THE COMPLEX VOLUME OF $\mathrm{SL}(n,\mathbb{C})$-REPRESENTATIONS OF 3-MANIFOLDS

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ABSTRACT. For a compact 3-manifold M with arbitrary (possibly empty) boundary, we give a parametrization of the set of conjugacy classes of boundary-unipotent representations of $\pi_1(M)$ into $\mathrm{SL}(n,\mathbb{C})$. Our parametrization uses Ptolemy coordinates, which are inspired by coordinates on higher Teichmüller spaces due to Fock and Goncharov. We show that a boundary-unipotent representation determines an element in Neumann's extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and use this to obtain an efficient formula for the Cheeger-Chern-Simons invariant, and in particular for the volume. Computations for the census manifolds show that boundary-unipotent representations are abundant, and numerical comparisons with census volumes, suggest that the volume of a representation is an integral linear combination of volumes of hyperbolic 3-manifolds. This is in agreement with a conjecture of Walter Neumann, stating that the Bloch group is generated by hyperbolic manifolds.

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1. Introduction

For a closed 3-manifold M, the Cheeger-Chern-Simons invariant [6, 7] of a representation ρ of $\pi_1(M)$ in $\mathrm{SL}(n,\mathbb{C})$ is given by the Chern-Simons integral

$$\widehat{c}(\rho)=\frac{1}{2}\int_M s^*\big(\mathrm{Tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)\big)\in\mathbb{C}/4\pi^2\mathbb{Z}, \tag{1.1}$$

where A is the flat connection in the flat $\mathrm{SL}(n,\mathbb{C})$-bundle E_ρ with holonomy ρ, and $s\colon M\to E_\rho$ is a section of E_ρ. Since $\mathrm{SL}(n,\mathbb{C})$ is 2-connected a section always exists, and a different choice of section changes the value of the integral by a multiple of $4\pi^2$.

When $n=2$, the imaginary part of the Cheeger-Chern-Simons invariant equals the hyperbolic volume of ρ. More precisely, if $D\colon\widetilde{M}\to\mathbb{H}^3$ is a developing map for ρ and $\nu_{\mathbb{H}^3}$ is the hyperbolic volume form, $\mathrm{Im}(\widehat{c}(\rho))$ equals the integral of $D^*(\nu_\rho)$ over a fundamental domain for M. In particular, if $M=\mathbb{H}^3/\Gamma$ is a hyperbolic manifold, and ρ is a lift to $\mathrm{SL}(2,\mathbb{C})$ of the geometric representation $\rho_{\mathrm{geo}}\colon\pi_1(M)\to\mathrm{PSL}(2,\mathbb{C})$, the imaginary part equals the volume of M. In fact, in this case we have

$$\widehat{c}(\rho)=i(\mathrm{Vol}(M)+i\mathrm{CS}(M)), \tag{1.2}$$

where $\mathrm{CS}(M)$ is the Chern-Simons invariant of M (with the Riemannian connection). Although this result is known to experts, no proof seems to be available (see [8, 21] for discussions). We give a proof in Section 2. The invariant $\mathrm{Vol}(M)+i\mathrm{CS}(M)$ is often referred to as *complex volume*. Motivated by this, we define the complex volume $\mathrm{Vol}_{\mathbb{C}}$ of a representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})$ by

$$\widehat{c}(\rho)=i\,\mathrm{Vol}_{\mathbb{C}}(\rho) \tag{1.3}$$

and define the *volume* of ρ to be the real part of the complex volume, i.e. the imaginary part of the Cheeger-Chern-Simons invariant. Surprisingly, as we shall see, the relationship to hyperbolic volume seems to persist even when $n > 2$.

The set of $\mathrm{SL}(n,\mathbb{C})$-representations is a complex variety with finitely many components, and the complex volume is constant on components. This follows from the fact that representations in the same component have cohomologous Chern-Simons forms. Hence, for any M, the set of complex volumes is a finite set.

We show that the definition of the Cheeger-Chern-Simons invariant naturally extends to compact manifolds with boundary, and representations $\rho\colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$ that are *boundary-unipotent*, i.e. take peripheral subgroups to a conjugate of the unipotent group N of upper triangular matrices with 1's on the diagonal. We formulate all our results in this more general setup.

The main result of the paper is a concrete algorithm for computing the set of complex volumes. The idea is that the set of (conjugacy classes of) boundary-unipotent representations can be parametrized by a variety, called the *Ptolemy variety*, which is defined by homogeneous polynomials of degree 2. The Ptolemy variety depends on a choice of triangulation, but if the triangulation is sufficiently fine, every representation is detected by the Ptolemy variety. We show that a point c in the Ptolemy variety naturally determines an element $\lambda(c)$ in Neumann's extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, such that if ρ is the representation corresponding to c, we have

$$R(\lambda(c)) = i\,\mathrm{Vol}_{\mathbb{C}}(\rho), \tag{1.4}$$

where $R\colon \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/4\pi^2\mathbb{Z}$ is a Rogers dilogarithm.

There is a canonical group homomorphism

$$\phi_n\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C}) \tag{1.5}$$

defined by taking a matrix A to its $(n-1)$th symmetric power (see Section 11). The map ϕ_n preserves unipotent elements, and we show that composing a boundary-unipotent representation in $\mathrm{SL}(2,\mathbb{C})$ with ϕ_n multiplies the complex volume by $\binom{n+1}{3}$. If $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, the geometric representation ρ_{geo} always lifts to a representation in $\mathrm{SL}(2,\mathbb{C})$, but if M has cusps, lifts are not necessarily boundary-unipotent. In fact, by a result of Calegari [5], if M has a single cusp, any lift of the geometric representation takes a longitude to an element with trace -2. When n is even, we shall thus, more generally, be interested in boundary-unipotent representations in

$$p\,\mathrm{SL}(n,\mathbb{C}) = \mathrm{SL}(n,\mathbb{C})\big/\langle \pm I\rangle. \tag{1.6}$$

Such representations have a complex volume defined modulo $\pi^2 i$, and our algorithm computes these as well. By studying representations in $p\,\mathrm{SL}(n,\mathbb{C})$, we make sure that when M is hyperbolic, there is always at least one representation with non-trivial complex volume, namely $\phi_n \circ \rho_{\mathrm{geo}}$.

Walter Neumann has conjectured that every element in the Bloch group $\mathcal{B}(\mathbb{C})$ is an integral linear combination of Bloch group elements of hyperbolic 3-manifolds. Since the extended Bloch group equals the Bloch group up to torsion, Neumann's conjecture would imply that all complex volumes are, up to rational multiples of $i\pi^2$, integral linear combinations of complex volumes of hyperbolic 3-manifolds. In particular, the volumes should all be integral linear combinations of volumes of hyperbolic manifolds.

Our algorithm has been implemented by Matthias Goerner. The algorithm uses Magma [3] to compute a primary decomposition of the Ptolemy variety, and then uses (1.4) to compute the complex volumes. For $n = 2$, we have computed primary decompositions of the Ptolemy varieties for all census manifolds with ≤ 8 simplices (these usually finish within a fraction of a second) and all link complements with ≤ 16 simplices in the SnapPy census [9] of knots with up to 11 crossings and links with up to 10 crossings. When there are more than 16 simplices some of the computations don't terminate. For $n = 3$, computations are feasible for many manifolds with up to 4 simplices, but for $n = 4$ the computations run out of memory for all manifolds with more than 2 simplices. It would be interesting to perform numerical calculations for $n \geq 4$. Our computations have revealed numerous (numerical) examples of linear combinations as predicted by Neumann's conjecture. To the best of our knowledge, our examples are the first concrete computations (the first of which were carried out in 2009) of the Cheeger-Chern-Simons invariant (complex volume) for $n > 2$.

1.1. **Statement of our results.** This section gives a brief summary of our main results. More details can be found in the paper.

1.1.1. *The Ptolemy variety.* Let M be a compact, oriented 3-manifold with (possibly empty) boundary, and let K be a closed 3-cycle (triangulated complex; see Definition 4.1) homeomorphic to the space obtained from M by collapsing each boundary component to a point. We identify each of the simplices of K with a standard simplex

$$\Delta_n^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \;\middle|\; 0 \leq x_i \leq n, \quad x_0 + x_1 + x_2 + x_3 = n\right\}. \tag{1.7}$$

Let $\Delta_n^3(\mathbb{Z})$ be the set of points in Δ_n^3 with integral coordinates, and let $\dot{\Delta}_n^3(\mathbb{Z})$ be $\Delta_n^3(\mathbb{Z})$ with the 4 vertex points removed.

Definition 1.1. A *Ptolemy assignment* on Δ_n^3 is an assignment $\dot{\Delta}_n^3(\mathbb{Z}) \to \mathbb{C}^*$, $t \mapsto c_t$, of a non-zero complex number c_t to each (non-vertex) integral point t of Δ_n^3 such that for each $\alpha \in \Delta_{n-2}^3(\mathbb{Z})$, the *Ptolemy relation*

$$c_{\alpha_{03}} c_{\alpha_{12}} + c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}} \tag{1.8}$$

is satisfied. Here, α_{ij} denotes the integral point $\alpha + e_i + e_j$. A Ptolemy assignment on K is a Ptolemy assignment c^i on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces.

Remark 1.2. The name is inspired by the resemblance of (1.8) with the Ptolemy relation between the lengths of the sides and diagonals of an inscribed quadrilateral (see Figure 1). In the work of Fock and Goncharov [14], the Ptolemy relations appear as relations between coordinates on the higher Teichmüller space when the triangulation of a surface is changed by a flip.

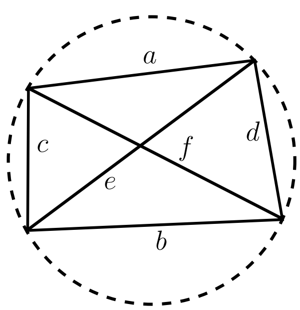

FIGURE 1. A quadrilateral is inscribed in a circle if and only if $ab + cd = ef$.

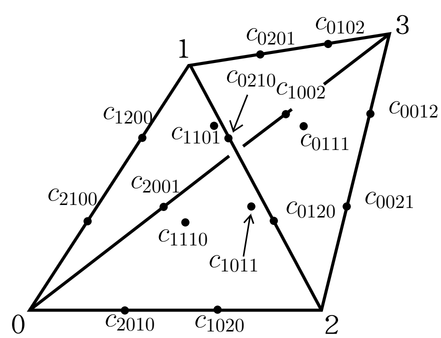

FIGURE 2. Ptolemy assignment for $n = 3$. The Ptolemy relation for $\alpha = 1000$ is $c_{2001}c_{1110} + c_{2100}c_{1011} = c_{2010}c_{1101}$.

It follows immediately from the definition that the set of Ptolemy assignments on K is an algebraic set $P_n(K)$, which we shall refer to as the *the Ptolemy variety*.

The extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ is generated by tuples $(u,v)\in\mathbb{C}^2$ with $e^u+e^v=1$, and the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})\subset\widehat{\mathcal{P}}(\mathbb{C})$ is the kernel of the map $\widehat{\mathcal{P}}(\mathbb{C})\to\wedge^2(\mathbb{C})$ taking (u,v) to $u\wedge v$. We refer to Section 3 for a review. Using (1.8), we obtain that a Ptolemy assignment c on Δ_n^3 gives rise to an element

$$\lambda(c)=\sum_{\alpha\in T^3(n-2)}(\widetilde{c}_{\alpha_{03}}+\widetilde{c}_{\alpha_{12}}-\widetilde{c}_{\alpha_{02}}-\widetilde{c}_{\alpha_{13}},\widetilde{c}_{\alpha_{01}}+\widetilde{c}_{\alpha_{23}}-\widetilde{c}_{\alpha_{02}}-\widetilde{c}_{\alpha_{13}})\in\widehat{\mathcal{P}}(\mathbb{C}),\tag{1.9}$$

where the tilde denotes a branch of logarithm (the particular choice is inessential). We thus have a map

$$\lambda\colon P_n(K)\to\widehat{\mathcal{P}}(\mathbb{C}),\qquad c\mapsto\sum_i\epsilon_i\lambda(c^i),\tag{1.10}$$

where the sum is over the simplices of K. Let $R_{\mathrm{SL}(n,\mathbb{C}),N}(M)$ denote the set of conjugacy classes of boundary-unipotent representations $\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})$. The following theorem (as well as Theorem 1.12 below) gives an efficient algorithm for computing complex volumes. For numerous examples, see Section 10.

