix

$$V = \begin{pmatrix} I & I & \cdots & I \\ I & \omega I & \cdots & \omega^{n-1} I \\ \vdots & \vdots & \ddots & \vdots \\ I & \omega^{n-1} I & \cdots & \omega^{(n-1)(n-1)} I \end{pmatrix}, \quad \omega = e^{\frac{2\pi\sqrt{-1}}{n}}. \quad (21)$$

Explicitly, we have

$$VCV^{-1} = \begin{pmatrix} r(\omega^0) & & & \\ & r(\omega^1) & & \\ & & \ddots & \\ & & & r(\omega^{n-1}) \end{pmatrix}$$

where $r(t) = C_0 + C_1 t + \cdots + C_{n-1} t^{n-1}$ is the *representer* of C . It follows that one can recover the matrix C from its representer $r(t)$ by

$$C_i = \frac{1}{n} (r(\omega^0) + \omega^i r(\omega^1) + \cdots + \omega^{i(n-1)} r(\omega^{n-1})) \quad (22)$$

for $0 \leq i < n$. Note that if $r(t)$ is constant, i.e. $C_1 = \cdots = C_{n-1} = 0$, then C commutes with V .

4.2. Circulant structure of NZ matrices. In this section we prove Theorem 3.1 and its Corollary 3.2.

To begin the proof of Theorem 3.1, we first consider the case of $\gamma = \mu$. Let a_i (resp., b_i) be the i -th row of $\mathbf{A}^{(n)}$ (resp., $\mathbf{B}^{(n)}$) and a_{μ^n} (resp., b_{μ^n}) be the last row of $\mathbf{A}_{\mu}^{(n)}$ (resp., $\mathbf{B}_{\mu}^{(n)}$). Recall that a_{μ^n} and b_{μ^n} represent the completeness equation of μ^n , hence they are n -times repetitions of a_{μ} ($= a_{\mu^1}$) and b_{μ} ($= b_{\mu^1}$) respectively:

$$a_{\mu^n} = \underbrace{(a_{\mu} \cdots a_{\mu})}_n, \quad b_{\mu^n} = \underbrace{(b_{\mu} \cdots b_{\mu})}_n.$$

For a vector v we denote by $\mathbf{O}[v]$ a square matrix whose last row is v and the other rows are trivial. From the symplectic property of Neumann-Zagier matrices [NZ85, Theorem 2.2], we have $a_i b_j^T = b_i a_j^T$ and $a_{\mu^n} b_j^T = b_{\mu^n} a_j^T$. It follows that

$$(\mathbf{A}^{(n)} + \mathbf{O}[a_{\mu^n}]) (\mathbf{B}_{\mu}^{(n)})^T = (\mathbf{B}^{(n)} + \mathbf{O}[b_{\mu^n}]) (\mathbf{A}_{\mu}^{(n)})^T$$

and

$$(\mathbf{B}_\mu^{(n)})^{-1}\mathbf{A}_\mu^{(n)} = ((\mathbf{B}_\mu^{(n)})^{-1}\mathbf{A}_\mu^{(n)})^T = (\mathbf{B}^{(n)} + \mathbf{O}[b_{\mu^n}])^{-1}(\mathbf{A}^{(n)} + \mathbf{O}[a_{\mu^n}]). \quad (23)$$

On the other hand, a simple matrix computation shows that

$$V\mathbf{O}[a_{\mu^n}]V^{-1} = \begin{pmatrix} \mathbf{O}[a_\mu] & 0 & \cdots & 0 \\ \omega^{-1}\mathbf{O}[a_\mu] & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{-(n-1)}\mathbf{O}[a_\mu] & 0 & \cdots & 0 \end{pmatrix}$$

and thus

$$V(\mathbf{A}^{(n)} + \mathbf{O}[a_{\mu^n}])V^{-1} = \begin{pmatrix} \mathbf{A}(1) + \mathbf{O}[a_\mu] & 0 & \cdots & 0 \\ \omega^{-1}\mathbf{O}[a_\mu] & \mathbf{A}(\omega^1) & & \\ \vdots & & \ddots & \\ \omega^{-(n-1)}\mathbf{O}[a_\mu] & & & \mathbf{A}(\omega^{n-1}) \end{pmatrix}.$$

Similarly, we have

$$V(\mathbf{B}^{(n)} + \mathbf{O}[b_{\mu^n}])V^{-1} = \begin{pmatrix} \mathbf{B}(1) + \mathbf{O}[b_\mu] & 0 & \cdots & 0 \\ \omega^{-1}\mathbf{O}[b_\mu] & \mathbf{B}(\omega^1) & & \\ \vdots & & \ddots & \\ \omega^{-(n-1)}\mathbf{O}[b_\mu] & & & \mathbf{B}(\omega^{n-1}) \end{pmatrix}.$$

It follows that

$$V(\mathbf{B}^{(n)} + \mathbf{O}[b_{\mu^n}])^{-1}V^{-1} = \begin{pmatrix} (\mathbf{B}(1) + \mathbf{O}[b_\mu])^{-1} & 0 & \cdots & 0 \\ C_1 & \mathbf{B}(\omega^1)^{-1} & & \\ \vdots & & \ddots & \\ C_{n-1} & & & \mathbf{B}(\omega^{n-1})^{-1} \end{pmatrix}$$

where C_k ($1 \leq k < n$) is a matrix satisfying

$$C_k(\mathbf{B}(1) + \mathbf{O}[b_\mu]) + \omega^{-k}\mathbf{B}(\omega^k)^{-1}\mathbf{O}[b_\mu] = 0. \quad (24)$$

Combining the above, we obtain

$$\begin{aligned} V(\mathbf{B}_\mu^{(n)})^{-1}\mathbf{A}_\mu^{(n)}V^{-1} &= V(\mathbf{B}^{(n)} + \mathbf{O}[b_{\mu^n}])^{-1}(\mathbf{A}^{(n)} + \mathbf{O}[a_{\mu^n}])V^{-1} \\ &= \begin{pmatrix} D_0 & 0 & \cdots & 0 \\ D_1 & \mathbf{B}(\omega^1)^{-1}\mathbf{A}(\omega^1) & & \\ \vdots & & \ddots & \\ D_{n-1} & & & \mathbf{B}(\omega^{n-1})^{-1}\mathbf{A}(\omega^{n-1}) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} D_0 &= (\mathbf{B}(1) + \mathbf{O}[b_\mu])^{-1}(\mathbf{A}(1) + \mathbf{O}[a_\mu]), \\ D_k &= C_k(\mathbf{A}(1) + \mathbf{O}[a_\mu]) + \omega^{-k}\mathbf{B}(\omega^k)^{-1}\mathbf{O}[a_\mu], \quad k = 1, \dots, n-1. \end{aligned}$$

On the other hand, from Equation (23) we have

$$D_0 = (\mathbf{B}(1) + \mathbf{O}[b_\mu])^{-1}(\mathbf{A}(1) + \mathbf{O}[a_\mu]) = \mathbf{B}_\mu^{-1}\mathbf{A}_\mu$$

(recall that $\mathbf{A}^{(1)} = \mathbf{A}(1)$ and $\mathbf{B}^{(1)} = \mathbf{B}(1)$) and from Equation (24) we have

$$\begin{aligned} D_k &= C_k(\mathbf{A}(1) + \mathbf{O}[a_\mu]) + \omega^{-k}\mathbf{B}(\omega^k)^{-1}\mathbf{O}[a_\mu] \\ &= \omega^{-k}\mathbf{B}(\omega^k)^{-1}(-\mathbf{O}[b_\mu](\mathbf{B}(1) + \mathbf{O}[b_\mu])^{-1}(\mathbf{A}(1) + \mathbf{O}[a_\mu]) + \mathbf{O}[a_\mu]) \\ &= \omega^{-k}\mathbf{B}(\omega^k)^{-1}(-\mathbf{O}[b_\mu]\mathbf{B}_\mu^{-1}\mathbf{A}_\mu + \mathbf{O}[a_\mu]) \\ &= 0 \end{aligned}$$

where the last equation follows from the last row of an obvious identity $\mathbf{B}_\mu\mathbf{B}_\mu^{-1}\mathbf{A}_\mu = \mathbf{A}_\mu$, i.e., $b_\mu\mathbf{B}_\mu^{-1}\mathbf{A}_\mu = a_\mu$. This proves Theorem 3.1 for the case of $\gamma = \mu$.

We prove the case of $\gamma = \lambda$ similarly. Let a_λ and b_λ be the last row of $\mathbf{A}_\lambda^{(n)}$ and $\mathbf{B}_\lambda^{(n)}$, respectively. We may write a_λ and b_λ as

$$a_\lambda = (v_1 \cdots v_n) \text{ and } b_\lambda = (w_1 \cdots w_n)$$

where v_i and w_i are vectors of length N . Note that $\mathbf{v} := \sum_{i=1}^n v_i$ and $\mathbf{w} := \sum_{i=1}^n w_i$ represent the complete equation of λ in \mathcal{T} and thus

$$a_{\lambda^n} := (\underbrace{\mathbf{v} \cdots \mathbf{v}}_n) \text{ and } b_{\lambda^n} := (\underbrace{\mathbf{w} \cdots \mathbf{w}}_n)$$

represent n -parallel copies of λ in $\mathcal{T}^{(n)}$. It follows that

$$(\mathbf{A}^{(n)} + \mathbf{O}[a_{\lambda^n}])(\mathbf{B}_\lambda^{(n)})^T = (\mathbf{B}^{(n)} + \mathbf{O}[b_{\lambda^n}])(\mathbf{A}_\lambda^{(n)})^T$$

and

$$(\mathbf{B}_\lambda^{(n)})^{-1}\mathbf{A}_\lambda^{(n)} = \left((\mathbf{B}_\lambda^{(n)})^{-1}\mathbf{A}_\lambda^{(n)} \right)^T = (\mathbf{B}^{(n)} + \mathbf{O}[b_{\lambda^n}])^{-1}(\mathbf{A}^{(n)} + \mathbf{O}[a_{\lambda^n}]).$$

Then the rest computation is exactly same as the case of $\gamma = \mu$. \square

We now turn to Corollary 3.2. The block Vandermonde matrix V given as in (21) commutes with the diagonal matrix $\Delta_{z'}^{(n)}$. Thus Theorem 3.1 implies

$$V \left((\mathbf{B}_\gamma^{(n)})^{-1}\mathbf{A}_\gamma^{(n)} + \Delta_{z'}^{(n)} \right) V^{-1} = \begin{pmatrix} \mathbf{B}_\gamma^{-1}\mathbf{A}_\gamma + \Delta_{z'} & & \\ & \ddots & \\ & & \mathbf{B}(\omega^{n-1})^{-1}\mathbf{A}(\omega^{n-1}) + \Delta_{z'} \end{pmatrix}$$

and Corollary 3.2 follows. \square

4.3. From flows to loop invariants of cyclic covers. In this section we prove Lemma 3.5 and Theorem 3.7.

Let $e = (v, v')$ be an oriented edge of G and $-e = (v', v)$ be the same edge with reversed orientation. It follows from [GY, Theorem 1.2] that

$$\mathbf{B}(1/t)^{-1}\mathbf{A}(1/t) = (\mathbf{B}(t)^{-1}\mathbf{A}(t))^T$$

and hence $\Pi(1/t) = \Pi(t)^T$. In particular, for any $\mathbb{Z}/n\mathbb{Z}$ -flows φ on G we have

$$\Pi(\omega^{\varphi(e)})_{\iota(v), \iota(v')} = \Pi(\omega^{-\varphi(e)})_{\iota(v'), \iota(v)} = \Pi(\omega^{\varphi(-e)})_{\iota(v'), \iota(v)}.$$

Combining the above with the fact that Π_γ is symmetric, we obtain Lemma 3.5. \square

We now turn to Theorem 3.7. Corollary 3.2 shows that $\Pi_\gamma^{(n)}$ is a block circulant matrix whose first row is

$$\frac{1}{n} \left(\sum_{k=0}^{n-1} \Pi_k \quad \sum_{k=0}^{n-1} \omega^k \Pi_k \quad \cdots \quad \sum_{k=0}^{n-1} \omega^{(n-1)k} \Pi_k \right)$$

where $\Pi_0 = \Pi_\gamma$ and $\Pi_k = \Pi(\omega^k)$ for $1 \leq k \leq n-1$. In addition, its $(aN+i, bN+j)$ -entry, where $1 \leq i, j \leq N$ and $0 \leq a, b \leq n-1$, is

$$\left(\Pi_\gamma^{(n)} \right)_{aN+i, bN+j} = \frac{1}{n} \left(\sum_{k=0}^{n-1} \omega^{(b-a)k} \Pi_k \right)_{i,j}.$$

For a Feynman diagram G with vertex set $V = V(G)$ and edge set $E = E(G)$ we have

$$\begin{aligned} W_{\mathcal{T}^{(n)}}(G) &= \frac{1}{\sigma(G)} \sum_{\iota \in \llbracket 1, nN \rrbracket^V} W_{\mathcal{T}^{(n)}}(G; \iota) \\ &= \frac{1}{\sigma(G)} \sum_{\iota \in \llbracket 1, nN \rrbracket^V} \left(\prod_{v \in V} \Gamma_{\iota(v)}^{(d(v))} \prod_{(v, v') \in E} \left(\Pi_\gamma^{(n)} \right)_{\iota(v), \iota(v')} \right). \end{aligned} \quad (25)$$

where $\llbracket i, j \rrbracket^V$ denotes the set of maps from the vertex set V to $\{i, \dots, j\}$. Since $\Pi_\gamma^{(n)}$ is a symmetric matrix, we may regard $(v, v') \in E$ as an oriented edge. Writing $\iota(v) = p(v)N + \underline{\iota}(v)$ for $1 \leq \underline{\iota}(v) \leq N$, we can rewrite Equation (25) as

$$\begin{aligned} & \frac{1}{\sigma(G)} \sum_{\underline{\iota} \in \llbracket 1, N \rrbracket^V} \sum_{p \in \llbracket 0, n-1 \rrbracket^V} \left(\prod_{v \in V} \Gamma_{p(v)N + \underline{\iota}(v)}^{(d(v))} \prod_{(v, v') \in E} \left(\Pi_\gamma^{(n)} \right)_{p(v)N + \underline{\iota}(v), p(v')N + \underline{\iota}(v')} \right) \\ &= \frac{1}{\sigma(G)} \sum_{\underline{\iota} \in \llbracket 1, N \rrbracket^V} \prod_{v \in V} \Gamma_{\underline{\iota}(v)}^{(d(v))} \left(\sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{(v, v') \in E} \left(\Pi_\gamma^{(n)} \right)_{p(v)N + \underline{\iota}(v), p(v')N + \underline{\iota}(v')} \right) \\ &= \frac{1}{\sigma(G)} \sum_{\underline{\iota} \in \llbracket 1, N \rrbracket^V} \prod_{v \in V} \Gamma_{\underline{\iota}(v)}^{(d(v))} \left(\sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{(v, v') \in E} \left(\frac{1}{n} \sum_{k=0}^{n-1} \omega^{(p(v')-p(v))k} \Pi_k \right)_{\underline{\iota}(v), \underline{\iota}(v')} \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{(v, v') \in E} \left(\frac{1}{n} \sum_{k=0}^{n-1} \omega^{(p(v')-p(v))k} \Pi_k \right)_{\underline{\iota}(v), \underline{\iota}(v')} \\ &= \frac{1}{n^{|E|}} \sum_{p \in \llbracket 0, n-1 \rrbracket^V} \sum_{\varphi \in \llbracket 0, n-1 \rrbracket^E} \left(\prod_{(v, v') \in E} \omega^{(p(v')-p(v))\varphi(v, v')} \left(\Pi_{\varphi(v, v')} \right)_{\underline{\iota}(v), \underline{\iota}(v')} \right) \\ &= \frac{1}{n^{|E|}} \sum_{\varphi \in \llbracket 0, n-1 \rrbracket^E} \left(\sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{(v, v') \in E} \omega^{(p(v')-p(v))\varphi(v, v')} \right) \left(\prod_{(v, v') \in E} \left(\Pi_{\varphi(v, v')} \right)_{\underline{\iota}(v), \underline{\iota}(v')} \right) \end{aligned}$$

where $\llbracket i, j \rrbracket^E$ denotes the set of maps from the edge set E to $\{i, \dots, j\}$. We can rewrite the summation in the first parenthesis of the last equation as

$$\sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{(v, v') \in E} \omega^{(p(v') - p(v))\varphi(v, v')} = \sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{v \in V} \omega^{p(v)(O(v) - I(v))} \quad (26)$$

where

$$O(v) := \sum_{\substack{e \in E \\ e \text{ out of } v}} \varphi(e) \text{ and } I(v) := \sum_{\substack{e \in E \\ e \text{ into } v}} \varphi(e).$$

Also, we have

$$\begin{aligned} \sum_{p \in \llbracket 0, n-1 \rrbracket^V} \prod_{v \in V} \omega^{p(v)(O(v) - I(v))} &= \prod_{v \in V} (1 + \omega^{O(v) - I(v)} + \dots + \omega^{(n-1)(O(v) - I(v))}) \\ &= \begin{cases} n^{|V|} & \text{if } O(v) \equiv I(v) \pmod{n} \text{ for all } v \in V \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Namely, Equation (26) reduces to $n^{|V|}$ if φ is a $\mathbb{Z}/n\mathbb{Z}$ -flow and vanishes, otherwise. Combining all the above, we obtain

$$\begin{aligned} W_{\mathcal{T}^{(n)}}(G) &= \frac{1}{\sigma(G) n^{|E| - |V|}} \sum_{\underline{\iota} \in \llbracket 1, N \rrbracket^V} \sum_{\varphi} \left(\prod_{v \in V} \Gamma_{\underline{\iota}(v)}^{(d(v))} \prod_{(v, v') \in E} (\Pi_{\varphi(v, v')})_{\underline{\iota}(v) \underline{\iota}(v')} \right) \\ &= \frac{1}{\sigma(G)} \sum_{\underline{\iota} \in \llbracket 1, N \rrbracket^V} W_{\mathcal{T}}^{(n)}(G; \underline{\iota}) = W_{\mathcal{T}}^{(n)}(G). \end{aligned}$$

Here we use the fact that the first betti number of a connected graph G is given by $|E(G)| - |V(G)| + 1$. This completes the proof of Theorem 3.7 \square

5. DETERMINING THE SHAPE OF THE LOOP INVARIANTS OF CYCLIC COVERS

In this section we give proofs of theorem stated in Introduction. Two main ingredients are Theorem 3.7, which expresses the ℓ -loop invariants $\Phi_{\mathcal{T}^{(n)}, \ell}$ of cyclic covers in terms of sums over $\mathbb{Z}/n\mathbb{Z}$ -tori of the inverse of a product of evaluations of $\delta_{\mathcal{T}}(t)$ at monomials, and Lemma 5.1 below, which converts a sum over a $\mathbb{Z}/n\mathbb{Z}$ -torus into a short rational function. It is inspired by, and closely related to, the problem of counting lattice points in rational convex polyhedra, and to the problem of expressing lattice point generating series in terms of short rational functions [Bar94, BW03].

To phrase it, we fix a vector $c = (c_1, \dots, c_s)$ of nonzero complex numbers other than the complex roots of unity and let $\mathbb{Q}[c, 1/S]$ denote the ring where S is the set of $1 - \prod_{i=1}^s c_i^{n_i}$ for all integers n_i satisfying $\prod_{i=1}^s c_i^{n_i} \neq 1$.