Theorem 1.3 (Proof in Section 9.5)**.** *A Ptolemy assignment c uniquely determines a boundary-unipotent representation $\mathcal{R}(c)\in R_{\mathrm{SL}(n,\mathbb{C}),N}(M)$. The map λ has image in $\widehat{\mathcal{B}}(\mathbb{C})$, and we have a commutative diagram*

$$\begin{tikzcd} P_n(K) \arrow[r, "\lambda"] \arrow[d, "\mathcal{R}"] & \widehat{\mathcal{B}}(\mathbb{C}) \arrow[d, "R"] \\ R_{\mathrm{SL}(n,\mathbb{C}),N}(M) \arrow[r, "i\,\mathrm{Vol}_{\mathbb{C}}"] & \mathbb{C}/4\pi^2\mathbb{Z}. \end{tikzcd}\tag{1.11}$$

Moreover, if the triangulation is sufficiently fine (a single barycentric subdivision suffices), the map $\mathcal{R}$ is surjective. □

Remark 1.4. We show in Section 9 that there is a one-one correspondence between points in $P_n(K)$ and generically decorated (see Section 5) boundary-unipotent $\mathrm{SL}(n,\mathbb{C})$-representations. Under this correspondence, the map $\mathcal{R}$ is just the forgetful map ignoring the decoration. Note that $P_n(K)$ depends on the triangulation and may be empty.

Let $H\subset\mathrm{SL}(n,\mathbb{C})$ denote the group of diagonal matrices, and let h denote the number of boundary components of M. In Section 4.1 we define an action of H^h on $P_n(K)$. We denote the quotient by $P_n(K)_{\mathrm{red}}$. The action only changes the decoration, so $\mathcal{R}$ factors through $P_n(K)_{\mathrm{red}}$.

Definition 1.5. A boundary-unipotent representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})$ is *peripherally well behaved* if the image of each peripheral subgroup is either trivial or contains an element with a maximal Jordan block. If the latter condition holds for each peripheral subgroup, we say that ρ is *peripherally non-degenerate.*

Remark 1.6. When $n=2$ all representations are peripherally well behaved.

Theorem 1.7 (Proof in Section 9.5)**.** *The image of $\mathcal{R}\colon P_n(K)_{\mathrm{red}}\to R_{\mathrm{SL}(n,\mathbb{C}),N}(M)$ consists of the set of representations admitting a generic decoration (see Definition 5.2). If such a representation is peripherally non-degenerate, the preimage in $P_n(K)_{\mathrm{red}}$ is a single point. If ρ is peripherally well behaved, any two preimages of $\mathcal{R}$ have the same image in $\widehat{\mathcal{B}}(\mathbb{C})$.*

Corollary 1.8. A peripherally well behaved boundary-unipotent representation ρ in $\mathrm{SL}(n,\mathbb{C})$ determines an element $[\rho]\in\widehat{\mathcal{B}}(\mathbb{C})$ such that $R([\rho])=i\,\mathrm{Vol}_{\mathbb{C}}(\rho)$. □

Remark 1.9. In general the pre-image of a representation under $\mathcal{R}$ can have large dimension.

1.1.2. *Hyperbolic manifolds and $p\,\mathrm{SL}(n,\mathbb{C})$-representations.* Let $\phi_n\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ denote the canonical irreducible representation. Note that when n is odd ϕ_n factors through $\mathrm{PSL}(2,\mathbb{C})$. If a representation ρ is in the image of $P_n(K) \to R_{\mathrm{SL}(n,\mathbb{C}),N}(M)$, we say that $P_n(K)$ *detects* ρ.

Theorem 1.10 (Proof in Section 11.1)**.** *Suppose $M = \mathbb{H}^3/\Gamma$ is an oriented, hyperbolic manifold with finite volume and geometric representation $\rho_{\mathrm{geo}}\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$. If the triangulation of K has no non-essential edges, and if n is odd, $P_n(K)$ is non-empty and detects $\phi_n \circ \rho_{\mathrm{geo}}$.* □

When n is even, $\phi_n \circ \rho_{\mathrm{geo}}$ is only a representation in $p\,\mathrm{SL}(n,\mathbb{C}) = \mathrm{SL}(n,\mathbb{C})/\langle \pm I\rangle$.

Definition 1.11. Let $\sigma \in Z^2(\Delta^3_n;\mathbb{Z}/2\mathbb{Z})$ be a cocycle. A $p\,\mathrm{SL}(n,\mathbb{C})$-*Ptolemy assignment* on Δ^3_n with *obstruction cocycle* σ is an assignment of Ptolemy coordinates to the integral points of Δ^3_n such that

$$\sigma_2\sigma_3 c_{\alpha_{03}} c_{\alpha_{12}} + \sigma_0\sigma_3 c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}}. \tag{1.12}$$

Here $\sigma_i \in \mathbb{Z}/2\mathbb{Z} = \langle \pm 1\rangle$ is the value of σ on the face opposite the ith vertex of Δ^3_n. A $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignment on K with obstruction cocycle $\sigma \in Z^2(K;\mathbb{Z}/2\mathbb{Z})$ is a collection of $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignments c^i on Δ_i with obstruction class σ_{Δ_i} such that the Ptolemy coordinates agree on common faces.

The set of $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignments on K with obstruction cocycle σ is an algebraic set $P^\sigma_n(K)$, which up to canonical isomorphism, only depends on the cohomology class of σ. The obstruction class to lifting a boundary-unipotent representation in $p\,\mathrm{SL}(n,\mathbb{C})$ to a boundary-unipotent representation in $\mathrm{SL}(n,\mathbb{C})$ is a class in $H^2(M,\partial M;\mathbb{Z}/2\mathbb{Z}) = H^2(K;\mathbb{Z}/2\mathbb{Z})$. For $\sigma \in H^2(K;\mathbb{Z}/2\mathbb{Z})$, let $R^\sigma_{p\,\mathrm{SL}(n,\mathbb{C}),N}(M)$ denote the set of (conjugacy classes of) boundary-unipotent representations in $p\,\mathrm{SL}(n,\mathbb{C})$ with obstruction class σ. If M is hyperbolic we let $\sigma_{\mathrm{geo}} \in H^2(K;\mathbb{Z}/2\mathbb{Z})$ denote the obstruction class of the geometric representation.

Theorem 1.12 (Proof in Section 9.5)**.** *Let n be even. For each $\sigma \in H^2(K;\mathbb{Z}/2\mathbb{Z})$, we have a commutative diagram ($\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$ is defined in Section 3.2)*

$$\begin{tikzcd} P^\sigma_n(K) \arrow[r, "\lambda"] \arrow[d, "\mathcal{R}"] & \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}} \arrow[d, "R"] \\ R^\sigma_{p\,\mathrm{SL}(n,\mathbb{C}),N}(M) \arrow[r, "i\,\mathrm{Vol}_{\mathbb{C}}"] & \mathbb{C}/\pi^2\mathbb{Z}. \end{tikzcd} \tag{1.13}$$

If the triangulation of K is sufficiently fine, $\mathcal{R}$ is surjective. If $M = \mathbb{H}^3/\Gamma$ is hyperbolic, and if K has no non-essential edges, $P^{\sigma_{\mathrm{geo}}}_n(K)$ detects $\phi_n \circ \rho_{\mathrm{geo}}$. □

Remark 1.13. The analogue of Theorem 1.7 also holds, except that the preimage of a peripherally well behaved representation is now parametrized by $Z^1(K;\mathbb{Z}/2\mathbb{Z})$ (see Section**??**).

Remark 1.14. If the triangulation has a non-essential edge, all Ptolemy varieties are empty. Hence, if $P^\sigma_2(K)$ is non-empty for some σ, and if M is hyperbolic, the Ptolemy variety $P^{\sigma_{\mathrm{geo}}}(K)$ will detect the geometric representation.

Theorem 1.15 (Proof in Section 11)**.** *Let ρ be a peripherally well behaved representation in $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$. The extended Bloch group element of $\phi_n \circ \rho$ is $\binom{n+1}{3}$ times that of ρ. In particular, composition with ϕ_n multiplies complex volume by $\binom{n+1}{3}$.* □

1.1.3. *The Cheeger-Chern-Simons class.* The Cheeger-Chern-Simons invariant can be viewed as a characteristic class $H_3(\mathrm{SL}(n,\mathbb{C}))\to\mathbb{C}/4\pi^2\mathbb{Z}$, and the result underlying the proof of commutativity of (1.11) is Theorem 1.16 below, giving an explicit cocycle formula for the Cheeger-Chern-Simons class. The formula generalizes the formula in Goette-Zickert [17] for $n=2$. Recall that a homology class can be represented by a formal sum of tuples $(g_0,\dots,g_3)$. To such a tuple, we can assign a Ptolemy assignment $c(g_0,\dots,g_3)$ defined by

$$c(g_0,\dots,g_3)_t=\det\big(\{g_0\}_{t_0}\cup\dots\cup\{g_3\}_{t_3}\big),\quad t=(t_0,\dots,t_3),\tag{1.14}$$

where $\{g_i\}_{t_i}$ denotes the ordered set consisting of the first t_i column vectors of g_i. One can always represent a homology class by tuples, such that all the determinants in (1.14) are non-zero.

Theorem 1.16 (Proof in Section 8)**.** *The Cheeger-Chern-Simons class $\widehat{c}$ factors as*

$$H_3(\mathrm{SL}(n,\mathbb{C}))\xrightarrow{\lambda}\widehat{\mathcal{B}}(\mathbb{C})\xrightarrow{R}\mathbb{C}/4\pi^2\mathbb{Z}\ ,\tag{1.15}$$

where λ is induced by the map taking a tuple $(g_0,\dots,g_3)$ to $\lambda(c(g_0,\dots,g_3))\in\widehat{\mathcal{P}}(\mathbb{C})$. □

1.1.4. *Thurston's gluing equations.* When $n=2$, Thurston's gluing equation variety $V(K)$ is another variety, which is often used to compute volume. It is given by an equation for each edge of K and an equation for each generator of the fundamental groups of the boundary-components of M (see Section 12).

Theorem 1.17 (Proof in Section 12)**.** *Suppose M has h boundary components. There is a surjective regular map*

$$\coprod_{\sigma\in H^2(K;\mathbb{Z}/2\mathbb{Z})}P_2^\sigma(K)\to V(K)\tag{1.16}$$

with fibers disjoint copies of $(\mathbb{C}^*)^h$. □

Remark 1.18. The Ptolemy variety seems to offer significant computational advantage over the gluing equations, but according to Fabrice Rouillier (private communications) one can manipulate the gluing equations to mitigate this.

1.1.5. *Algebraic K-therory.* As shown in Zickert [30], the extended Bloch group can also be defined over a number field F, and we have a canonical isomorphism $\widehat{\mathcal{B}}(F)\cong K_3^{\mathrm{ind}}(F)$.

Theorem 1.19 (Proof in Section 13)**.** *Let F be a number field. A boundary-unipotent representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,F)$ determines an element of $\widehat{\mathcal{B}}(F)=K_3^{\mathrm{ind}}(F)$ such that for each embedding $\tau\colon F\to\mathbb{C}$, we have*

$$R(\tau([\rho]))=i\,\mathrm{Vol}_{\mathbb{C}}(\tau\circ\rho).\tag{1.17}$$

If ρ is irreducible, $[\rho]$ lies in $\widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$, where $\mathrm{Tr}(\rho)\subset F$ is the trace field of ρ. □

1.2. **Neumann's conjecture.** The fact that (1.10) has image in $\widehat{\mathcal{B}}(\mathbb{C})$ as opposed to $\widehat{\mathcal{P}}(\mathbb{C})$ has very interesting conjectural consequences. It is well known (see e.g. Suslin [27]) that the Bloch group $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space, and Walter Neumann has conjectured that it is generated by Bloch invariants of hyperbolic manifolds. More generally, Walter Neumann has proposed the following stronger conjecture [22]:

Conjecture 1.20. Let $F\subset\mathbb{C}$ be a concrete number field which is not in $\mathbb{R}$. The Bloch group $\mathcal{B}(F)$ is generated (integrally) modulo torsion by hyperbolic manifolds with invariant trace field contained in F.

Using Theorems 1.3 and 1.12, Conjecture 1.20 implies:

Conjecture 1.21. Let ρ be a boundary-unipotent representation of $\pi_1(M)$ in $\mathrm{SL}(n,\mathbb{C})$ or $p\,\mathrm{SL}(n,\mathbb{C})$. There exist hyperbolic 3-manifolds $M_1,\dots,M_k$ and integers $r_1,\dots,r_k$ such that

$$\mathrm{Vol}_{\mathbb{C}}(\rho)=\sum r_i\,\mathrm{Vol}_{\mathbb{C}}(M_i)\in\mathbb{C}/i\pi^2\mathbb{Q}. \tag{1.18}$$

In particular, $\mathrm{Vol}(\rho)=\sum r_i\,\mathrm{Vol}(M_i)\in\mathbb{R}$.

We give some examples in Section 10.

Remark 1.22. The Ptolemy coordinates may be considered as a 3-dimensional analogue of Fock and Goncharov's $\mathcal{A}$-coordinates [14]. They were defined for 3-manifolds in Zickert [30] (under the name *ideal cochain*), and have subsequently been studied by several other authors. These include Bergeron-Falbel-Guilloux [2], Garoufalidis-Goerner-Zickert [15] and Dimofte-Gabella-Goncharov [10].

1.3. **Overview of the paper.** Section 2 gives a detailed review of the Cheeger-Chern-Simons classes for flat bundles. Many details are included in order to give a self-contained proof of (1.2). Section 3 gives a brief review of the two variants of the extended Bloch group, and Section 4 reviews the theory, introduced in Zickert [31], of decorated representations and relative fundamental classes. In Section 5, we introduce the notion of *generic* decorations and define the Ptolemy variety $P_n(K)$. In Section 6, we construct a chain complex of Ptolemy assignments, and use it to construct a map from $H_3(\mathrm{SL}(n,\mathbb{C}),N)$ to $\widehat{\mathcal{B}}(\mathbb{C})$ commuting with stabilization. This shows that a decorated boundary-unipotent representation determines an element in the extended Bloch group, which is given explicitly in terms of the Ptolemy coordinates. In Section 7, we show that the extended Bloch group element of a decorated, peripherally well behaved representation is independent of the decoration, and in Section 8, we show that the Cheeger-Chern-Simons class is given as in Theorem 1.16. In Section 9, we show that the Ptolemy variety parametrizes generically decorated representations, and give an explicit formula for recovering a representation from its Ptolemy coordinates. In Section 10, we give some examples of computations, and list some interesting findings. Section 11 discusses the irreducible representations of $\mathrm{SL}(2,\mathbb{C})$, and Section 12 discusses the relationship to Thurston's gluing equations when $n=2$. Finally, Section 13 is a brief discussion of other fields.

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2. The Cheeger-Chern-Simons classes

The Cheeger-Chern-Simons classes [6, 7] are characteristic classes of principal bundles with connection. For general bundles, the characteristic classes are differential characters [6], but for flat bundles they reduce to ordinary (singular) cohomology classes. In this paper we will focus exclusively on flat bundles. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$, and let Λ be a proper subring of $\mathbb{F}$. Let G be a Lie group over $\mathbb{F}$ with finitely many components. There is a characteristic class $S_{P,u}$ for each pair (P,u) consisting of an invariant polynomial $P\in I^k(G;\mathbb{F})$ and a class $u\in H^{2k}(BG;\Lambda)$, whose image in $H^{2k}(BG;\mathbb{F})$ equals $W(P)$, where W is the Chern-Weil homomorphism

$$W\colon I^k(G;\mathbb{F})\to H^{2k}(BG;\mathbb{F}). \tag{2.1}$$

The characteristic class $S_{P,u}$ associates to each flat G-bundle $E \to M$ a cohomology class $S_{P,u}(E) \in H^{2k-1}(M;\mathbb{F}/\Lambda)$.

2.1. **Simply connected, simple Lie groups.** If G is simply connected and simple, $H^1(G;\mathbb{Z})$ and $H^2(G;\mathbb{Z})$ are trivial, and $H^3(G;\mathbb{Z}) \cong \mathbb{Z}$. Hence, by the Serre spectral sequence for the universal bundle, we have an isomorphism

$$S\colon H^4(BG;\mathbb{Z}) \cong H^3(G;\mathbb{Z}) \cong \mathbb{Z} \tag{2.2}$$

called the *suspension*. The Killing form on G defines an invariant polynomial $B \in I^2(G;\mathbb{F})$, and since B is real on the maximal compact subgroup K of G, $W(B)$ is a real class. Hence, there exists a unique positive real number α such that $W(\alpha B)$ is a generator of $H^4(BG;4\pi^2\mathbb{Z})$. We refer to αB as the *renormalized Killing form*, and denote the Cheeger-Chern-Simons class $S_{\alpha B, W(\alpha B)}$ by $\widehat{c}$.

Recall that every class in $H^3(G;\mathbb{F})$ can be represented by a G-invariant 3-form. The following is well known (see e.g. Kamber-Tondeur [19, (5.74) p. 116]).

Proposition 2.1. Let $P \in I^2(G;\mathbb{F})$. The suspension of $W(P)$ is represented by the invariant 3-form

$$\sigma(P) = -\frac{1}{6}P(\omega\wedge[\omega,\omega]) \in \Omega^3(G;\mathbb{F})^G \tag{2.3}$$

where ω is the Maurer-Cartan form on G. □

Let $E \to M$ be a G-bundle with flat connection θ. We can view θ as a map $\mathfrak{g}^* \to \Omega^1(E;\mathbb{F})$, so by taking exterior powers, θ induces a map

$$\theta\colon \Omega^3(G)^G = \wedge^3(\mathfrak{g}^*) \to \Omega^3(E;\mathbb{F}). \tag{2.4}$$

Note that $\theta(\sigma(P)) = -\frac{1}{6}P(\theta\wedge[\theta,\theta])$. In the following, P denotes the renormalized Killing form.

Proposition 2.2 ([6, Proposition 2.8])**.** Let $E \to M$ be a G-bundle, with flat connection θ, over a closed 3-manifold M. The cohomology class $\widehat{c}(E) \in H^3(M;\mathbb{F}/4\pi^2\mathbb{Z})$ satisfies

$$\widehat{c}(E)([M]) = \int_M s^*\big(\theta(\sigma(P))\big) \in \mathbb{F}/4\pi^2\mathbb{Z}, \tag{2.5}$$

where s is a section of E (which exists since G is 2-connected). □

Remark 2.3. Since $\sigma(P) \in H^3(G;4\pi^2\mathbb{Z})$ is a generator, it follows that a change of section changes the integral by a multiple of $4\pi^2\mathbb{Z}$.

Example 2.4. For $G = \mathrm{SL}(n,\mathbb{C})$, the renormalized Killing form P equals $\frac{1}{2}\mathrm{Tr}$, where Tr is the trace form $(A,B) \mapsto \mathrm{Tr}(AB)$. For a flat connection, $d\theta = -\frac{1}{2}[\theta,\theta] = -\theta\wedge\theta$, so (2.5) yields

$$\widehat{c}(E)([M]) = \frac{1}{2}\int_M s^*\big(\mathrm{Tr}(\theta\wedge d\theta + \frac{2}{3}\theta\wedge\theta\wedge\theta)\big) \in \mathbb{C}/4\pi^2\mathbb{Z} \tag{2.6}$$

recovering the Chern-Simons integral (1.1). Note that P also equals the (renormalized) second Chern-polynomial c_2. It thus follows that $\widehat{c} = \widehat{c}_2$.

2.2. **Complex groups and volume.** Recall that there is a 1-1 correspondence between flat G-bundles over M and representations $\pi_1(M) \to G$ up to conjugation. This correspondence takes a flat bundle to its holonomy representation. If $\rho\colon \pi_1(M) \to G$ is a representation, we let E_ρ denote the corresponding flat bundle. In the following G denotes a simply connected, simple, complex Lie group, and M a *closed*, oriented 3-manifold. The following definition is motivated by Theorem 2.8 below.

Definition 2.5. The *complex volume* $\mathrm{Vol}_{\mathbb{C}}(\rho)$ of a representation $\rho\colon \pi_1(M) \to G$ is defined by

$$\widehat{c}(E_\rho)([M]) = i\,\mathrm{Vol}_{\mathbb{C}}(\rho) \in \mathbb{C}/4\pi^2\mathbb{Z}. \tag{2.7}$$

The *volume* $\mathrm{Vol}(\rho)$ of ρ is the real part of $\mathrm{Vol}_{\mathbb{C}}(\rho)$.

The bundle E_ρ is isomorphic to $\widetilde{M}\times_\rho G$, and we thus have a 1-1 correspondence between sections of E_ρ and ρ-equivariant maps $\widetilde{M} \to G$ such that $f\colon \widetilde{M} \to G$ corresponds to the section $s(x) = [\widetilde{x}, f(\widetilde{x})]$.

Lemma 2.6. For any ρ-equivariant map $f\colon \widetilde{M} \to G$, we have $i\,\mathrm{Vol}_{\mathbb{C}}(\rho) = \int_D f^*(\sigma(P))$, where D is a fundamental domain for M in $\widetilde{M}$.

Proof. For any invariant form $\eta \in \Omega^3(G)^G$, the form $\theta(\eta) \in \Omega^3(E_\rho;\mathbb{F})$ is induced by the pullback of η under the projection $\widetilde{M} \times G \to G$. Letting $\eta = \sigma(P)$, the result follows from (2.5). □

Let $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ be hyperbolic 3-space. We identify the orthonormal frame bundle $F(\mathbb{H}^3)$ of $\mathbb{H}^3$ with $\mathrm{PSL}(2,\mathbb{C})$.

Lemma 2.7. For $G = \mathrm{SL}(2,\mathbb{C})$, $\sigma(P) = -h^* \wedge e^* \wedge f^*$, where $h = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$ are the standard generators of $\mathfrak{sl}(2,\mathbb{C})$ over $\mathbb{C}$.

Proof. As in Example 2.4, $P = \frac{1}{2}\mathrm{Tr}$. Using the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it follows from (2.3) that $\sigma(P) \in \Omega^3(G)^G = \wedge^3(\mathfrak{g}^*)$ is given by

$$\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (A,B,C) \mapsto -\frac{1}{2}\mathrm{Tr}(A[B,C]). \tag{2.8}$$

A simple computation shows that if $A = \left(\begin{smallmatrix}a_1 & a_2\\ a_3 & -a_1\end{smallmatrix}\right)$, $B = \begin{pmatrix}b_1 & b_2\\ b_3 & -b_1\end{pmatrix}$ and $C = \left(\begin{smallmatrix}c_1 & c_2\\ c_3 & -c_1\end{smallmatrix}\right)$

$$-\frac{1}{2}\mathrm{Tr}(A[B,C]) = -\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix} = -h^* \wedge e^* \wedge f^*(A,B,C). \tag{2.9}$$

This proves the result. □

Theorem 2.8. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold, and let $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ be a lift of the geometric representation. We have*

$$\widehat{c}(E_\rho)([M]) = i(\mathrm{Vol}(M) + i\mathrm{CS}(M)) \text{ in } \mathbb{C}/2\pi^2\mathbb{Z}, \tag{2.10}$$

where $\mathrm{CS}(M) = 2\pi^2\,\mathrm{cs}(M)$, and $\mathrm{cs}(M)$ is the (Riemannian) Chern-Simons invariant [7, (6.2)].

Proof. The fact that the imaginary part equals volume is well known, and follows from the fact (see Dupont [12]) that the imaginary part of $\sigma(P)$ is cohomologous to the pullback of the hyperbolic volume form. Yoshida [29, Lemma 3.1] shows that the real part of the form $h^* \wedge e^* \wedge f^*$ equals $2\pi^2$ cs, where cs is the Riemannian Chern-Simons form on $F(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$ (pulled back to $\mathrm{SL}(2,\mathbb{C})$. Note that the Riemannian connection on $F(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$ descends to the Riemannian connection on $F(M) = \mathrm{PSL}(2,\mathbb{C})/\Gamma$. If $f\colon \widetilde{M} \to \mathrm{SL}(2,\mathbb{C})$ is ρ-equivariant, the composition

$$\widetilde{M} \xrightarrow{f} \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{PSL}(2,\mathbb{C}) \longrightarrow \mathrm{PSL}(2,\mathbb{C})/\Gamma = F(M) \tag{2.11}$$

is ρ-invariant, and thus descends to a section of $F(M)$. The result now follows from Yoshida's result together with Lemma 2.7 and Lemma 2.6. □

Remark 2.9. Note that Theorem 2.8 implies that modulo $2\pi^2$, the complex volume of a representation lifting the geometric representation only depends on M and not on the choice of lift.

Remark 2.10. Since P is real on K, the imaginary part of $\sigma(P)$ is cohomologous to an invariant 3-form on G/K. Since $H^3(\mathfrak{g},\mathfrak{k};\mathbb{R})=\mathbb{R}$, there is a unique such form up to scaling. We may thus think of $\operatorname{Im}(\sigma(P))$ as a volume form.

2.3. **The universal classes and group cohomology.** The Cheeger-Chern-Simons classes are also defined for the universal flat bundle $EG^\delta\to BG^\delta$. For an explicit construction, we refer to Dupont-Kamber [13] or Dupont-Hain-Zucker [11]. In particular, we have a class $\widehat{c}\in H^3(BG^\delta;\mathbb{C}/4\pi^2\mathbb{Z})$. If $\rho\colon\pi_1(M)\to G$ is a representation, with classifying map $B\rho\colon M\to BG^\delta$, we thus have

$$\widehat{c}(B\rho_*([M]))=i\operatorname{Vol}_{\mathbb{C}}(\rho). \tag{2.12}$$

It is well known that the homology of BG^δ is the homology of the chain complex $C_*\otimes_{\mathbb{Z}[G]}\mathbb{Z}$, where C_* is any free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. A convenient choice of free resolution is the complex C_*, generated in degree n by tuples $(g_0,\dots,g_n)$, and with boundary map given by

$$\partial(g_0,\dots,g_n)=\sum(-1)^i(g_0,\dots,\widehat{g}_i,\dots,g_n). \tag{2.13}$$

The homology of $C_*\otimes_{\mathbb{Z}[G]}\mathbb{Z}$ is denoted $H_*(G)$, so $H_*(G)=H_*(BG^\delta)$. Theorem 1.16 gives a concrete cocycle formula for $\widehat{c}\colon H_3(\mathrm{SL}(n,\mathbb{C}))\to\mathbb{C}/4\pi^2\mathbb{Z}$.