Lemma 5.1. Let T_0, \dots, T_s be Laurent monomials in variables t_1, \dots, t_d . If $T_i = t_i$ for $i = 1, \dots, d$ (hence $s \geq d$) and the exponents of T_{d+1}, \dots, T_s are in $\{0, \pm 1\}$, then there exists a polynomial

$$p(x_1, \dots, x_s, y) \in \mathbb{Q}[c, 1/S][x_1, \dots, x_s, y]$$

of x_i -degree at most 1 and y -degree at most $s - d$ such that

$$\sum_{t_1^n = \dots = t_d^n = 1} \frac{T_0}{(1 - c_1 T_1) \cdots (1 - c_s T_s)} = n^d p \left(\frac{1}{1 - c_1^n}, \dots, \frac{1}{1 - c_s^n}, n \right) \quad (27)$$

for all but finitely many n .

Proof. For the sake of exposition we first consider the case $T_0 = 1$. Since we take the sum over $t_1^n = \dots = t_d^n = 1$, the left-hand side of (27) equals to

$$\begin{aligned} & \frac{1}{\prod_i (1 - c_i^n)} \sum_{t_1^n = \dots = t_d^n = 1} \prod_{i=1}^s \frac{1 - c_i^n T_i^n}{1 - c_i T_i} \\ &= \frac{1}{\prod_i (1 - c_i^n)} \sum_{t_1^n = \dots = t_d^n = 1} \prod_{i=1}^s (1 + c_i T_i + \dots + (c_i T_i)^{n-1}) \\ &= \frac{1}{\prod_i (1 - c_i^n)} \sum_{k_1, \dots, k_s=0}^{n-1} \sum_{t_1^n = \dots = t_d^n = 1} c_1^{k_1} \dots c_s^{k_s} T_1^{k_1} \dots T_s^{k_s}. \end{aligned} \quad (28)$$

On the other hand, for any Laurent monomial T in t_1, \dots, t_d , we have

$$\sum_{t_1^n = \dots = t_d^n = 1} T = \begin{cases} n^d & \text{if the exponents of } T \text{ are multiple of } n \\ 0 & \text{otherwise} \end{cases}.$$

From the fact that $T_i = t_i$ for $i = 1, \dots, d$, there are exactly n^{s-d} monomials in (28) whose exponents are multiple of n . Indeed, if we choose $0 \leq k_{d+1}, \dots, k_s < n$ freely, there is unique $0 \leq k_1, \dots, k_d < n$ such that the exponents of $T_1^{k_1} \dots T_s^{k_s}$ are multiple of n . Thus we may write Equation (28) as

$$\frac{n^d}{\prod_i (1 - c_i^n)} \sum_{k_{d+1}, \dots, k_s=0}^{n-1} c_1^{k_1} \dots c_s^{k_s}, \quad (29)$$

regarding k_1, \dots, k_d as functions in k_{d+1}, \dots, k_s . Precisely, these functions are given as

$$k_i = [R_i(k_{d+1}, \dots, k_s)]_n, \quad i = 1, \dots, d$$

where $R_i(k_{d+1}, \dots, k_s)$ is a linear combination of k_{d+1}, \dots, k_s with coefficients in $\{0, \pm 1\}$ and $0 \leq [x]_n < n$ denotes the integer congruent to x in modulo n .

Let $P_n = \{(k_{d+1}, \dots, k_s) \in \mathbb{Z}^{s-d} \mid 0 \leq k_{d+1}, \dots, k_s \leq n - 1\}$ and partition P_n with respect to $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ by letting

$$P_{n,r} = \{(k_{d+1}, \dots, k_s) \in P \mid r_i n \leq R_i(k_{d+1}, \dots, k_s) \leq (r_i + 1)n - 1, i = 1, \dots, d\}.$$

Note that $P_{n,r}$ is empty for all but finitely many $r \in \mathbb{Z}^d$ and is an integral polytope, since the coefficients of R_1, \dots, R_d are in $\{0, \pm 1\}$. Also, we have

$$\begin{aligned} \sum_{(k_{d+1}, \dots, k_s) \in P_{n,r}} c_1^{k_1} \dots c_s^{k_s} &= c_1^{-nr_1} \dots c_d^{-nr_d} \sum_{(k_{d+1}, \dots, k_s) \in P_{n,r}} c_1^{R_1} \dots c_d^{R_d} c_{d+1}^{k_{d+1}} \dots c_s^{k_s} \\ &= c_1^{-nr_1} \dots c_d^{-nr_d} \sum_{(k_{d+1}, \dots, k_s) \in P_{n,r}} q_{d+1}^{k_{d+1}} \dots q_s^{k_s} \end{aligned} \quad (30)$$

for some $q_i \in \mathbb{C}$ given by a product of c_i and some of $c_1^{\pm 1}, \dots, c_d^{\pm 1}$. On the other hand, the formula of Brion (see e.g [BP99]) allows us to compute the generating function of $P_{n,r}$

$$F(P_{n,r}) = \sum_{(k_{d+1}, \dots, k_s) \in P_{n,r}} t_{d+1}^{k_{d+1}} \cdots t_s^{k_s} \quad (31)$$

in terms of the cones associated to the vertices of $P_{n,r}$. If we translate these cones so that their vertices become the origin, they depend only on the coefficients of R_1, \dots, R_d ; in particular, does not depend on n . We thus obtain from the formula of Brion that

$$F(P_{n,r}) = \sum_{v: \text{vertex of } P_{n,r}} f_v(t_{d+1}, \dots, t_s) g_v(t_{d+1}^n, \dots, t_s^n) \quad (32)$$

where f_v is a rational function (not depending on n) and g_v is a Laurent polynomial (coming from the vertex translation). The denominator of f_v is of the form $\prod_i (1 - t_{d+1}^{a_{i,d+1}} \cdots t_s^{a_{i,s}})$ for integers $a_{i,j}$ coming from the rays of the cones of $P_{n,r}$ at its vertices.

Substituting $t_{d+1} = q_{d+1}, \dots, t_s = q_s$ to Equations (31) and (32), we deduce that (30) is a polynomial in $c_1^{\pm n}, \dots, c_s^{\pm n}$ and n with coefficients in $\mathbb{Q}[c, 1/S]$ where the the exponent of n is at most dimension of $P_{n,r}$, hence at most $s - d$. This proves that (29) is also a polynomial in $c_1^{\pm n}, \dots, c_s^{\pm n}$ and n with coefficients in $\mathbb{Q}[c, 1/S]$, i.e.,

$$\frac{n^d}{\prod_i (1 - c_i^n)} \sum_{k_{d+1}, \dots, k_s=0}^{n-1} c_1^{k_1} \cdots c_s^{k_s} = \frac{n^d q(c_1^n, \dots, c_s^n, n)}{\prod_i (1 - c_i^n)} \quad (33)$$

for some polynomial $q(x_1, \dots, x_s, y) \in \mathbb{Q}[c, 1/S][x_1^{\pm 1}, \dots, x_s^{\pm 1}, y]$ with y -degree at most $s - d$. Comparing the degree of the above equation with respect to c_i^n , we deduce that q is a polynomial in x_i with x_i -degree at most 1. Then rewriting the right-hand side of (33) as a polynomial in $1/(1 - c_i^n)$, we obtain a desired polynomial p satisfying (27). This completes the proof when $T_0 = 1$. When $T_0 \neq 1$, the same argument holds for large n such that the exponents of T_0 are in between $-n$ and n . \square

The following example illustrates the above lemma.

Example 5.2. When $s = 1$, we have

$$\sum_{t^n=1} \frac{1}{1 - ct} = \frac{n}{1 - c^n}. \quad (34)$$

Using the above identity and the partial fraction decomposition

$$\frac{1}{(1 - at)(1 - bt)} = \frac{1}{1 - b/a} \frac{1}{1 - at} + \frac{1}{1 - a/b} \frac{1}{1 - bt} \quad a \neq b$$

of $1/((1 - at)(1 - bt))$ with respect to t , we obtain that

$$\sum_{t^n=1} \frac{1}{(1 - at)(1 - bt)} = \frac{1}{1 - b/a} \frac{1}{1 - a^n} + \frac{1}{1 - a/b} \frac{1}{1 - b^n} \quad a \neq b.$$

Taking the limit of the above equality when b tends to a , we obtain that

$$\sum_{t^n=1} \frac{1}{(1 - at)^2} = \frac{n^2}{(1 - a^n)^2} + \frac{n - n^2}{1 - a^n}. \quad (35)$$

Likewise, we have

$$\begin{aligned} \sum_{t^n=1} \frac{1}{(1-at)^3} &= \frac{n^3}{(1-a^n)^3} - \frac{3(n^3-n^2)}{2(1-a^n)^2} + \frac{n^3-3n^2+2n}{2(1-a^n)}, \\ \sum_{t^n=1} \frac{1}{(1-at)^4} &= \frac{n^4}{(1-a^n)^4} + \frac{2(-n^4+n^3)}{(1-a^n)^3} + \frac{7n^4-18n^3+11n^2}{6(1-a^n)^2} - \frac{n^4-6n^3+11n^2-6n}{6(1-a^n)}. \end{aligned} \quad (36)$$

5.1. **Proof of Theorem 1.2.** Recall Theorem 3.7 that the ℓ -loop invariant $\Phi_{\mathcal{T}^{(n),\ell}}$ is given as a finite sum over ℓ -loop Feynman diagrams. The weight of each Feynman diagram is a sum over $\mathbb{Z}/n\mathbb{Z}$ -flows of a product of entries $\Gamma_i^{(k)}$, which are \mathbb{Q} -linear combinations of shape parameters and their inverses, times entries of the propagator matrix $\Pi(\omega)$ where ω is a complex n -th root of unity.

We claim that the entries of $\Pi(t)$ are in $\delta_{\mathcal{T}}(t)^{-1}F[t^{\pm 1}]$. Clearly, entries of $\mathbf{B}(t)^{-1}\mathbf{A}(t)$ are in $\det \mathbf{B}(t)^{-1}\mathbb{Z}[t^{\pm 1}]$. Recall that $\det \mathbf{B}(t)$ has a factor $t-1$ and that $\mathbf{B}(t)^{-1}\mathbf{A}(t)$ equals to $\mathbf{B}_{\lambda}^{-1}\mathbf{A}_{\lambda}$ at $t=1$ (see Remark 3.3). Thus, in fact, entries of $\mathbf{B}(t)^{-1}\mathbf{A}(t)$ are in $\frac{t-1}{\det \mathbf{B}(t)}\mathbb{Z}[t^{\pm 1}]$. Combining this with the definition of the twisted 1-loop invariant $\delta_{\mathcal{T}}(t)$, namely,

$$(t-1)\delta_{\mathcal{T}}(t) = f \det(\mathbf{A}(t) - \mathbf{B}(t)\Delta_{\mathcal{Z}'}) = f \det \mathbf{B}(t) \det \Pi(t)^{-1}$$

for some $f \in F$, we deduce that entries of $\Pi(t)$ are in $\delta_{\mathcal{T}}(t)^{-1}F[t^{\pm 1}]$.