2.4. **Compact manifolds with boundary.** In Section 6.1 below, we construct a natural extension of $\widehat{c}\colon H_3(\mathrm{SL}(n,\mathbb{C}))\to\mathbb{C}/4\pi^2\mathbb{Z}$ to a homomorphism

$$\widehat{c}\colon H_3(\mathrm{SL}(n,\mathbb{C}),N)\to\mathbb{C}/4\pi^2\mathbb{Z}, \tag{2.14}$$

where N is the subgroup of upper triangular matrices with 1's on the diagonal.

Definition 2.11. Let $\rho\colon\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})$ be a boundary-unipotent representation. The *complex volume* of ρ is defined by

$$\widehat{c}(B\rho_*([M,\partial M]))=i\operatorname{Vol}_{\mathbb{C}}(\rho), \tag{2.15}$$

where $B\rho\colon(M,\partial M)\to(B\,\mathrm{SL}(n,\mathbb{C})^\delta,BN^\delta)$ is a classifying map for ρ.

Remark 2.12. Unlike when M is closed, the classifying map is not uniquely determined by ρ; it depends on a choice of decoration (see Section 4). The complex volume, however, is independent of this choice (See Remark 8.5).

2.5. **Central elements of order** 2. For any simple complex Lie group G, there is a canonical homomorphism (defined up to conjugation)

$$\phi_G\colon\ \mathrm{SL}(2,\mathbb{C})\to G. \tag{2.16}$$

The element $s_G=\phi_G(-I)$ is a central element of G of order dividing 2, and equals $(-I)^{n+1}$ if $G=\mathrm{SL}(n,\mathbb{C})$ (see e.g. Fock-Goncharov [14, Corollary 2.1]). Let

$$pG=G/\langle s_G\rangle. \tag{2.17}$$

Note that ϕ_G descends to a homomorphism $\mathrm{PSL}(2,\mathbb{C})\to pG$. The following follows easily from the Serre spectral sequence.

Proposition 2.13. Suppose s_G has order 2. The canonical map $p^*\colon H^4(BpG;\mathbb{Z})\to H^4(BG;\mathbb{Z})$ is surjective with kernel of order dividing 4. $\square$

Corollary 2.14. There is a canonical characteristic class $\widehat{c}\colon H_3(pG)\to\mathbb{C}/\pi^2\mathbb{Z}$.

Proof. By Proposition 2.13, there exists a canonical class $u\in H^4(BpG;\pi^2\mathbb{Z})$ such that $p^*(u)=W(P)\in H^4(BG;\pi^2\mathbb{Z})$. Define $\widehat{c}=S_{P,u}$. $\square$

In Section 6.3, we construct a homomorphism

$$\widehat{c}\colon H_3(p\,\mathrm{SL}(n,\mathbb{C}),N) \to \mathbb{C}/\pi^2\mathbb{Z}, \tag{2.18}$$

which extends $\widehat{c}$ to a characteristic class of bundles with boundary-unipotent holonomy. The complex volume of a representation in $p\,\mathrm{SL}(n,\mathbb{C})$ is defined as in Definition 2.11.

3. The extended Bloch group

We use the conventions of Zickert [30]; the original reference is Neumann [21].

Definition 3.1. The *pre-Bloch group* $\mathcal{P}(\mathbb{C})$ is the free abelian group on $\mathbb{C}\setminus\{0,1\}$ modulo the *five term relation*

$$x - y + \frac{y}{x} - \frac{1-x^{-1}}{1-y^{-1}} + \frac{1-x}{1-y} = 0, \quad \text{for } x \neq y \in \mathbb{C}\setminus\{0,1\}. \tag{3.1}$$

The *Bloch group* is the kernel of the map $\nu\colon \mathcal{P}(\mathbb{C}) \to \wedge^2(\mathbb{C}^*)$ taking z to $z \wedge (1-z)$.

Definition 3.2. The *extended pre-Bloch group* $\widehat{\mathcal{P}}(\mathbb{C})$ is the free abelian group on the set

$$\widehat{\mathbb{C}} = \big\{(e,f) \in \mathbb{C}^2 \;\big|\; \exp(e) + \exp(f) = 1\big\} \tag{3.2}$$

modulo the *lifted five term relation*

$$(e_0,f_0) - (e_1,f_1) + (e_2,f_2) - (e_3,f_3) + (e_4,f_4) = 0 \tag{3.3}$$

if the equations

$$\begin{gathered} e_2 = e_1 - e_0, \quad e_3 = e_1 - e_0 - f_1 + f_0, \quad f_3 = f_2 - f_1 \\ e_4 = f_0 - f_1, \quad f_4 = f_2 - f_1 + e_0 \end{gathered} \tag{3.4}$$

are satisfied. The *extended Bloch group* is the kernel of the map $\widehat{\nu}\colon \widehat{\mathcal{P}}(\mathbb{C}) \to \wedge^2(\mathbb{C})$ taking (e,f) to $e \wedge f$.

An element $(e,f) \in \widehat{\mathbb{C}}$ with $\exp(e) = z$ is called a *flattening* with *cross-ratio* z. Letting $\mu_{\mathbb{C}}$ denote the roots of unity in $\mathbb{C}^*$, we have a commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mu_{\mathbb{C}} \arrow[r, "2\log"] \arrow[d, "\chi"] & \mathbb{C}/4\pi i\mathbb{Z} \arrow[r] \arrow[d, "\chi"] & \mathbb{C}^*/\mu_{\mathbb{C}} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathcal{B}}(\mathbb{C}) \arrow[r] \arrow[d, "\pi"] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[r, "\widehat{\nu}"] \arrow[d, "\pi"] & \wedge^2(\mathbb{C}) \arrow[r] \arrow[d] & K_2(\mathbb{C}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{B}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r, "\nu"] \arrow[d] & \wedge^2(\mathbb{C}^*) \arrow[r] \arrow[d] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.5}$$

The map π is induced by the map taking a flattening to its cross-ratio, and χ is the map taking $e \in \mathbb{C}/4\pi i\mathbb{Z}$ to $(e, f+2\pi i) - (e,f)$, where $f \in \mathbb{C}$ is any element such that $(e,f) \in \widehat{\mathbb{C}}$.

3.1. **The regulator.** By fixing a branch of logarithm, we may write a flattening with cross-ratio z as $[z;p,q]=\big(\log(z)+p\pi i,\log(1-z)+q\pi i\big)$, where $p,q\in\mathbb{Z}$ are *even* integers. There is a well defined *regulator map*

$$\begin{aligned} R\colon \widehat{\mathcal{P}}(\mathbb{C})&\to\mathbb{C}/4\pi^2\mathbb{Z},\\ [z;p,q]&\mapsto \mathrm{Li}_2(z)+\frac{1}{2}(\log(z)+p\pi i)(\log(1-z)-q\pi i)-\pi^2/6. \end{aligned}\tag{3.6}$$

3.2. **The $\mathrm{PSL}(2,\mathbb{C})$-variant of the extended Bloch group.** There is another variant of the extended Bloch group using flattenings $[z;p,q]$, where p and q are allowed to be odd. This group is defined as above using the set

$$\widehat{\mathbb{C}}_{\mathrm{odd}}=\left\{(e,f)\in\mathbb{C}^2\;\middle|\;\pm\exp(e)\pm\exp(f)=1\right\},\tag{3.7}$$

and fits in a diagram similar to (3.5). We use a subscript PSL to denote the variant allowing odd flattenings. We have an exact sequence

$$0\longrightarrow\mathbb{Z}/4\mathbb{Z}\longrightarrow\widehat{\mathcal{B}}(\mathbb{C})\longrightarrow\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}\longrightarrow 0.\tag{3.8}$$

For odd flattenings, the regulator (3.6) is well defined modulo $\pi^2\mathbb{Z}$.

Theorem 3.3 (Neumann [21], Goette-Zickert [17])**.** *There are natural isomorphisms*

$$H_3(\mathrm{PSL}(2,\mathbb{C}))\cong\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}},\quad H_3(\mathrm{SL}(2,\mathbb{C}))\cong\widehat{\mathcal{B}}(\mathbb{C})\tag{3.9}$$

such that the Cheeger-Chern-Simons classes agree with the regulators. □

The following result is needed in Section 7. The first part is proved in Zickert [30, Lemma 3.16], and the second has a similar proof, which we leave to the reader.

Lemma 3.4. For $(e,f)\in\widehat{\mathbb{C}}$ and $p,q\in\mathbb{Z}$, we have

$$(e+2\pi ip,f+2\pi iq)-(e,f)=\chi(qe-pf+2pq\pi i)\in\widehat{\mathcal{P}}(\mathbb{C}),\tag{3.10}$$

$$(e+\pi ip,f+\pi iq)-(e,f)=\chi(qe-pf+pq\pi i)\in\widehat{\mathcal{P}}(\mathbb{C})_{\mathrm{PSL}}.\tag{3.11}$$

□

3.3. **Arbitrary fields.** In Zickert [30], extended Bloch groups $\widehat{\mathcal{B}}_E(F)$ and $\widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}}$ are defined for an arbitrary field F and a primitive extension E of F^* by $\mathbb{Z}$. The definitions are as above using the sets

$$\widehat{E}_F=\left\{(e,f)\in E^2\;\middle|\;\pi(e)+\pi(f)=1\right\},\quad (\widehat{E}_F)_{\mathrm{odd}}=\left\{(e,f)\in E^2\;\middle|\;\pm\pi(e)\pm\pi(f)=1\right\}.\tag{3.12}$$

If F is a number field, the extended Bloch groups are up to canonical isomorphism independent of the choice of extension, so we may omit the subscript E.

Theorem 3.5 (Zickert [30, Theorem 1.1])**.** *Let F be a number field. There is a natural isomorphism*

$$K_3^{\mathrm{ind}}(F)\cong\widehat{\mathcal{B}}(F)\tag{3.13}$$

respecting Galois actions. □

Corollary 3.6 (Zickert [30, Corollary 7.14])**.** For each embedding $\tau\colon F\to\mathbb{C}$, the induced map $\tau\colon\widehat{\mathcal{B}}(F)\to\widehat{\mathcal{B}}(\mathbb{C})$ is injective. □

Corollary 3.7 (Galois descent; Zickert [30, Corollary 7.15])**.** Let $F_2:F_1$ be an extension of number fields. An element in $\widehat{\mathcal{B}}(F_2)$ is in $\widehat{\mathcal{B}}(F_1)$ if and only if it is invariant under all automorphisms of F_2 over F_1. □

4. Decorations of representations

In this section we review the notion of decorated representations introduced in Zickert [31]. Throughout the section, G denotes an arbitrary group, not necessarily a Lie group. Let H be subgroup of G. An *ordered simplex* is a simplex with a fixed vertex ordering.

Definition 4.1. A *closed* 3*-cycle* is a cell complex K obtained from a finite collection of ordered 3-simplices Δ_i by gluing together pairs of faces using order preserving simplicial attaching maps. We assume that all faces have been glued, and that the space $M(K)$, obtained by truncating the Δ_i's before gluing, is an oriented 3-manifold with boundary. Let ϵ_i be a sign indicating whether or not the orientation of Δ_i given by the vertex ordering agrees with the orientation of $M(K)$.

Note that up to removing disjoint balls (which does not effect the fundamental group), the manifold $M(K)$ only depends on the underlying topological space of K, and not on the choice of 3-cycle structure. Also note that for any compact, oriented 3-manifold M with (possibly empty) boundary, the space $\widehat{M}$ obtained from M by collapsing each boundary component to a point has a structure of a closed 3-cycle K such that $M = M(K)$.

Let K be a closed 3-cycle, and let $M = M(K)$. Let L denote the space obtained from the universal cover $\widetilde{M}$ of M by collapsing each boundary component to a point. The 3-cycle structure of K induces a triangulation of L, and also a triangulation of M by truncated simplices. The covering map extends to a map $L \to K$, and the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations extends to an action on L, which is determined by fixing, once and for all, a base point in M together with one of its lifts. Note that the stabilizer of each zero cell is a *peripheral* subgroup of $\pi_1(M)$, i.e. a subgroup induced by inclusion of a boundary component.

Definition 4.2. Let H be a subgroup of G. A representation $\rho\colon \pi_1(M) \to G$ is a (G,H)-*representation* if the image of each peripheral subgroup lies in a conjugate of H.

Definition 4.3. Let ρ be a (G,H)-representation. A *decoration* (on K) of ρ is a ρ-equivariant map

$$D\colon L^{(0)} \to G/H, \tag{4.1}$$

where $L^{(0)}$ is the zero skeleton of L.

Note that if $D(e) = gH$, we have $g^{-1}\rho(\mathrm{Stab}(e))g \subset H$, where $\mathrm{Stab}(e)$ is the stabilizer of e. Since D is ρ-equivariant, it follows that D determines subgroup of H for each boundary component which is well defined up to conjugation in H.

Definition 4.4. Two decorations of ρ are *equivalent* for each boundary component of M the corresponding subgroups of H are conjugate (in H).

Remark 4.5. If D is a decoration of ρ, then gD is a decoration of $g\rho g^{-1}$. Since we are only interested in representations up to conjugation, we consider such two decorations to be equal.

Proposition 4.6. Let E be a flat G-bundle over M whose holonomy representation is a (G,H)-representation ρ. There is a 1-1 correspondence between decorations of ρ up to equivalence, and reductions of $E_{\partial M}$ to an H-bundle over ∂M.

Proof. For each boundary component S_i of M, choose a base point in S_i and a path to the base point of M. This determines a lift e_i in L of the vertex of K corresponding to S_i, and an identification of $\pi_1(S_i)$ with $\mathrm{Stab}(e_i) \subset \pi_1(M)$. If F is a reduction of $E_{\partial M}$, the holonomy representations $\rho_i\colon \pi_1(S_i) \to H$ of F_{S_i} are conjugate to ρ, so there exist $g_i \in G$ such that $g_i^{-1}\rho g_i = \rho_i$. Assigning the coset g_iH to e_i yields a decoration, which up to equivalence is independent of the choice of g_i's.

On the other hand, a decoration assigns cosets g_iH to e_i such that $g_i^{-1}\rho(\mathrm{Stab}(e_i))g_i \subset H$. Hence, g_i defines an isomorphism of E_{S_i} with an H-bundle, which up to isomorphism only depends on the equivalence class of the decoration. □

4.1. **The diagonal action.** Let $N_G(H)$ denote the normalizer of H in G, and h the number of boundary components of M. There is an action of $(N_G(H)/H)^h$ on the set of equivalence classes of decorations given by right multiplication. More precisely, $(x_1,\dots,x_h)$ acts by taking a decoration D to the decoration D' defined as follows: If D takes a lift v of the ith boundary component to gH, then D' takes v to gx_iH. If $H=N$ and $G=\mathrm{SL}(n,\mathbb{C})$, $N_G(H)/H$ is the group of diagonal matrices. We thus refer to the action as the *diagonal action*.

Proposition 4.7. If a boundary-unipotent representation ρ is peripherally well behaved, the diagonal action on the set of equivalence classes of decorations of ρ is transitive.

Proof. It is enough to prove this in the case where there is only one boundary component. In this case, the image of the peripheral subgroup is either trivial or contains an element with a maximal Jordan block. In the first case, all decorations are equivalent, and in the second case, the result follows from the fact that if a subgroup A of N contains an element with a maximal Jordan form, the normalizer of A in $\mathrm{SL}(n,\mathbb{C})$ equals the normalizer of N. □

4.2. **The fundamental class of a decorated representation.** A flat G-bundle over M determines a classifying map $M\to BG^\delta$, where the δ indicates that G is regarded as a discrete group. It thus follows from Proposition 4.6 that a decorated representation $\rho\colon \pi_1(M)\to G$ determines a map

$$B\rho\colon (M,\partial M)\to (BG^\delta, BH^\delta). \tag{4.2}$$

In particular, ρ gives rise to a fundamental class

$$[\rho]=B\rho_*([M,\partial M])\in H_3(G,H), \tag{4.3}$$

where, by definition, $H_*(G,H)=H_*(BG^\delta,BH^\delta)$. Note that the fundamental class is independent of the particular 3-cycle structure on K.

Recall that M is triangulated by truncated simplices. By restriction, a (G,H) cocycle on M determines a (G,H)-cocycle on each truncated simplex $\overline{\Delta_i}$. Let $\overline{B}_*(G,H)$ denote the chain complex generated in degree n by (G,H)-cocycles on a truncated n-simplex. As proved in Zickert [31, Section 3], $\overline{B}_*(G,H)$ computes the homology groups $H_3(G,H)$. Note that a (G,H)-cocycle on M determines (up to conjugation) a decorated (G,H)-representation.

Proposition 4.8 (Zickert [31, Proposition 5.10])**.** Let τ be a (G,H)-cocycle on M representing a decorated (G,H)-representation ρ. The cycle

$$\sum \epsilon_i\tau_{\overline{\Delta}_i}\in \overline{B}_3(G,H), \tag{4.4}$$

represents the fundamental class of ρ. □

5. Generic decorations and Ptolemy coordinates

In all of the following, $G=\mathrm{SL}(n,\mathbb{C})$, and N is the subgroup of upper triangular matrices with 1's on the diagonal. A (G,N)-representation $\rho\colon\pi_1(M)\to G$ is called *boundary-unipotent*. For a matrix $g\in G$ and a positive integer $i\le n\in\mathbb{N}$, let $\{g\}_i$ be the ordered set consisting of the first i column vectors of g.

Definition 5.1. A tuple $(g_0N,\dots,g_kN)$ of N-cosets is *generic* if for each tuple $t=(t_0,\dots,t_k)$ of non-negative integers with sum n, we have

$$c_t := \det\left(\bigcup_{i=0}^{k}\{g_i\}_{t_i}\right)\neq 0, \tag{5.1}$$

where the determinant is viewed as a function on ordered sets of n vectors in $\mathbb{C}^n$. The numbers c_t are called *Ptolemy coordinates*.

Definition 5.2. A decoration of a boundary-unipotent representation is *generic* if for each simplex Δ of L, the tuple of cosets assigned to the vertices of Δ is generic.

For a set X, let $C_*(X)$ be the acyclic chain complex generated in degree k by tuples $(x_0,\dots,x_k)$. If X is a G-set, the diagonal G-action makes $C_*(X)$ into a complex of $\mathbb{Z}[G]$-modules. Let $C_*^{\mathrm{gen}}(G/N)$ be the subcomplex of $C_*(G/N)$ generated by generic tuples.

Proposition 5.3. The complex $C_*^{\mathrm{gen}}(G/N)\otimes_{\mathbb{Z}[G]}\mathbb{Z}$ computes the relative homology. If $\rho\colon\pi_1(M)\to G$ is a generically decorated representation, the fundamental class of ρ is represented by

$$\sum \epsilon_i(g_0^iN,g_1^iN,g_2^iN,g_3^iN)\in C_3^{\mathrm{gen}}(G/N), \tag{5.2}$$

where $(g_0^iN,\dots,g_3^iN)$ are the cosets assigned to lifts $\widetilde{\Delta}_i$ of the Δ_i's. $\square$

Proposition 5.3 is proved in Section 9. The idea is that a generic tuple canonically determines a (G,N)-cocycle on a truncated simplex. Hence, $C_*^{\mathrm{gen}}(G/N)\otimes_{\mathbb{Z}[G]}\mathbb{Z}$ is isomorphic to a subcomplex of $\overline{B}_3(G,N)$, and the representation (5.2) of the fundamental class is then an immediate consequence of (4.4).

Proposition 5.4. After a single barycentric subdivision of K, every decoration of a boundary-unipotent representation $\rho\colon\pi_1(M)\to G$ is equivalent to a generic one.

Proof. After a barycentric subdivision of K, every simplex Δ of K has distinct vertices and at least three vertices of Δ are interior (link is a sphere). Fix lifts $e_i\in L$ of each interior vertex of K. Since the stabilizer of a lift of an interior vertex is trivial, assigning any coset g_iH to e_i yields an equivalent decoration. Since the g_i's can be chosen arbitrarily, the result follows. $\square$

5.1. **The geometry of the Ptolemy coordinates.** We canonically identify each ordered k-simplex with a standard simplex

$$\Delta_n^k=\left\{(x_0,\dots,x_k)\in\mathbb{R}^{k+1}\;\middle|\;0\le x_i\le n,\ \sum_{i=0}^{k}x_i=n\right\}. \tag{5.3}$$

Recall that a tuple $(g_0N,\dots,g_kN)$ has a Ptolemy coordinate for each tuple of $k+1$ non-negative integers summing to n. In other words, there is a Ptolemy coordinate for each integral point of Δ_n^k. We denote the set of integral points in Δ_n^k by $\Delta_n^k(\mathbb{Z})$.

Definition 5.5. A *Ptolemy assignment* on Δ_n^k is an assignment of a non-zero complex number c_t to each integral point t of Δ_n^k such that the c_t's are the Ptolemy coordinates of some tuple $(g_0N,\dots,g_kN)\in C_k^{\mathrm{gen}}(G/N)$. A Ptolemy assignment on K is a Ptolemy assignment on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces.

Note that a generically decorated boundary-unipotent representation determines a Ptolemy assignment on K. In Section 9, we show that every Ptolemy assignment is induced by a unique decorated representation.

Lemma 5.6. The number of elements in $\Delta^k_l(\mathbb{Z})$ is $\binom{l+k}{k}$.

Proof. The map $(a_0,\dots,a_k)\mapsto\{a_0+1,a_0+a_1+2,\dots,a_0+\dots+a_{k-1}+k\}$ gives a bijection between $T^k(l)$ and subsets of $\{1,\dots,l+k\}$ with k elements. $\square$

Let e_i, $0\le i\le k$, be the ith standard basis vector of $\mathbb{Z}^{k+1}$. For each $\alpha\in\Delta^k_{n-2}(\mathbb{Z})$, the points $\alpha+2e_i$ in Δ^k_n span a simplex $\Delta^k(\alpha)$, whose integral points are the points $\alpha_{ij}:=\alpha+e_i+e_j$, see Figure 3. We refer to $\Delta^k(\alpha)$ as a *subsimplex* of Δ^k_n. By Lemma 5.6, Δ^3_n has $\binom{n+3}{3}$ integral points and $\binom{n+1}{3}$ subsimplices.

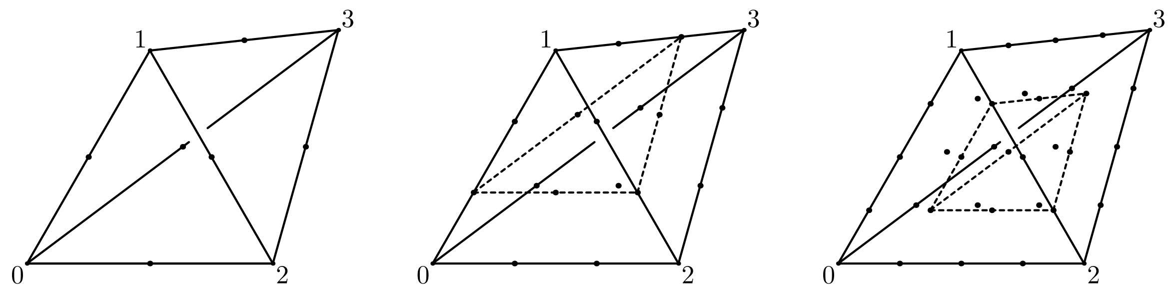

FIGURE 3. The integral points on Δ^3_n for $n=2$, 3 and 4. The indicated subsimplices correspond to $\alpha=(0,1,0,0)$ and $\alpha=(0,1,1,0)$.

Proposition 5.7 (Fock-Goncharov [14, Lemma 10.3])**.** The Ptolemy coordinates of a generic tuple (g_0N,g_1N,g_2N,g_3N) satisfy the *Ptolemy relations*

$$c_{\alpha_{03}}c_{\alpha_{12}}+c_{\alpha_{01}}c_{\alpha_{23}}=c_{\alpha_{02}}c_{\alpha_{13}},\quad \alpha\in\Delta^3_{n-2}(\mathbb{Z}). \tag{5.4}$$

Proof. Let $\alpha=(a_0,a_1,a_2,a_3)\in\Delta^3_{n-2}(\mathbb{Z})$. By performing row operations, we may assume that the first $n-2$ rows of the $n\times(n-2)$ matrix

$$\big(\{g_0\}_{a_0},\{g_1\}_{a_1},\{g_2\}_{a_2},\{g_3\}_{a_3}\big) \tag{5.5}$$

are the standard basis vectors. Letting x_i and y_i denote the last two entries of $(g_i)_{a_i+1}$, the Ptolemy relation for α is then equivalent to the (Plücker) relation

$$\det\begin{pmatrix}x_0&x_3\\y_0&y_3\end{pmatrix}\det\begin{pmatrix}x_1&x_2\\y_1&y_2\end{pmatrix}+\det\begin{pmatrix}x_0&x_1\\y_0&y_1\end{pmatrix}\det\begin{pmatrix}x_2&x_3\\y_2&y_3\end{pmatrix}=\det\begin{pmatrix}x_0&x_2\\y_0&y_2\end{pmatrix}\det\begin{pmatrix}x_1&x_3\\y_1&y_3\end{pmatrix}, \tag{5.6}$$

which is easily verified. $\square$

Note that the Ptolemy coordinate assigned to the ith vertex of Δ^k_n is $\det(\{g_i\}_n)=\det(g_i)=1$. We shall thus often ignore the vertex points. Let $\dot{\Delta}^k_n(\mathbb{Z})$ denote the non-vertex integral points of Δ^k_n. The following is proved in Section 9.