Fix an ℓ -loop Feynman diagram G . Recall that the set of all $\mathbb{Z}/n\mathbb{Z}$ -flows on G is an abelian group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^d$ where d is the first betti number of G . This can be proved by choosing a spanning tree of G , assigning an arbitrary element of $\mathbb{Z}/n\mathbb{Z}$ to each edge of G not in the tree, and then extending this assignment in a unique way to a flow on G . It follows that if we denote by t_1, \dots, t_d the values of a flow on the edges not in the tree, then the values of a flow on the edges of G are Laurent monomials in t_1, \dots, t_d with exponents 0, 1 or -1 . Then Theorem 3.7 together with the above claim shows that the contribution of G to the ℓ -loop invariant $\Phi_{\mathcal{T}^{(n),\ell}}$ is a linear combination of

$$\sum_{t_1^n=\dots=t_d^n=1} \frac{T_0}{\delta_{\mathcal{T}}(T_1) \cdots \delta_{\mathcal{T}}(T_s)} \quad (37)$$

where s is the number of edges of G and T_i are Laurent monomials in t_1, \dots, t_d satisfying the condition of Lemma 5.1. Note that $s \leq 3\ell - 3$, as ℓ -loop Feynman diagrams have at most $3\ell - 3$ edges.

Remark 5.3. Applying Lemma 5.1 directly to (37), we obtain the existence of a polynomial $p_{\mathcal{T},\ell} \in \mathcal{F}_{(3\ell-3)r}E[x_1, \dots, x_r][y]$ satisfying Equation (6). The non-resonance assumption is not required here, but the degree bound for $p_{\mathcal{T},\ell}$ may not be optimal.

From the non-resonance assumption, we have $\lambda_i^{\pm 1} \neq \lambda_j^{\pm 1}$ if $i \neq j$. Therefore, by using the partial fraction decomposition, we can write $1/\delta_{\mathcal{T}}(t)$ as an $E[t^{\pm 1}]$ -linear combination of $1/(1-\lambda_i^{\pm 1}t)$ for $i=1, \dots, r$ so that (37) is given as an $E[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ -linear combination of terms of the form

$$\sum_{t_1^n=\dots=t_d^n=1} \frac{1}{(1-\lambda_{i_1}^{\pm 1}T_1) \cdots (1-\lambda_{i_s}^{\pm 1}T_s)}. \quad (38)$$

Lemma 5.4. Let S be a subset of the edge set of G such that for each odd-degree vertex of G , not all adjacent edges are contained in S . Then $|S| \leq 2\ell - 2$ and $|S| - (d - 1) \leq \ell - 1$.

Proof. Let n_k be the number of k -vertices in G . Note that $\#(\text{edges}) = \frac{1}{2} \sum_{k \geq 1} k n_k$ and that $G \in \mathcal{G}_\ell$ implies $n_1 + n_2 + \#(\text{loops}) \leq \ell$. From the condition on S , we have

$$|S| \leq \#(\text{edges}) - \frac{1}{2} \sum_{k:\text{odd}} n_k = n_2 + n_3 + 2n_4 + 2n_5 + \dots \quad (39)$$

On the other hand, the connectedness of G implies that

$$\#(\text{loops}) - 1 = \#(\text{edges}) - \#(\text{vertices}) = \sum_{k \geq 1} \frac{(k-2)}{2} n_k$$

and thus

$$\#(\text{loops}) - 1 + \frac{1}{2} n_1 = \frac{1}{2} n_3 + n_4 + \frac{3}{2} n_5 + \dots \quad (40)$$

Comparing the coefficients of n_k for $k \geq 3$ in (39) and (40), we have

$$|S| \leq n_1 + n_2 + 2\#(\text{loops}) - 2 \leq 2\ell - 2$$

and

$$|S| - (d - 1) = |S| - \#(\text{loops}) + 1 \leq n_1 + n_2 + \#(\text{loops}) - 1 \leq \ell - 1. \quad \square$$

Lemma 5.4 implies that if $s > 2\ell - 2$, then there exists an odd-degree vertex in G . Let T_1, \dots, T_{2m+1} be the values of a flow on the adjacent edges of that vertex. From the definition of flow, the product of some of T_1, \dots, T_{2m+1} should be equal to the product of the others. Namely, we have (after re-labeling T_j)

$$T_1 \cdots T_k = T_{k+1} \cdots T_{2m+1}$$

for some $1 \leq k \leq 2m + 1$, hence

$$\prod_{j=1}^k (\lambda_{i_j}^{-1} (\lambda_{i_j} T_j - 1) + \lambda_{i_j}^{-1}) = \prod_{j=k+1}^{2m+1} (\lambda_{i_j}^{-1} (\lambda_{i_j} T_j - 1) + \lambda_{i_j}^{-1}).$$

Expanding the above equation, the constant term $\prod_{j=1}^k \lambda_{i_j}^{-1} - \prod_{j=k+1}^{2m+1} \lambda_{i_j}^{-1}$ which is non-zero due to the non-resonance assumption, is given by as a linear combination of $1 - \lambda_{i_1} T_1, \dots, 1 - \lambda_{i_{2m+1}} T_{2m+1}$ (and their products). It follows that we can write

$$\frac{1}{(1 - \lambda_{i_1} T_1) \cdots (1 - \lambda_{i_{2m+1}} T_{2m+1})}$$

as a linear combination of

$$\frac{1}{(1 - \lambda_{i_1} T_1) \cdots (\widehat{1 - \lambda_{i_j} T_j}) \cdots (1 - \lambda_{i_{2m+1}} T_{2m+1})}, \quad j = 1, \dots, 2m + 1$$

where the hat means that the j -th factor is excluded. This explains that we can decompose (38) into terms of the same form with $s \leq 2\ell - 2$. Then the existence of a polynomial

$$p_{\mathcal{T}, \ell}(x_1, \dots, x_r, y) \in \mathcal{F}_{2\ell-2} E[x_1, \dots, x_r][y] \quad (41)$$

satisfying Equation (6) follows from Lemma 5.1 where the last inequality of Lemma 5.4 shows that the y -degree of $p_{\mathcal{T},\ell}$ is at most $\ell - 1$ (and at least 1; see Equations (17) and (27)). \square

Remark 5.5. If one uses Neumann-Zagier datum with respect the meridian, then we need to consider the contribution (37) coming from all subgraphs of ℓ -loop Feynman diagrams. This does not affect on the existence of the polynomial $p_{\mathcal{T},\ell}$, but it may be a Laurent polynomial in the variable y .

Remark 5.6. The above proof gives an integrality statement for the coefficients of $p_{\mathcal{T},\ell}$, namely we can replace the field E in (41) by the ring $\frac{1}{d_\ell}O_{F,S}$ where O_F denotes the ring of integers of invariant trace field F , S denotes the set of all nonzero numbers of the form $\prod_i \lambda_i^{n_i} - 1$, as well as the denominators of the shapes z , z' and z'' and $O_{F,S} = O_F[1/S]$ denotes the localization of O_F with respect to S . Finally, d_ℓ is a universal denominator that comes from the greatest common divisor of the inverse automorphism factor of the ℓ -loop Feynman diagrams given explicitly in [GZb, Sec.9].

5.2. Proof of Theorem 1.3. As a quadratic $\delta_{\mathcal{T}}(t)$ automatically satisfies the non-resonance condition, we obtain from Theorem 1.2 a Laurent polynomial

$$p_{\mathcal{T},\ell}(x, y) \in \mathcal{F}_{2\ell-2}E[x][y]$$

such that for all but finitely many n , we have

$$\Phi_{\mathcal{T}^{(n)},\ell} = p_{\mathcal{T},\ell}\left(\frac{1}{1-\lambda^n}, n\right) \quad (42)$$

where λ is a root of $\delta_{\mathcal{T}}(t)$.

Lemma 5.7. For a given $k \geq 1$ there exist polynomials $\alpha_{k,i}(x) \in \mathbb{Q}[\lambda, \frac{1}{\lambda^2-1}][x]$ of degree at most k for $i = 0, \dots, k$ such that

$$\sum_{t^n=1} \frac{1}{\delta(t)^k} = \alpha_{k,0}(n) + \frac{\alpha_{k,1}(n)}{1-\lambda^n} + \dots + \frac{\alpha_{k,k}(n)}{(1-\lambda^n)^k} \quad (43)$$

for all $n \geq 1$. In addition, $\alpha_{k,k}(x) = 2\lambda^k(\lambda^2 - 1)^{-k}x^k$.