Proposition 5.8. For every assignment $c\colon\dot{\Delta}^3_n(\mathbb{Z})\to\mathbb{C}^*$, $t\mapsto c_t$ satisfying the Ptolemy relations (5.4), there is a unique Ptolemy assignment on Δ^3_n whose Ptolemy coordinates are c_t. $\square$

Corollary 5.9. The set of Ptolemy assignments on K is an algebraic set $P_n(K)$ called the *Ptolemy variety*. Its ideal is generated by the Ptolemy relations (5.4) (together with an extra equation making sure that all Ptolemy coordinates are non-zero). $\square$

Remark 5.10. It thus follows that Definition 5.5 agrees with Definition 1.1 when $k=3$. When $k>3$ and $n>2$ there are further relations among the Ptolemy coordinates. We shall not need these here.

5.2. **The diagonal action and the reduced Ptolemy variety.** If $d_0,\dots,d_3$ are diagonal matrices with $d_i = \operatorname{diag}(d_{i0},\dots d_{i,n-1})$, it follows from (5.1) that if the Ptolemy coordinates of a tuple $(g_0N,\dots,g_3N)$ are c_t, the Ptolemy coordinates c'_t of the tuple $(g_0d_0N,\dots,g_3d_3N)$ are given by

$$c'_t = c_t\prod_{k=0}^{t_0} d_{0k}\prod_{k=0}^{t_1} d_{1k}\prod_{k=0}^{t_2} d_{2k}\prod_{k=0}^{t_3} d_{3k}. \tag{5.7}$$

We therefore have an action of H^h on $P_n(K)$, which agrees with the action in Section 4.1. The quotient $P_n(K)_{\mathrm{red}}$ is called the *reduced Ptolemy variety.*

5.3. $p\,\mathrm{SL}(n,\mathbb{C})$**-Ptolemy coordinates.** When n is even, a $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignment on Δ_n^k may be defined as in Definition 5.5. Note, however, that the Ptolemy coordinates are now only defined up to a sign. Since we are mostly interested in 3-cycles, the following definition is more useful.

Definition 5.11. Let $\Delta = \Delta_n^3$, and let $\sigma \in Z^2(\Delta;\mathbb{Z}/2\mathbb{Z})$ be a cellular 2-cocycle. A $p\,\mathrm{SL}(n,\mathbb{C})$-*Ptolemy assignment* on Δ with *obstruction cocycle* σ is an assignment $c\colon \dot\Delta_n^3(\mathbb{Z}) \to \mathbb{C}^*$ satisfying the $p\,\mathrm{SL}(n,\mathbb{C})$-*Ptolemy relations*

$$\sigma_2\sigma_3 c_{\alpha_{03}}c_{\alpha_{12}} + \sigma_0\sigma_3 c_{\alpha_{01}}c_{\alpha_{23}} = c_{\alpha_{02}}c_{\alpha_{13}}. \tag{5.8}$$

Here $\sigma_i \in \mathbb{Z}/2\mathbb{Z} = \langle\pm1\rangle$ is the value of σ on the face opposite the ith vertex of Δ. A $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignment on K with obstruction cocycle $\sigma \in Z^2(K;\mathbb{Z}/2\mathbb{Z})$ is a $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy-assignment c^i on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces, and such that the obstruction cocycle of c^i is σ_{Δ_i}.

Note that for each $\sigma \in Z^2(K;\mathbb{Z}/2\mathbb{Z})$, the set of $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy-assignments on K form a variety $P_n^\sigma(K)$. We show in Section 9 that this variety only depends on the cohomology class of σ in $H^2(K;\mathbb{Z}/2\mathbb{Z}) = H^2(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ and that the Ptolemy variety parametrizes generically decorated boundary-unipotent $p\,\mathrm{SL}(n,\mathbb{C})$-representations whose obstruction class to lifting to a boundary-unipotent $\mathrm{SL}(n,\mathbb{C})$-representation is σ. The diagonal action (5.7) is defined on $P_n^\sigma(K)$ as well, and the quotient is denoted by $P_n^\sigma(K)_{\mathrm{red}}$. Note that when σ is the trivial cocycle taking all 2-cells to 1, $P^\sigma(K) = P(K)$.

5.4. **Cross-ratios and flattenings.** For $x \in \mathbb{C}\setminus\{0\}$, let $\widetilde{x} = \log(x)$, where log is some fixed (set theoretic) section of the exponential map.

Given a Ptolemy assignment c on $\Delta_{n=2}^3$, we endow $\Delta_{n=2}^3$ with the shape of an ideal simplex with cross-ratio $z = \frac{c_{03}c_{12}}{c_{02}c_{13}}$ and a flattening

$$\lambda(c) = (\widetilde{c}_{03} + \widetilde{c}_{12} - \widetilde{c}_{02} - \widetilde{c}_{13}, \widetilde{c}_{01} + \widetilde{c}_{23} - \widetilde{c}_{02} - \widetilde{c}_{13}) \in \widehat{\mathcal{P}}(\mathbb{C}). \tag{5.9}$$

By Propositions 5.7 and 5.8, a Ptolemy assignment on Δ_n^3 induces a Ptolemy assignment c_α on each subsimplex $\Delta^3(\alpha)$. We thus have a map

$$\lambda\colon P_n(K) \to \widehat{\mathcal{P}}(\mathbb{C}), \qquad c \mapsto \sum_i \epsilon_i \sum_{\alpha\in\Delta_{n-2}^3(\mathbb{Z})} \lambda(c_\alpha^i). \tag{5.10}$$

Similarly, we have a map $P_n^\sigma(K) \to \widehat{\mathcal{P}}(\mathbb{C})_{\mathrm{PSL}}$ defined by the same formula. We next prove that these maps have image in the respective extended Bloch groups.

Remark 5.12. The shapes associated to a Ptolemy assignment satisfy equations resembling Thurston's gluing equations. This is studied in Garoufalidis-Goerner-Zickert [15].

6. A chain complex of Ptolemy assignments

Let Pt^n_k be the free abelian group on Ptolemy assignments on Δ^k_n. The usual boundary map induces a boundary map $Pt^n_k \to Pt^n_{k-1}$ and the natural map $C^{\mathrm{gen}}_*(G/N) \to Pt^n_*$ taking a tuple $(g_0N,\dots,g_kN)$ to its Ptolemy assignment is a chain map. The result below is proved in Section 9.

Proposition 6.1. A generic tuple is determined up to the diagonal G-action by its Ptolemy coordinates. □

Corollary 6.2. The natural map induces an isomorphism

$$C^{\mathrm{gen}}_*(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong Pt^n_*. \tag{6.1}$$

In particular, $H_*(G,N) = H_*(Pt^n_*)$. □

Lemma 6.3. Let $c \in Pt^n_k$ be a Ptolemy assignment, and let $\alpha \in \Delta^k_{n-2}(\mathbb{Z})$. The Ptolemy coordinates $c_{\alpha_{ij}}$, $i \neq j$ are the Ptolemy coordinates of a unique Ptolemy assignment c_α on the subsimplex $\Delta^k(\alpha)$.

Proof. For $1 \leq k \leq 3$, this follows from Proposition 5.8. For $k > 3$, the result follows by induction, using the fact that 5 Ptolemy coordinates on Δ^3_2 determines the last. □

A Ptolemy assignment c on Δ^k_n thus induces a Ptolemy assignment c_α on each subsimplex. We thus have maps

$$J^n_k\colon Pt^n_k \to Pt^2_k, \quad c \mapsto \sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})} c_\alpha. \tag{6.2}$$

For a Ptolemy assignment $c \in Pt^n_k$ let $c_{\underline{i}} \in Pt^n_{k-1}$ be the induced Ptolemy assignment on the ith face of Δ^k_n, i.e. we have $\partial(c) = \sum_{i=0}^k (-1)^i c_{\underline{i}}$. Note that

$$(c_{\underline{i}})_{(a_0,\dots,a_{k-1})} = c_{(a_0,\dots,a_{i-1},0,a_i,\dots a_{k-1})_{\underline{i}}} \in Pt^2_{k-1}. \tag{6.3}$$

For $\beta \in \Delta^k_{n-3}(\mathbb{Z})$, let $c_{\beta^i} = c_{(\beta+e_i)_{\underline{i}}} \in Pt^2_{k-1}$, and define $\partial_\beta(c) \in Pt^2_{k-1}$ by

$$\partial_\beta(c) = \sum_{i=0}^k (-1)^i c_{\beta^i} \in Pt^2_{k-1}. \tag{6.4}$$

The geometry is explained in Figure 4.

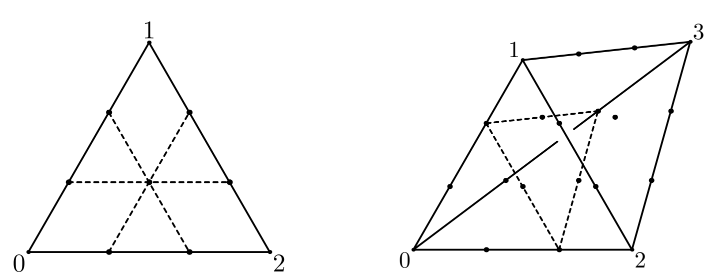

Figure 4. The dotted lines in the left figure indicate c_{β^0}, c_{β^1} and c_{β^2} for $k = 2$. The triangle in the right figure indicates c_{β^0} for $k = 3$. Here, $n = 3$ and $\beta = 0$.

Proposition 6.4. Let $c \in Pt_k^n$. We have

$$\partial(J_k^n(c)) - J_{k-1}^n(\partial(c)) = \sum_{\beta\in\Delta_{n-3}^k(\mathbb{Z})} \partial_\beta(c) \in Pt_{k-1}^2. \tag{6.5}$$

Proof. By (6.3), we have

$$\begin{aligned}\partial(J_k^n(c)) - J_{k-1}^n(\partial(c)) &= \sum_{i=0}^k (-1)^i \sum_{\alpha\in\Delta_{n-2}^k(\mathbb{Z})} c_{\alpha_{\underline{i}}} - \sum_{i=0}^k (-1)^i \sum_{\substack{\alpha\in\Delta_{n-2}^k(\mathbb{Z})\\ a_i=0}} c_{\alpha_{\underline{i}}}\\ &= \sum_{i=0}^k (-1)^i \sum_{\substack{\alpha\in\Delta_{n-2}^k(\mathbb{Z})\\ a_i>0}} c_{\alpha_{\underline{i}}}\\ &= \sum_{\beta\in\Delta_{n-3}^k(\mathbb{Z})} \sum_{i=0}^k (-1)^i c_{(\beta+e_i)_{\underline{i}}}\\ &= \sum_{\beta\in\Delta_{n-3}^k(\mathbb{Z})} \partial_\beta(c)\end{aligned} \tag{6.6}$$

as desired. □

6.1. **The map to the extended Bloch group.** We wish to define a map

$$\lambda\colon H_3(\mathrm{SL}(n,\mathbb{C}),N) \to \widehat{\mathcal{B}}(\mathbb{C}). \tag{6.7}$$

Letting $\widetilde{x}$ denote a logarithm of x, we consider the maps

$$\lambda\colon Pt_3^2 \to \mathbb{Z}[\widehat{\mathbb{C}}], \quad c \mapsto (\widetilde{c}_{03} + \widetilde{c}_{12} - \widetilde{c}_{02} - \widetilde{c}_{13}, \widetilde{c}_{01} + \widetilde{c}_{23} - \widetilde{c}_{02} - \widetilde{c}_{13}) \tag{6.8}$$

$$\mu\colon Pt_2^2 \to \wedge^2(\mathbb{C}), \quad c \mapsto -\widetilde{c}_{01}\wedge\widetilde{c}_{02} + \widetilde{c}_{01}\wedge\widetilde{c}_{12} - \widetilde{c}_{02}\wedge\widetilde{c}_{12} + \widetilde{c}_{02}\wedge\widetilde{c}_{02}. \tag{6.9}$$

Remark 6.5. The term $\widetilde{c}_{02}\wedge\widetilde{c}_{02}$ vanishes in $\wedge^2(\mathbb{C})$, but over general fields this term is needed. General fields are discussed in Section 13.