Proof. For $k = 1$ one easily checks that

$$\sum_{t^n=1} \frac{1}{\delta(t)} = -\frac{\lambda n}{\lambda^2 - 1} + \frac{2\lambda n}{(\lambda^2 - 1)(1 - \lambda^n)}.$$

Suppose that there are polynomials $\alpha_{k,0}(x), \dots, \alpha_{k,k}(x)$ with $\deg \alpha_{k,i} \leq k$ satisfying the equation (43). We then take the derivative both sides of (43) with respect to λ . From the left-hand side, we obtain

$$\begin{aligned} \frac{d}{d\lambda} \left(\sum_{t^n=1} \frac{1}{\delta(t)^k} \right) &= \sum_{t^n=1} \left(\frac{k t^k}{(t-\lambda)^{k+1}(t-\lambda^{-1})^k} - \frac{k t^k \lambda^{-2}}{(t-\lambda)^k(t-\lambda^{-1})^{k+1}} \right) \\ &= \frac{k}{1-\lambda^{-2}} \sum_{t^n=1} \frac{1}{\delta(t)^{k+1}}. \end{aligned}$$

From the right-hand side, we obtain

$$\begin{aligned} & \sum_{i=0}^k \frac{d}{d\lambda} \alpha_{k,i}(n) + \sum_{i=1}^k \frac{i n \lambda^{n-1} \alpha_{k,i}(n)}{(1-\lambda^n)^{i+1}} \\ &= \sum_{i=0}^k \frac{d}{d\lambda} \alpha_{k,i}(n) + \sum_{i=1}^k \left(\frac{i n \lambda^{-1} \alpha_{k,i}(n)}{(1-\lambda^n)^{i+1}} - \frac{i n \lambda^{-1} \alpha_{k,i}(n)}{(1-\lambda^n)^i} \right). \end{aligned}$$

Comparing the above two equations, we obtain polynomials $\alpha_{k+1,0}(x), \dots, \alpha_{k+1,k+1}(x)$ satisfying

$$\sum_{t^n=1} \frac{1}{\delta(t)^{k+1}} = \alpha_{k+1,0}(n) + \frac{\alpha_{k+1,1}(n)}{1-\lambda^n} + \dots + \frac{\alpha_{k+1,k+1}(n)}{(1-\lambda^n)^{k+1}}.$$

In particular, $\alpha_{k+1,k+1}(x) = x \alpha_{k,k}(x) / (\lambda - \lambda^{-1})$ and $\deg \alpha_{k+1,i} \leq k+1$. This completes the proof. \square

Lemma 5.8. For a given $k \geq 1$ there exist polynomials $\beta_{k,i}(x) \in \mathbb{Q}[\lambda^{\pm 1}, \frac{1}{\lambda^2-1}][x]$ of degree at most k for $i = 0, \dots, k$ satisfying

$$\frac{1}{(1-\lambda^n)^k} = \sum_{t^n=1} \left(\beta_{k,0}(n^{-1}) + \frac{\beta_{k,1}(n^{-1})}{\delta(t)} + \dots + \frac{\beta_{k,k}(n^{-1})}{\delta(t)^k} \right)$$

for all $n \geq 1$.

Proof. Writing Lemma 5.7 as a linear combination of $1, (1-\lambda^n)^{-1}, \dots, (1-\lambda^n)^{-k}$, we obtain a $(k+1) \times (k+1)$ matrix $\alpha(x)$ with entries in $\mathbb{Q}[\lambda, \frac{1}{\lambda^2-1}][x]$ such that

$$\alpha(n) \begin{pmatrix} 1 \\ (1-\lambda^n)^{-1} \\ \vdots \\ (1-\lambda^n)^{-k} \end{pmatrix} = \begin{pmatrix} 1 \\ \sum_{t^n=1} \delta(t)^{-1} \\ \vdots \\ \sum_{t^n=1} \delta(t)^{-k} \end{pmatrix}. \quad (44)$$

Moreover, $\alpha(x)$ is a lower-triangular matrix whose i -th row consists of polynomials of degree at most i and whose diagonal is $(2\lambda^k(\lambda^2-1)^{-k}x^k, \dots, 2\lambda(\lambda^2-1)^{-1}x, 1)$. It follows that entries of the inverse $\alpha(x)^{-1}$ are in $\mathbb{Q}[\lambda^{\pm 1}, \frac{1}{\lambda^2-1}][x^{-1}]$. We then obtain the lemma by multiplying $\alpha(n)^{-1}$ on both sides of Equation (44). \square

Applying Lemma 5.8 to Equation (42), we obtain the existence of Laurent polynomials $q_{\mathcal{T},\ell}(x, y) \in \mathcal{F}_{2\ell-2}E[x][y^{\pm 1}]$ satisfying for all but finitely many n

$$\Phi_{\mathcal{T}^{(n)},\ell} = \sum_{t^n=1} q_{\mathcal{T},\ell} \left(\frac{1}{\delta_{\mathcal{T}}(t)}, \frac{1}{n} \right). \quad (45)$$

From the fact that the asymptotic of $\Phi_{\mathcal{T}^{(n)},\ell}$ is a linear in n (see Theorem 1.1), we deduce that $q_{\mathcal{T},\ell}$ is a polynomial in y . It remains to show that the coefficients of $q_{\mathcal{T},\ell}$ lie in the field $F(\lambda + \lambda^{-1}) = F$. This follows from the fact that the left-hand side of (45) is invariant under $\lambda \mapsto \lambda^{-1}$, and hence so is the right-hand side. This completes the proof of Theorem 1.3.

Remark 5.9. Theorem 1.3 requires only that $\delta_{\mathcal{T}}(t)$ has no complex roots of unity as roots.

5.3. Proof of Theorem 1.1. In this section we give a proof of our main Theorem 1.1 by combining Theorem 3.7 and Theorem 1.2.

Theorem 3.7 expresses the ℓ -loop invariant $\Phi_{\mathcal{T}^{(n),\ell}}^c$ as a Riemann sum, which is asymptotic to a Riemann integral as n tends to infinity. This makes sense since the function to be summed or integrated is a sum of products of $1/\delta_{\mathcal{T}}(t)$ and the latter is nonzero on the unit circle by assumption. The corrections to the approximation of the Riemann sum by a Riemann integral are given by the Euler-Maclaurin summation formula and vanish to all orders in $1/n$, since the functions to be integrated are periodic. This proves the existence of $\Psi_{\mathcal{T},\ell}^c$ given by sums of multidimensional integrals over tori satisfying Equation (3) up to $O(1/n^\infty)$. Since $(\mathcal{T}^{(n)})^{(m)} = \mathcal{T}^{(nm)}$ for all integers m and $n \geq 1$, Equation (3) implies that $\Psi_{\mathcal{T},\ell}^c$ are multiplicative under cyclic covers, i.e., satisfy Equation (4).

It remains to improve the estimate $O(1/n^\infty)$ in Equation (3) to a sharp exponential estimate. For that, we use the explicit shape of the Riemann sums given in Theorem 1.2. We can choose roots $\lambda_1, \dots, \lambda_r$ of $\delta_{\mathcal{T}}(t)$ so that they are strictly inside the unit disk (since they come in pairs $\lambda, 1/\lambda$ and by assumption, none is on the unit circle). Then the difference between the n -Riemann sum and the Riemann integral is $O(p(n)|\lambda_j|^n)$ as n tends to infinity where λ_j is the smallest, in absolute value, root of $\delta_{\mathcal{T}}(t)$ and $p(n)$ is a polynomial of n of bounded degree.

We finally give a Feynman diagram definition of $\Psi_{\mathcal{T},\ell}^c$ using S^1 -flows. Our definition is analogous to the $\mathbb{Z}/n\mathbb{Z}$ flows used in Section 3.3 and in Theorem 3.7 to describe the connected ℓ -loop invariants of n -cyclic covers. An S^1 -flow on a Feynman diagram G is a map $\varphi : E(G) \rightarrow S^1$ such that for all $v \in V(G)$

$$\prod_{\substack{e \in E(G) \\ e \text{ into } v}} \varphi(e) = \prod_{\substack{e \in E(G) \\ e \text{ out of } v}} \varphi(e). \quad (46)$$

The set of S^1 -flows on G is isomorphic to (as a multiplicative set) the d -dimensional torus $(S^1)^d$ where d is the first betti number of G . In addition, the value T_e of an S^1 -flow on each edge of G is a Laurent monomial in d variables, say t_1, \dots, t_d . For a vertex-labeling $\iota : V(G) \rightarrow \{1, \dots, N\}$ we define

$$W_{\mathcal{T}}^{(\infty)}(G; \iota) := \prod_{v \in V(G)} \Gamma_{\iota(v)}^{(d(v))} \int_{(S^1)^d} \prod_{(v,v') \in E(G)} \Pi(T_{(v,v')} \iota(v), \iota(v')) \frac{dt_1 \cdots dt_d}{t_1 \cdots t_d} \quad (47)$$

and let

$$W_{\mathcal{T}}^{(\infty)}(G) := \frac{1}{\sigma(G)} \sum_{\iota} W_{\mathcal{T}}^{(\infty)}(G; \iota) \quad (48)$$

where the sum is over all vertex-labeling $\iota : V(G) \rightarrow \{1, \dots, N\}$. For instance, we have

$$W_{\mathcal{T}}^{(\infty)}(G) = \frac{1}{8} \sum_{1 \leq i, j \leq N} \Gamma_i^{(3)} \Gamma_j^{(3)} \int_{(S^1)^2} \Pi(t_1)_{ij} \Pi(t_2)_{ij} \Pi(t_1^{-1} t_2^{-1})_{ij} \frac{dt_1 dt_2}{t_1 t_2} \quad (49)$$

for a Feynman diagram given as in Figure 1 (cf. Equation (19)). The leading term $\Psi_{\mathcal{T},\ell}^c$ is given by

$$\Psi_{\mathcal{T},\ell}^c = \text{coeff} \left[\sum_{G \in \mathcal{G}_\ell} W_{\mathcal{T}}^{(\infty)}(G), \hbar^{\ell-1} \right] + \Gamma^{(0)}. \quad (50)$$

Note that contrast to the ℓ -loop invariant $\Phi_{\mathcal{T},\ell}^c$, a choice of a peripheral curve is not used in the above formula for $\Psi_{\mathcal{T},\ell}^c$. This completes the proof of Theorem 1.1. \square

Remark 5.10. Theorem 1.1 holds under the assumption that $\delta_{\mathcal{T}}(t)$ has no roots on the unit circle, otherwise the limit does not exist. Also, the exponentially small bound is optimal. Compare with the following toy example when λ is not a complex root of unity

$$\sum_{t^n=1} \frac{t}{(t-\lambda)(t-\lambda^{-1})} = \frac{n}{\lambda-\lambda^{-1}} \left(\frac{1}{1-\lambda^n} - \frac{1}{1-\lambda^{-n}} \right) = \begin{cases} n \frac{\lambda}{\lambda^2-1} + O(n\lambda^n) & \text{if } |\lambda| < 1 \\ n \frac{\lambda}{1-\lambda^2} + O(n\lambda^{-n}) & \text{if } |\lambda| > 1 \end{cases}$$

as n tends to infinity, whereas the limit does not exist if $|\lambda| = 1$.