Lemma 6.6 (Zickert [30, Lemma 6.9])**.** Let $\mathbb{Z}[\widehat{\mathrm{FT}}]$ be the subgroup of $\mathbb{Z}[\widehat{\mathbb{C}}]$ generated by the lifted five term relations. There is a commutative diagram

$$\begin{tikzcd} Pt_4^2 \arrow[r, "\partial"] \arrow[d, "\lambda\circ\partial"] & Pt_3^2 \arrow[r, "\partial"] \arrow[d, "\lambda"] & Pt_2^2 \arrow[d, "\mu"] \\ \mathbb{Z}[\widehat{\mathrm{FT}}] \arrow[r, hook] & \mathbb{Z}[\widehat{\mathbb{C}}] \arrow[r, "\widehat{\nu}"] & \wedge^2(\mathbb{C}). \end{tikzcd} \tag{6.10}$$

□

It follows that λ induces a map $\lambda\colon H_3(\mathrm{SL}(2,\mathbb{C}),N) \to \widehat{\mathcal{B}}(\mathbb{C})$. This map equals the map defined in Zickert [31, Section 7]. The fact that λ is independent of the choice of logarithm is proved in Zickert [31, Remark 6.11], and also follows from Proposition 7.7 below.

Lemma 6.7. For each $c \in Pt_4^n$ and each $\beta \in \Delta_{n-3}^4(\mathbb{Z})$, we have

$$\lambda(\partial_\beta(c)) = 0 \in \widehat{\mathcal{P}}(\mathbb{C}). \tag{6.11}$$

Proof. Let $(e_i,f_i)=\lambda(c_{\beta^i})$ be the flattening associated to c_{β^i}. We prove that the flattenings satisfy the five term relation by proving that the equations (3.4) are satisfied. We have

$$\begin{aligned} e_0 &= \widetilde{c}_{\beta+(1,1,0,0,1)}+\widetilde{c}_{\beta+(1,0,1,1,0)}-\widetilde{c}_{\beta+(1,1,0,1,0)}-\widetilde{c}_{\beta+(1,0,1,0,1)} \\ e_1 &= \widetilde{c}_{\beta+(1,1,0,0,1)}+\widetilde{c}_{\beta+(0,1,1,1,0)}-\widetilde{c}_{\beta+(1,1,0,1,0)}-\widetilde{c}_{\beta+(0,1,1,0,1)} \\ e_2 &= \widetilde{c}_{\beta+(1,0,1,0,1)}+\widetilde{c}_{\beta+(0,1,1,1,0)}-\widetilde{c}_{\beta+(1,0,1,1,0)}-\widetilde{c}_{\beta+(0,1,1,0,1)} \end{aligned} \tag{6.12}$$

and it follows that $e_2=e_1-e_0$ as desired. The other 4 equations are proved similarly. $\square$

Lemma 6.8. For each $c\in Pt_3^n$ and each $\beta\in\Delta^3_{n-3}(\mathbb{Z})$, $\mu(\partial_\beta(c))=0\in\wedge^2(\mathbb{C})$.

Proof. We have

$$\begin{aligned} \mu(c_{\beta^0}) = &-\widetilde{c}_{\beta+(1,1,1,0)}\wedge\widetilde{c}_{\beta+(1,1,0,1)}+\widetilde{c}_{\beta+(1,1,1,0)}\wedge\widetilde{c}_{\beta+(1,0,1,1)} \\ &-\widetilde{c}_{\beta+(1,1,0,1)}\wedge\widetilde{c}_{\beta+(1,0,1,1)}+\widetilde{c}_{\beta+(1,1,0,1)}\wedge\widetilde{c}_{\beta+(1,1,0,1)}. \end{aligned} \tag{6.13}$$

Using this together with the similar formulas for $\mu(c_{\beta^i})$, we obtain that

$$\sum(-1)^i\mu(c_{\beta^i})=0\in\wedge^2(\mathbb{C}),$$

proving the result. $\square$

Corollary 6.9. The map $\lambda\circ J_3^n$ induces a map

$$\lambda\colon H_3(\mathrm{SL}(n,\mathbb{C}),N)\to\widehat{\mathcal{B}}(\mathbb{C}). \tag{6.14}$$

Proof. Using Proposition 6.4, this follows from Lemma 6.7 and Lemma 6.8. $\square$

Remark 6.10. For $n=3$, this map agrees with the map considered in Zickert [30].

Definition 6.11. The *extended Bloch group element* of a decorated (G,N)-representation ρ is defined by $\lambda([\rho])$, where $[\rho]\in H_3(\mathrm{SL}(n,\mathbb{C}),N)$ is the fundamental class of ρ.

Note that if the decoration of ρ is generic, and c is the corresponding Ptolemy assignment, the extended Bloch group element is given by $\lambda(c)$, where $\lambda\colon P_n(K)\to\widehat{\mathcal{P}}(\mathbb{C})$ is given by (5.10).

Proposition 6.12. The map $\lambda\colon P_n(K)\to\widehat{\mathcal{P}}(\mathbb{C})$ has image in $\widehat{\mathcal{B}}(\mathbb{C})$.

Proof. If $c\in P_n(K)$ is a Ptolemy assignment on K, we have a cycle $\alpha=\sum_i\epsilon_ic^i\in Pt_3^n$, and one easily checks that $\lambda(c)$ as defined in (5.10) equals $\lambda([\alpha])$. This proves the result. $\square$

6.2. **Stabilization.** We now prove that the map $\lambda\colon H_3(\mathrm{SL}(n,\mathbb{C}),N)\to\widehat{\mathcal{B}}(\mathbb{C})$ respects stabilization. We regard $\mathrm{SL}(n-1,\mathbb{C})$ as a subgroup of $\mathrm{SL}(n,\mathbb{C})$ via the standard inclusion adding a 1 as the upper left entry.

Let $\pi\colon M(n,\mathbb{C})\to M(n-1,\mathbb{C})$ be the map sending a matrix to the submatrix obtained by removing the first row and last column. The subgroup $D_k(\mathrm{SL}(n,\mathbb{C})/N)$ of $C_k^{\mathrm{gen}}(\mathrm{SL}(n,\mathbb{C})/N)$ generated by tuples $(g_0N,\dots,g_kN)$ such that the upper left entry of each g_i is 1 and such that

$$(\pi(g_0)N,\dots,\pi(g_k)N)\in C_k^{\mathrm{gen}}(\mathrm{SL}(n-1,\mathbb{C})/N) \tag{6.15}$$

form an $\mathrm{SL}(n-1,\mathbb{C})$-complex. Consider the $\mathrm{SL}(n-1,\mathbb{C})$-invariant chain maps

$$\pi\colon D_*(\mathrm{SL}(n,\mathbb{C})/N)\to Pt_*^{n-1} \tag{6.16}$$

$$i\colon D_*(\mathrm{SL}(n,\mathbb{C})/N)\to Pt_*^{n}, \tag{6.17}$$

where the first map is induced by π and the second is induced by the inclusion $D_*(\mathrm{SL}(n,\mathbb{C})/N)\to C_*^{\mathrm{gen}}(\mathrm{SL}(n,\mathbb{C})/N$. Let $D_k=D_k(\mathrm{SL}(n,\mathbb{C})/N)\otimes_{\mathbb{Z}[\mathrm{SL}(n-1,\mathbb{C})]}\mathbb{Z}$.

Lemma 6.13. The maps $\lambda\circ\pi$ and $\lambda\circ i$ from D_3 to $\widehat{\mathcal{P}}(\mathbb{C})$ agree on cycles.

Proof. Let $c\in D_k$ be induced by a tuple $(g_0N,\dots,g_kN)\in D_k(\mathrm{SL}(n,\mathbb{C})/N)$, and let c^I be the collection of Ptolemy coordinates associated to $(N,g_0N,\dots,g_kN)$. Since some Ptolemy coordinates may be zero, c^I is not necessarily a Ptolemy assignment. Note, however, that c^I_α *is* a Ptolemy assignment for each $(a_0,\dots,a_{k+1})\in\Delta^{k+1}_{n-2}(\mathbb{Z})$ with $a_0=0$. Note also that $c^I_\alpha\in Pt^2_{k+1}$ only depends on c. Hence, there is a map

$$P\colon D_k\to Pt^2_{k+1},\quad c\mapsto\sum_{\substack{\alpha\in\Delta^{k+1}_{n-2}(\mathbb{Z})\\ a_0=0}}c^I_\alpha. \tag{6.18}$$

We wish to prove the following:

$$\partial P(c)+P\partial(c)=J^n_k(i(c))-J^{n-1}_k(\pi(c))+\sum_{\substack{\beta\in\Delta^{k+1}_{n-3}(\mathbb{Z})\\ b_0=0}}\partial_\beta(c^I)\in Pt^2_{k+1}. \tag{6.19}$$

Given this, the result follows immediately from Lemma 6.7.

One easily verifies that

$$c^I_{(\underline{1},b_0,\dots,b_k)}=\pi(c)_{(b_0,\dots,b_k)}\in Pt^{n-1}_k,\quad (b_0,\dots,b_k)\in\Delta^k_{n-3}(\mathbb{Z}). \tag{6.20}$$

$$c^I_{(\underline{0},a_0,\dots,a_k)}=i(c)_{(a_0,\dots,a_k)},\quad (a_0,\dots,a_k)\in\Delta^k_{n-2}(\mathbb{Z}). \tag{6.21}$$

Using this, one has

$$\begin{aligned}\partial P(c)+P\partial(c)&=\sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})}i(c)_\alpha+\sum_{i=1}^{k+1}(-1)^i\sum_{\substack{\alpha\in\Delta^{k+1}_{n-2}(\mathbb{Z})\\ a_0=0}}c^I_{\alpha_{\underline{i}}}+\sum_{i=0}^{k}(-1)^i\sum_{\substack{\alpha\in\Delta^{k+1}_{n-2}(\mathbb{Z})\\ a_0=0,a_{i+1}=0}}c^I_{\alpha_{\underline{i+1}}}\\
&=\sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})}i(c)_\alpha+\sum_{i=1}^{k+1}(-1)^i\sum_{\substack{\alpha\in\Delta^{k+1}_{n-2}(\mathbb{Z})\\ a_0=0,a_i>0}}c^I_{\alpha_{\underline{i}}}\\
&=\sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})}i(c)_\alpha+\sum_{\substack{\beta\in\Delta^{k+1}_{n-3}(\mathbb{Z})\\ b_0=0}}\sum_{i=1}^{k+1}(-1)^ic^I_{\beta^i}\\
&=\sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})}i(c)_\alpha-\sum_{\substack{\beta\in\Delta^{k+1}_{n-3}(\mathbb{Z})\\ b_0=0}}c^I_{\beta^0}+\sum_{\substack{\beta\in\Delta^{k+1}_{n-3}(\mathbb{Z})\\ b_0=0}}\partial_\beta(c^I)\\
&=J^n_k(i(c))-J^{n-1}_k(\pi(c))+\sum_{\substack{\beta\in\Delta^{k+1}_{n-3}(\mathbb{Z})\\ b_0=0}}\partial_\beta(c^I).\end{aligned} \tag{6.22}$$

This proves (6.19), hence the result. □

Proposition 6.14. The map $\lambda\colon H_3(\mathrm{SL}(n,\mathbb{C}),N)\to\widehat{\mathcal{B}}(\mathbb{C})$ respects stabilization.