5.4. Generalized power sums. In this section we review briefly the *generalized power sums* and their properties, following [vdP89, EvdPSW03]. The latter are sequences of the form

$$a_n = \sum_{j=1}^m A_j(n) \lambda_j^n \quad (51)$$

with roots λ_j for $1 \leq j \leq m$ distinct complex numbers and coefficients $A_j(n)$ polynomials of degree $d_j - 1$ for positive integers d_j for $1 \leq j \leq m$. The order of the generalized power sum is $d = \sum_{j=1}^m d_j$. Generalized power sums are solutions to linear recursions with constant coefficients, explicitly,

$$a_{n+d} = s_1 a_{n+d-1} + \cdots + s_d a_n, \quad n = 0, 1, 2, \dots \quad (52)$$

where $s(t) = \prod_{j=1}^m (1 - \lambda_j t)^{d_j} = 1 - s_1 t - \cdots - s_d t^d$, and their generating series

$$\sum_{n=0}^{\infty} a_n t^n = \frac{r(t)}{s(t)}$$

is a rational function of negative t -degree. Note that if the roots of a generalized power sum $a(n)$ are known, then (a_n) is determined by its first d values, as follows from recursion (52).

A special but important example of generalized power sums are the quasipolynomials, whose roots are complex roots of unity. Quasipolynomials play a key role in the lattice point counting in rational convex polyhedra [Ehr62, BV97, BP99, BR15].

Note that the vector space of generalized power sums is a ring with respect to pointwise multiplication of sequences.

We next recall the Lech–Mahler–Skolem theorem, whose statement is elementary and whose proof requires p -adic analysis.

Theorem 5.11. [Sko35, Mah35, Lec53] *The zero set $\{n \in \mathbb{N} \mid a_n = 0\}$ of a generalized power sum (a_n) is the union of a finite set and a finite set of arithmetic progressions.*

Moreover, if the roots of a generalized power sum are multiplicatively independent, then its zero set is either finite or all the natural numbers.

We now come to the proof of Proposition 1.4. By putting Equation (5) of Theorem 1.2 in a common denominator and abbreviating $p_{\mathcal{T},\ell}$ by p , it follows that

$$((1 - \lambda_1^n) \dots (1 - \lambda_r^n))^{2\ell-2} \Phi_{\mathcal{T}^{(n)},\ell} = \tilde{p}(\lambda_1^n, \dots, \lambda_r^n, n) \quad (53)$$

where $\tilde{p}(x_1, \dots, x_r, y) = (1 - x_1)^{2\ell-2} \dots (1 - x_r)^{2\ell-2} p(\frac{1}{1-x_1}, \dots, \frac{1}{1-x_r}, y) \in \mathcal{F}_{r(2\ell-2)} E[x][y]$. It is easy to see that the linear map $\mathcal{F}_{2\ell-2} E[x][y] \rightarrow \mathcal{F}_{r(2\ell-2)} E[x][y]$ that sends p to \tilde{p} is injective and that the sequence $(\tilde{p}(\lambda_1^n, \dots, \lambda_r^n, n))$ is a generalized power sum. Since the dimension of $\mathcal{F}_s E[x]$ is $\binom{r+s}{s}$ and the y -degree of $\tilde{p}(x, y)$ is at least 1 and at most $\ell - 1$, the non-resonance assumptions of Λ imply that the set of roots of this generalized power sum are monomials in $\lambda_1, \dots, \lambda_r$ (in total at most $\binom{r+r(2\ell-2)}{r}$) of y -degree at least 1 and at most $\ell - 1$. It follows that the degree of the generalized power sum (53) is at most $(\ell - 1) \binom{r+r(2\ell-2)}{r}$. Thus, if Λ is known, then the first $(\ell - 1) \binom{r+r(2\ell-2)}{r}$ many initial values of it determine it completely, and moreover determine \tilde{p} . Since the map $p \mapsto \tilde{p}$ is injective, the above discussion together with (53) imply that the polynomial $p_{\mathcal{T},\ell}$ is determined by Λ and by $(\ell - 1) \binom{r+r(2\ell-2)}{r}$ many initial values of $\Phi_{\mathcal{T}^{(n)},\ell}$.

It remains to reduce the number of initial values of $\Phi_{\mathcal{T}^{(n)},\ell}$ from $(\ell - 1) \binom{r+r(2\ell-2)}{r}$ to $d_{\ell,r} := (\ell - 1) \binom{r+2\ell-2}{r}$. This is possible, because \tilde{p} lies in a $d_{\ell,r}$ -dimensional subspace of $\mathcal{F}_{r(2\ell-2)} E[x][y]$, but the details are more delicate.

To prove this, we write $p(x, y)$ in terms of its monomials as $p(x, y) = \sum_{(\alpha,\beta) \in \mathcal{C}} c_{\alpha,\beta} x^\alpha y^\beta$ where $\alpha = (\alpha_1, \dots, \alpha_r)$, $\beta \in \mathbb{N}$, $x = (x_1, \dots, x_r)$ and $x^\alpha = x_1^{\alpha_1} \dots x_r^{\alpha_r}$. Then, $|\alpha| = \alpha_1 + \dots + \alpha_r \leq 2\ell - 2$, $\beta \leq \ell - 1$ and $|\mathcal{C}| = d_{\ell,r}$.

Consider Equation (6) for $n = n, n + 1, \dots, n + d_{\ell,r} - 1$ as a system of linear equations with unknowns $c_{\alpha,\beta}$. The corresponding square matrix $(\frac{1}{(1-\lambda_j^n)^\alpha} (n + j)^\beta)$ has rows indexed by $j = 0, \dots, d_{\ell,r} - 1$ and columns indexed by $(\alpha, \beta) \in \mathcal{C}$. After putting the matrix into a common denominator, its numerator is a generalized power sum, whose roots are monomials in λ_j . If this generalized power sum is not identically zero, the Lech-Mahler-Skolem theorem and the non-resonance assumption on λ_j implies that the zero set of the generalized power sum is finite, hence we can solve the system of linear equations using $d_{\ell,r}$ consecutive values of $\Phi_{\mathcal{T}^{(n)},\ell}$ to recover the coefficients of $p(x, y)$.

Thus, we need to prove that the generalized power sum is not identically zero. Assume otherwise. Using the multiplicative independence of Λ , it follows that that each polynomial $A_j(n)$ (in the notation of (51)) is identically zero and this implies that the determinant $\Delta_{\mathcal{C}}(y, \lambda, n) := \det(M_{\mathcal{C}}(y, \lambda, n))$ of the matrix $M_{\mathcal{C}}(y, \lambda, n) := ((\frac{1}{(1-y\lambda_j)^\alpha} (n + j)^\beta))$ vanishes identically for all $y = (y_1, \dots, y_r)$ and all n . Let us totally order \mathcal{C} by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $|\alpha| > |\alpha'|$ or $|\alpha| = |\alpha'|$ and $\beta \geq \beta'$. We extend this to a partial order for subsets \mathcal{C}' of \mathcal{C} by $\mathcal{C}' \geq \mathcal{C}''$ if the maximum element of \mathcal{C}' is greater than or equal to the maximum element of \mathcal{C}'' .

Let $\mathcal{S} = \{\mathcal{C}' \mid \Delta_{\mathcal{C}'}(y, \lambda, n) = 0 \text{ for all } y, n\}$. Note that \mathcal{S} is nonempty since it contains \mathcal{C} . Let $\mathcal{C}' \in \mathcal{S}$ denote an element of \mathcal{S} with $|\mathcal{C}'|$ minimum. Let (α', β') denote the maximum element of \mathcal{C}' . We distinguish two cases.

Case 1. If $\beta' > 0$, then after doing column operations on the matrix $M_{\mathcal{C}'}(y, \lambda, n)$ we can assume that the $j = 0$ row has vanishing entries for (α', β) except at $\beta = 0$. Then, the matrix obtained from $M_{\mathcal{C}'}(y, \lambda, n)$ by removing the $(\alpha', 0)$ row and column is $M_{\mathcal{C}' - \{\alpha', \beta'\}}(\lambda y, \lambda, n + 1)$.

Expanding the determinant $\Delta_{\mathcal{C}'}(y, \lambda, n)$ with respect to the 0-th row, it follows that

$$0 = \Delta_{\mathcal{C}'}(y, \lambda, n) = \frac{1}{(1-y)^{\alpha'}} \Delta_{\mathcal{C}' - \{\alpha', \beta'\}}(\lambda y, \lambda, n + 1) + (\text{other terms}). \quad (54)$$

Both sides of the above equation are rational functions of y , and each term of the determinants and of the other terms are products of $\frac{1}{(1-y\lambda^j)^{\alpha''}} = \prod_{i=1}^r \frac{1}{(1-y_i\lambda_i)^{\alpha''_i}}$. On the other hand, the other terms do not have a singularity $(1-y)^{\alpha'}$. It follows that $\Delta_{\mathcal{C}' - \{\alpha', \beta'\}}(\lambda y, \lambda, n + 1) = 0$. Thus, $\mathcal{C}' - \{\alpha', \beta'\} \in \mathcal{S}$ but $|\mathcal{C}' - \{\alpha', \beta'\}| < |\mathcal{C}'|$ a contradiction.

Case 2. If $\beta' = 0$, then expanding the determinant $\Delta_{\mathcal{C}'}(y, \lambda, n)$ with respect to the 0-th row. Equation (54) still holds and the same reasoning as in the first case implies that $\Delta_{\mathcal{C}' - \{\alpha', \beta'\}}(\lambda y, \lambda, n + 1) = 0$ giving a contradiction once again.

This concludes the proof of the second part of the proposition.

For the first part, observe that the n -th cyclic resultant $N_n(\delta_{\mathcal{T}})$ of the twisted 1-loop invariant $\delta_{\mathcal{T}}(t)$ is given by

$$N_n(\delta_{\mathcal{T}}) = \prod_{\omega^n=1} \delta_{\mathcal{T}}(\omega) = \prod_{j=1}^r (1 - \lambda_j^n)(1 - \lambda_j^{-n}),$$

which equals to $(1 - \lambda_1^n)^2 \dots (1 - \lambda_r^n)^2$ times the n -th power of a signed monomial. This and Equation (53) imply that the sequence $(N_n(\delta_{\mathcal{T}})^{\ell-1} \Phi_{\mathcal{T}^{(n)}, \ell})$ is a generalized power sum, hence its generating series (9) is rational. This concludes the proof of Proposition 1.4. \square

5.5. Asymptotics. The rest of the section is devoted to the proof of Proposition 1.5. Fix a rational function $R(x, y) \in \mathbb{C}(x)[y]$ regular at $x = 0$, where $x = (x_1, \dots, x_r)$. Then, we can consider the image of $R(x, y)$ in the completed power series ring $\mathbb{C}[[x]][y]$

$$R(x, y) = \sum_{k \in \mathbb{N}^r} a_k(y) x^k \quad (55)$$

where $k = (k_1, \dots, k_r) \in \mathbb{N}^r$ and $x^k = x_1^{k_1} \dots x_r^{k_r}$, and where $a_k(y)$ are polynomials in y of degree at most d . Fix a set $\Lambda_+ = \{\lambda_j \mid j = 1, \dots, r\}$ of multiplicatively independent nonzero complex numbers with $|\lambda_j| < 1$ for all $j = 1, \dots, r$. Consider the set $\mathcal{L} = \{\lambda^k \mid k \in \mathbb{N}^r\}$ where $\lambda^k = \lambda_1^{k_1} \dots \lambda_r^{k_r}$, and let \mathcal{E} denote the set of the absolute values of the elements of \mathcal{L} . Since $0 < |\lambda_j| < 1$ for all j , it follows that \mathcal{E} is a discrete subset of $(0, 1)$ with 0 its only limit point. Hence, we can write $\mathcal{E} = \{L_m \mid m \in \mathbb{N}\}$ where $1 > L_0 > L_1 > L_2 > \dots$. Using this, we can partition $\mathcal{E} = \sqcup_{m \in \mathbb{N}} \mathcal{E}_m$, where \mathcal{E}_m is the set of λ^k with $|\lambda^k| = L_m$. The assumptions on $\{\lambda_j\}$ imply that each \mathcal{E}_m is a finite set and that the map $\{x^k \mid k \in \mathbb{N}\} \rightarrow \mathcal{L}$ that sends x^k to λ^k is 1-1. Thus, Equation (55) can be written in the form

$$R(x, y) = \sum_{m=0}^{\infty} R_m(x, y), \quad R_m(x, y) = \sum_{k \in \mathcal{E}_m} a_k(y) x^k \quad (56)$$

where $R_m(x, y)$ is a finite sum. Letting $a_{m,n} = R_m(\lambda^n, n)$, it follows that for each fixed m , the sequence $n \mapsto a_{m,n}$ is a generalized power sum that satisfies $a_{m,n} = O(n^d L_m^n)$, that the

series

$$a_n = \sum_{m=0}^{\infty} a_{m,n} \quad (57)$$

is absolutely convergent and that the partial sums satisfy

$$a_n - \sum_{m=0}^{M-1} a_{m,n} = a_{M,n} + O(n^d L_{M+1}^n). \quad (58)$$

It follows by induction on the natural number M that the left hand side of (58) is a generalized power sum which is a sequence of Nilsson type [Gar11], whose asymptotic expansion to all orders in $1/n$ is terminating and given exactly by $a_{M,n}$. This implies by induction that a_n (or infinitely many evaluations of it), determine $a_{m,n} = R_m(\lambda^n, n)$ for all m . This implies in turn that a_n determines the polynomial $R_m(x, y)$ and the set \mathcal{E}_m for all m , and hence determines the image of $R(x, y)$ in $\mathbb{C}[[x]][y]$ by (55) as well as the set \mathcal{E} (and hence Λ_+). Since the map $\mathbb{C}(x)[y] \rightarrow \mathbb{C}[[x]][y]$ (partially defined on rational functions which are regular at $x = 0$) is injective, this completes the first part of Proposition 1.5.

For the second part, assume that $\delta_{\mathcal{T}}(t)$ is non-resonant with no roots on the unit circle. Without loss of generality, we can choose λ_j for $j = 1, \dots, r$ with $|\lambda_j| < 1$ for all j such that the roots of $\delta_{\mathcal{T}}(t)$ are $\{\lambda_1^{\pm 1}, \dots, \lambda_r^{\pm 1}\}$. Note that $p_{\mathcal{T}, \ell}$ lies in $E[x][y]$ which is a subspace of $E(x)_{\text{loc}}[y]$ (where $E(x)_{\text{loc}}$ denotes the ring of rational functions on x which are regular at $x = 0$). This fact, together with part (a) of the lemma, concludes the proof of the proposition. \square

6. EXAMPLES

The twisted loop invariants, defined as formal Gaussian integrals, are explicitly computed algebraically in terms of the NZ-datum of an ideal triangulation. Likewise, the twisted loop invariants can be computed algebraically using the twisted version of the NZ-datum of [GY]. In other words, Theorem 3.7 leads to an effective computation of the twisted loop invariants.

In this section we illustrate our theorems by using an exact computation of the ℓ -loop invariants of n -cyclic covers for $\ell = 2, 3$ and various values of n depending on the complexity of the knot, i.e., on the number of tetrahedra, the degree of its invariant trace field and the degree of its twisted 1-loop invariant $\delta(t)$.

Recall that we can reconstruct the polynomial $p_{\mathcal{T}, \ell}(x, y)$ from \mathcal{L} and $(\ell - 1) \binom{r+2\ell-2}{r}$ values of the ℓ -loop invariants of the n -cyclic covers. Note the relation $r \leq 3g - 2$ (see [DFJ12, Thm.1.5]) between the degree of the adjoint torsion polynomial and the genus g of the knot, namely the minimum genus of all Seifert surfaces of the knot. In all of our examples, equality $r = 3g - 2$ is attained.

The next table summarizes the number of values of n -cyclic covers needed to determine the loop invariants for all n , when Λ is known.

6.1. Genus 1 examples: 4_1 and 5_2 knots. In this section we give two examples of genus 1 knots, namely the two simplest hyperbolic knots, the 4_1 and the 5_2 knot. For the knot 4_1 , we computed the 2 and 3-loop invariants of n -cyclic covers for $n = 1, \dots, 100$. Note that these invariants are elements of the invariant trace field $\mathbb{Q}(\sqrt{-3})$, but due to the chirality of the

g	r	#2-loop values	#3-loop values
1	1	3	10
2	4	15	140
g	$3g - 2$	$\frac{3}{2}g(3g - 1)$	$\frac{1}{4}(3g + 2)(3g + 1)g(3g - 1)$

TABLE 1. The number of values needed to determine the 2 and 3-loop invariants of cyclic covers when Λ is known.

knot, they are essentially elements of \mathbb{Q} . We computed those rational numbers numerically, and after multiplying them by the expected denominators (coming from the n -cyclic resultant of a small power of the twisted 1-loop invariant), we lifted the nearly rational numbers to exact rational numbers, and checked that they agree within the precision of the computation (about 1000 digits). For the knot 5_2 , the 2 and 3-loop invariants of the n -cyclic cover is an element of the cubic invariant trace field of discriminant -23 . In this case, we numerically computed the invariants using Theorem 3.7 for each embedding of the shapes in the complex numbers, and then took the product thus numerically computing numerically the coefficients of the minimal polynomial of the invariants. The latter has rational coefficients, which as before can be numerically computed and then lifted to exact rational numbers. Having done do, we converted the geometric root of the minimal polynomial back to the invariant trace field, thus getting exact value of the 2 and 3-loop invariants for n -cyclic covers of 5_2 for $n = 1, \dots, 60$. Using this data, we then interpolated numerically to find the formulas presented below. Once the formulas were found, an exact computation can verify them for the computed values of n .

The answer found agrees with the algebraic computation of Theorem 1.3 and gives evidence to the conjecture that Theorem 1.2 works for all natural numbers n , as opposed to all but finitely many n .

Although we computed the 2 and 3-loop invariants for cyclic covers for NZ data that uses both the meridian and the longitude, we will present our results using the longitude only.

We now present our computations for the 4_1 knot, whose invariant trace field is $F = \mathbb{Q}(\sqrt{-3})$ is a subfield of \mathbb{C} . The twisted 1-loop invariant is $\delta(t) = t - 5 + t^{-1}$, and has coefficients in the real part of F , namely \mathbb{Q} (this is an accident because 4_1 is amphichiral). Let us denote the sum of the evaluation of a function at complex roots of unity by $\text{Av}_n(f(t)) = \sum_{\omega^n=1} f(\omega)$. We have $\Phi_{\mathcal{T}^{(n),\ell}}^c = \text{Av}_n(\varphi_\ell^c(t, n))$ where $\delta = \delta(t)$ and

$$\begin{aligned} \varphi_2^c(t, n) &= \left(\frac{4}{3n\delta^2} + \frac{20}{63n\delta} + \frac{55}{1512} \right) \sqrt{-3} \\ \varphi_3^c(t, n) &= -\frac{80}{3n^2\delta^4} - \frac{1976}{315n^2\delta^3} + \left(-\frac{8}{189} + \frac{916}{1323n^2} \right) \frac{1}{\delta^2} + \left(\frac{473}{26460} + \frac{2036}{19845n^2} \right) \frac{1}{\delta}. \end{aligned} \quad (59)$$

This illustrates Theorem 1.3. The primes 67 and 103 that appear in the denominators of the above expressions come from the n -cyclic resultant of $\delta(t)$ when one determines the coefficients of $n^j \delta^j$ from few initial values of n .

After doing a partial fraction decomposition of the rational functions that appear in (59) and using Equations (34)-(36) of Example 5.2, we obtain explicit formulas for the invariants

in terms of generalized power sums illustrating Theorem 1.2

$$\begin{aligned}\Phi_{\mathcal{T}^{(n)},2}^c &= -\frac{n(55\lambda^n + 82 + 55\lambda^{-n})\sqrt{-3}}{1512(1-\lambda^n)(1-\lambda^{-n})} \\ \Phi_{\mathcal{T}^{(n)},3}^c &= \frac{-n^2(\frac{32}{1323}\lambda^n + \frac{32}{441} + \frac{32}{1323}\lambda^{-n}) + n\sqrt{21}(-\frac{317}{238140}\lambda^{2n} - \frac{1985}{166698}\lambda^n + \frac{1985}{166698}\lambda^{-n} + \frac{317}{238140}\lambda^{-2n})}{(1-\lambda^n)^2(1-\lambda^{-n})^2}\end{aligned}\quad (60)$$

where $\lambda = \frac{1}{2}(5 + \sqrt{21}) \approx 4.7912$ is one of the two roots of $\delta(t)$, the other one being $\lambda^{-1} = \frac{1}{2}(5 - \sqrt{21}) \approx 0.2087$.

It follows from either (59) or (60) that the leading asymptotics of $\Phi_{\mathcal{T}^{(n)},\ell}^c$ for $\ell = 2, 3$ are given by

$$\begin{aligned}\Phi_{\mathcal{T}^{(n)},2}^c &= \frac{55\sqrt{-3}}{1512}n + O(n|\lambda|^{-n}) \\ \Phi_{\mathcal{T}^{(n)},3}^c &= -\frac{317\sqrt{21}}{238140}n + O(n^2|\lambda|^{-n})\end{aligned}\quad (61)$$

illustrating Theorem 1.1. Equations (60) imply that the generating series of Proposition 1.4 are given by

$$\sum_{n=0}^{\infty} (1-\lambda^n)(1-\lambda^{-n})\Phi_{\mathcal{T}^{(n)},2}^c t^n = -\sqrt{-3} \frac{t(119 - 530t + 1068t^2 - 530t^3 + 119t^4)}{504(-1+t)^2(1-5t+t^2)^2} \quad (62)$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} ((1-\lambda^n)(1-\lambda^{-n}))^2 \Phi_{\mathcal{T}^{(n)},3}^c t^n &= \frac{t(1+t)}{588(-1+t)^3(1-23t+t^2)^2(1-5t+t^2)^3} (343 - 15565t + 249432t^2 - 1448727t^3 \\ &\quad + 4346901t^4 - 6772800t^5 + 4346901t^6 - 1448727t^7 + 249432t^8 - 15565t^9 + 343t^{10}).\end{aligned}$$

The factor $1 - 23t + t^2$ which appears in the above denominator equals to $(1 - \lambda^2 t)(1 - \lambda^{-2} t)$. This concludes the discussion of the 4_1 knot.

The next hyperbolic knot is the 5_2 knot, whose invariant trace field is the cubic field of discriminant -23 generated by the root $\xi \approx -0.662 - 0.562i$ of the equation $\xi^3 - \xi - 1 = 0$. The twisted 1-loop invariant is

$$\delta(t) = (2 + 4\xi + 2\xi^2)t - 5 - 2\xi + 3\xi^2 + (2 + 4\xi + 2\xi^2)t^{-1}. \quad (63)$$

We have $\Phi_{\mathcal{T}(n),\ell}^c = \text{Av}_n(\varphi_\ell^c(t, n))$ where $\delta = \delta(t)$ and

$$\begin{aligned} \varphi_2^c(t, n) &= -\frac{4(-16228 - 3232\xi + 8679\xi^2)}{7705n} \frac{1}{\delta^2} \\ &\quad + \left(\frac{39 - 56\xi - 24\xi^2}{46} - \frac{4(-26127539 + 15044839\xi + 3721992\xi^2)}{94440185n} \right) \frac{1}{\delta} \\ &\quad + \frac{123094133 - 446744448\xi + 259344006\xi^2}{2266564440} \\ \varphi_3^c(t, n) &= \left(\frac{144171776}{516235} + \frac{86345584\xi}{516235} - \frac{136288528\xi^2}{516235} \right) \frac{1}{n^2\delta^4} \\ &\quad \left(\left(\frac{2021650619247678416}{12919632159420875} - \frac{194429261261137656\xi}{12919632159420875} - \frac{1412467704798780848\xi^2}{12919632159420875} \right) \frac{1}{n} \right. \\ &\quad + \left. \frac{362208}{7705} - \frac{57728\xi}{7705} - \frac{243704\xi^2}{7705} \right) \frac{1}{n\delta^3} \\ &\quad \left(\frac{19464170555699}{8731939505100} - \frac{1654507907596\xi}{2182984876275} + \frac{661858625444\xi^2}{727661625425} \right. \\ &\quad + \left. \left(\frac{43685924340213}{2910646501700} - \frac{13472722690929\xi}{1455323250850} + \frac{2008555368111\xi^2}{2910646501700} \right) \frac{1}{n} \right. \\ &\quad + \left. \left(\frac{161627626755245606632}{8963543285548396125} - \frac{119998551075098128112\xi}{8963543285548396125} + \frac{127286031414479468\xi^2}{2987847761849465375} \right) \frac{1}{n^2} \right) \frac{1}{\delta^2} \\ &\quad \left(\left(\frac{-19378724062777204444}{475067794134064994625} - \frac{474092286600322084396\xi}{475067794134064994625} + \frac{128349517059147927744\xi^2}{158355931378021664875} \right) \frac{1}{n^2} \right. \\ &\quad + \left. \left(\frac{1746056639554239}{35675794171336900} - \frac{31637787802490587\xi}{17837897085668450} + \frac{11693862723463677\xi^2}{8918948542834225} \right) \frac{1}{n} \right. \\ &\quad + \left. \left(\frac{59134987864619182444}{475067794134064994625} - \frac{1527155471544628788041\xi}{1900271176536259978500} + \frac{3826051934183205772\xi^2}{6885040494696594125} \right) \right) \frac{1}{\delta} \\ &\quad \left(-\frac{428855832942393}{8918948542834225} - \frac{2998162280908073\xi}{35675794171336900} + \frac{1615737458359533\xi^2}{17837897085668450} \right). \end{aligned}$$

The leading asymptotic values are given by $\Phi_{\mathcal{T}(n),\ell}^c = n\Psi_{\mathcal{T},\ell}^c + O(n^{\ell-1}|\lambda|^n)$ for $\ell = 2, 3$ where $\lambda \approx 0.0502 - 0.1704i$ is a root of $\delta(t)$ and satisfies the equation $8\lambda^6 - 28\lambda^5 + 270\lambda^4 - 109\lambda^3 + 270\lambda^2 - 28\lambda + 8 = 0$ and

$$\begin{aligned} \Psi_{\mathcal{T}(n),2}^c &= \frac{1}{1160480993280} (-16601383280 + 239466164328\lambda - 30998500743\lambda^2 + 51073175277\lambda^3 \\ &\quad - 2600093877\lambda^4 + 384393303\lambda^5) \\ \Psi_{\mathcal{T}(n),3}^c &= \frac{1}{26122110666289424422956544000} (3763333983996990578027312 - 27832672813601695938777064\lambda \\ &\quad + 98732772027957178344155\lambda^2 + 1194221340324541487037559\lambda^3 - 453984084634619809746255\lambda^4 \\ &\quad + 29998843726647510986933\lambda^5). \end{aligned} \tag{64}$$

6.2. Higher genus examples: the 6_2 and the $(-2, 3, 7)$ pretzel knots. The above two examples illustrate our main theorems when $\delta(t)$ is quadratic and the Seifert genus g is 1. We now present two further examples of higher genus illustrating Theorem 1.2 and Equation (7). The coefficients c_{ij} , c_i and c_0 in Equation (7) can be determined from $(r+1)(r+2)/2$ values of $\Phi_{\mathcal{T}(n),2}$ and lie in the splitting field E of $\delta(t)$.

The first example is the 6_2 knot where $g = 2$ and $r = 4$. Its default SnapPy triangulation has 5 tetrahedra and its invariant trace field has degree 5. We computed 120 exact values of the 2-loop invariant of the cyclic covers. To do so, we numerically computed these values for all 5 embeddings of the shapes to the complex numbers, and from that we computed

the minimal polynomial (whose coefficients are integers computed approximately to high precision and then recognized). Once we knew the minimal polynomial, we converted its chosen root to the fixed embedding of the invariant trace field to the complex numbers. Having done so, we used 15 values of $n = 1, \dots, 15$ to numerically compute the above coefficients to 1000 digits, and then used the remaining 105 values of n to check our numerical answer, which agreed to all 1000 digits of precision. Note that the coefficients c_{ij} , c_i and c_0 that appear in (7) are elements of the splitting field of $\delta(t)$, an explicit number field of degree 120. Although we can numerically compute the coefficients to arbitrary high precision, eg. 10000 digits, it is not likely that we can express them explicitly by elements of the splitting field.

Our second example is the $(-2, 3, 7)$ pretzel knot where $g = 5$ and $r = 13$. Its default SnapPy triangulation has 3 ideal tetrahedra and its invariant trace field is cubic (and equal to that of the 5_2 knot). Working as above, we were able to compute the first 140 values of the 2-loop invariant of its n -cyclic cover. We used 105 values to determine the coefficients of (7) to the precision of 1000 digits, and then 35 further values to check our prediction. Once again, Equation (7) worked to all the accuracy of 1000 digits. In this case, the complexity of the splitting field is prohibitive, and it is unlikely that one will be able to compute the exact values of the 105 constants in Equation (7).

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INTERNATIONAL CENTER FOR MATHEMATICS, DEPARTMENT OF MATHEMATICS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHENZHEN, CHINA

<http://people.mpim-bonn.mpg.de/stavros>

Email address: stavros@mpim-bonn.mpg.de

INTERNATIONAL CENTER FOR MATHEMATICS, DEPARTMENT OF MATHEMATICS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHENZHEN, CHINA

<https://sites.google.com/view/seokbeom>

Email address: sbyoon15@gmail.com