Proof. First note that π induces an isomorphism $D^0(\mathrm{SL}(n,\mathbb{C})/N)\cong C^0(\mathrm{SL}(n-1)/N)$. Using a standard cone argument, one easily checks that $D_*(\mathrm{SL}(n,\mathbb{C})/N)$ is a free $\mathrm{SL}(n-1,\mathbb{C})$-resolution of $\mathrm{Ker}(D^0(\mathrm{SL}(n,\mathbb{C})/N)\to\mathbb{Z})$. Hence, D_* computes $H_*(\mathrm{SL}(n-1,\mathbb{C}),N)$, and the result follows from Lemma 6.13. □

6.3. $p\,\mathrm{SL}(n,\mathbb{C})$**-Ptolemy assignments.** When n is even, define pPt^n_* to be the complex of Ptolemy coordinates of generic tuples in $p\,\mathrm{SL}(n,\mathbb{C})/N$. The Ptolemy coordinates are defined as in (5.1), and take values in $\mathbb{C}^*/\langle\pm1\rangle$. As in (6.1), we have an isomorphism $C^{\mathrm{gen}}_*(p\,\mathrm{SL}(n,\mathbb{C})/N)_{p\,\mathrm{SL}(n,\mathbb{C})}\cong pPt^n_*$. For $c\in\mathbb{C}^*/\langle\pm1\rangle$ let $\widetilde{c}\in\mathbb{C}$ be the image of some fixed set theoretic section of $\mathbb{C}\xrightarrow{\exp}\mathbb{C}^*\to\mathbb{C}^*/\langle\pm1\rangle$, e.g. $\frac{1}{2}\log(x^2)$ (the particular choice is inessential). The map

$$\lambda\colon pPt^2_3\to\mathbb{Z}[\widehat{\mathbb{C}}_{\mathrm{odd}}],\quad c\mapsto(\widetilde{c}_{03}+\widetilde{c}_{12}-\widetilde{c}_{02}-\widetilde{c}_{13},\widetilde{c}_{01}+\widetilde{c}_{23}-\widetilde{c}_{02}-\widetilde{c}_{13}) \tag{6.23}$$

induces a map $H_3(\mathrm{PSL}(2,\mathbb{C}),N)\to\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$, which agrees with the map constructed in Zickert [31, Section 3]. By precomposing λ with the map $pJ^n_3\colon pPt^n_3\to pPt^2_3$ defined as in (6.2) we obtain a map

$$\lambda\colon H_3(p\,\mathrm{SL}(n,\mathbb{C}),N)\to\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}, \tag{6.24}$$

which commutes with stabilization. This proves that a decorated boundary-unipotent representation in $p\,\mathrm{SL}(n,\mathbb{C})$ determines an element in $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$. The proofs of the above assertions are word by word identical to their $\mathrm{SL}(n,\mathbb{C})$-analogs.

7. Invariance under the diagonal action

We now show that the extended Bloch group element of a decorated representation is invariant under the diagonal action. We first prove that we can choose logarithms of the Ptolemy coordinates independently, without affecting the extended Bloch group element.

Definition 7.1. Let $c\colon\dot{\Delta}^k_n(\mathbb{Z})\to\mathbb{C}^*$ be a Ptolemy assignment. A *lift* of c is an assignment $\widetilde{c}\colon\dot{\Delta}^k_n(\mathbb{Z})\to\mathbb{C}$ such that $\exp(\widetilde{c})=c$.

For any lift $\widetilde{c}$ of a Ptolemy assignment c on Δ^3_2, we have a flattening

$$\lambda(\widetilde{c})=(\widetilde{c}_{03}+\widetilde{c}_{12}-\widetilde{c}_{02}-\widetilde{c}_{13},\widetilde{c}_{01}+\widetilde{c}_{23}-\widetilde{c}_{02}-\widetilde{c}_{13})\in\widehat{\mathbb{C}}. \tag{7.1}$$

Definition 7.2. The *log-parameters* of a flattening $(e,f)\in\widehat{\mathbb{C}}$ are defined by

$$w_{ij}=\begin{cases}e & \text{if } ij=01 \text{ or } ij=23\\ -f & \text{if } ij=12 \text{ or } ij=03\\ -e+f & \text{if } ij=02 \text{ or } ij=13.\end{cases} \tag{7.2}$$

Lemma 7.3. Let $\widetilde{c}\colon\dot{\Delta}^3_2(\mathbb{Z})\to\mathbb{C}$ be a lifted Ptolemy assignment, and let w_{ij} be the log-parameters of $\lambda(\widetilde{c})$. Fix $i<j\in\{0,\dots,3\}$ and let $\widetilde{c}'$ be the lifted Ptolemy assignment obtained from $\widetilde{c}$ by adding $2\pi\sqrt{-1}$ to $\widetilde{c}_{ij}$. Then

$$\lambda(\widetilde{c}')-\lambda(\widetilde{c})=\chi(w_{ij}+2\pi\sqrt{-1}\delta_{ij}), \tag{7.3}$$

where δ_{ij} is 1 if $ij=02$ or 13 and 0 otherwise.

Proof. Denote the flattening $\lambda(\widetilde{c})$ by (e,f). If $ij=03$ or 12, it follows from (7.1) that $\lambda(\widetilde{c}')=(e+2\pi\sqrt{-1},f)$. Similarly, $\lambda(\widetilde{c}')=(e,f+2\pi\sqrt{-1})$ if $ij=01$ or 23, and $\lambda(\widetilde{c}')=(e-2\pi\sqrt{-1},f-2\pi\sqrt{-1})$ if $ij=02$ or 13. By Lemma 3.4,

$$\begin{aligned}(e+2\pi\sqrt{-1},f)-(e,f)&=\chi(-f)\\ (e,f+2\pi\sqrt{-1})-(e,f)&=\chi(e)\\ (e-2\pi\sqrt{-1},f-2\pi\sqrt{-1})&=\chi(-e+f+2\pi\sqrt{-1}).\end{aligned} \tag{7.4}$$

This proves the result. ∎

Let $\widetilde{c}$ be a lift of a Ptolemy assignment c. For each $\alpha \in \Delta^3_{n-2}(\mathbb{Z})$, $\widetilde{c}$ induces a lift $\widetilde{c}_\alpha$ of c_α. Consider the element

$$\tau = \sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})} \lambda(\widetilde{c}_\alpha) \in \widehat{\mathcal{P}}(\mathbb{C}). \tag{7.5}$$

Fix a point $t_0 \in \dot{\Delta}^k_n(\mathbb{Z})$. We wish to understand the effect on τ of adding $2\pi\sqrt{-1}$ to $\widetilde{c}_{t_0}$. This changes τ into an element $\tau' \in \widehat{\mathcal{P}}(\mathbb{C})$. Let $w_{ij}(\alpha)$ denote the log-parameters of $\lambda(\widetilde{c}_\alpha)$. Note that t_0 either lies on an edge, on a face, or in the interior of Δ^3_n.

Lemma 7.4. Suppose t_0 is on the edge ij of Δ^3_n. Then

$$\tau' - \tau = \chi(w_{ij}(\alpha) + 2\pi\sqrt{-1}\delta_{ij}), \tag{7.6}$$

where $\alpha = t - e_i - e_j$, (i.e. α is such that t_0 is an edge point of $\Delta^3(\alpha)$).

Proof. This follows immediately from Lemma 7.3. □

Lemma 7.5. Suppose t_0 is on a face opposite vertex i. Then $\tau' - \tau = (-1)^i\chi(\kappa + 2\pi\sqrt{-1})$, where κ is given by

$$\kappa = \widetilde{c}_{\eta_i(0,-1,1)} - \widetilde{c}_{\eta_i(0,1,-1)} - \big(\widetilde{c}_{\eta_i(-1,0,1)} - \widetilde{c}_{\eta_i(1,0,-1)}\big) + \widetilde{c}_{\eta_i(-1,1,0)} - \widetilde{c}_{\eta_i(1,-1,0)}, \tag{7.7}$$

where η_i inserts a zero as the ith vertex.

Proof. For simplicity assume $i = 0$. The other cases are proved similarly. There are exactly three α's for which t_0 is an edge point of $\Delta^3(\alpha)$. These are

$$\alpha_0 = t_0 - (0,0,1,1), \quad \alpha_1 = t_0 - (0,1,0,1), \quad \alpha_2 = t_0 - (0,1,1,0). \tag{7.8}$$

Note that $\widetilde{c}_t = (\widetilde{c}_{\alpha_0})_{23} = (\widetilde{c}_{\alpha_1})_{13} = (\widetilde{c}_{\alpha_2})_{12}$. Since adding $2\pi\sqrt{-1}$ to $\widetilde{c}_{t_0}$ leaves $\widetilde{c}_\alpha$ unchanged unless $\alpha \in \{\alpha_0, \alpha_1, \alpha_2\}$, Lemma 7.3 implies that

$$\tau' - \tau = \chi(w_{23}(\alpha_0)) + \chi(w_{13}(\alpha_1) + 2\pi\sqrt{-1}) + \chi(w_{12}(\alpha_2)). \tag{7.9}$$

One easily checks that

$$\begin{aligned} w_{23}(\alpha_0) &= \widetilde{c}_{(1,0,-1,0)} + \widetilde{c}_{(0,1,0,-1)} - \widetilde{c}_{(1,0,0,-1)} - \widetilde{c}_{(0,1,-1,0)} \\ w_{13}(\alpha_1) &= \widetilde{c}_{(1,0,0,-1)} + \widetilde{c}_{(0,-1,1,0)} - \widetilde{c}_{(1,-1,0,0)} - \widetilde{c}_{(0,0,1,-1)} \\ w_{12}(\alpha_2) &= \widetilde{c}_{(1,-1,0,0)} + \widetilde{c}_{(0,0,-1,1)} - \widetilde{c}_{(1,0,-1,0)} - \widetilde{c}_{(0,-1,0,1)}, \end{aligned} \tag{7.10}$$

from which the result follows. □

Lemma 7.6. If t_0 is an interior point, $\tau' = \tau$.

Proof. If t_0 is an interior point, there are six α's for which t_0 is an edge point of $\Delta^3(\alpha)$. These are α_0, α_1 and α_2 as defined in (7.8) as well as

$$\alpha_3 = t_0 - (1,1,0,0), \quad \alpha_4 = t_0 - (1,0,1,0), \quad \alpha_5 = t_0 - (1,0,0,1). \tag{7.11}$$

Again, by Lemma 7.3

$$\begin{aligned} \tau' - \tau = \chi(w_{23}(\alpha_0)) + \chi(w_{13}(\alpha_1) + 2\pi\sqrt{-1}) + \chi(w_{12}(\alpha_2)) + \\ \chi(w_{01}(\alpha_3)) + \chi(w_{02}(\alpha_4) + 2\pi\sqrt{-1}) + \chi(w_{03}(\alpha_5)). \end{aligned} \tag{7.12}$$

Using (7.10) as well as

$$\begin{aligned} w_{01}(\alpha_3) &= \widetilde{c}_{(0,-1,0,1)} + \widetilde{c}_{(-1,0,1,0)} - \widetilde{c}_{(0,-1,1,0)} - \widetilde{c}_{(-1,0,0,1)} \\ w_{02}(\alpha_4) &= \widetilde{c}_{(0,1,-1,0)} + \widetilde{c}_{(-1,0,0,1)} - \widetilde{c}_{(0,0,-1,1)} - \widetilde{c}_{(-1,1,0,0)} \\ w_{03}(\alpha_5) &= \widetilde{c}_{(0,0,1,-1)} + \widetilde{c}_{(-1,1,0,0)} - \widetilde{c}_{(0,1,0,-1)} - \widetilde{c}_{(-1,0,1,0)} \end{aligned} \tag{7.13}$$

we see that all terms in (7.12) cancel out. Hence, $\tau'=\tau$. ☐

Proposition 7.7. Let c be a Ptolemy assignment on K. For any lift $\widetilde{c}$ of c, the element

$$\lambda(\widetilde{c})=\sum_i\sum_{\alpha\in\Delta^k_{n-2}(\mathbb{Z})}\epsilon_i\lambda(\widetilde{c}^i_\alpha)\in\widehat{\mathcal{P}}(\mathbb{C}) \tag{7.14}$$

is independent of the choice of lift. In particular, if c is the Ptolemy assignment of a decorated representation ρ, $\lambda(\widetilde{c})$ is the extended Bloch group element of ρ.

Proof. Let $\widetilde{c}$ and $\widetilde{c}'$ be lifts of c. Let $t_0\in\dot{\Delta}^3_n(\mathbb{Z})$. We wish to prove that $\lambda(\widetilde{c})=\lambda(\widetilde{c}')$. It is enough to prove this when $\widetilde{c}'$ is obtained from $\widetilde{c}$ by adding $2\pi\sqrt{-1}$ to $\widetilde{c}_t$. If t_0 is an interior point, the result follows immediately from Lemma 7.6. If t_0 is a face point, t_0 lies in exactly two simplices of K, and it follows from Lemma 7.5 that the two contributions to the change in $\lambda(\widetilde{c})$ appear with opposite signs (by (3.5), $2\chi(2\pi\sqrt{-1})=0$). Suppose t_0 is an edge point. Let C be the 3-cycle obtained by gluing together all the $\Delta^3(\alpha)$'s having t_0 as an edge point, using the face pairings induced from K. Let e be the (interior) 1-cell of C containing t_0. The argument in Zickert [31, Theorem 6.5] shows that the total log-parameter around e is zero. It thus follows from Lemma 7.4 that adding $2\pi\sqrt{-1}$ to $\widetilde{c}_{t_0}$ changes $\lambda(\widetilde{c})$ by 2-torsion which is trivial if and only if the number n of simplices in C for which t is a 02 edge or a 13 edge is even. Consider a curve λ in C encircling e. The vertex ordering induces an orientation on each face of each simplex of C, such that when λ passes through two faces of a simplex in C, the two orientations agree unless e is a 02 edge or a 13 edge. Since M is orientable, it follows that n is even. The second statement follows by letting $\widetilde{c}=\log c$. ☐

Proposition 7.8. The extended Bloch group element of a decorated boundary-unipotent representation is invariant under the diagonal action.

Proof. The argument is local. Let c be a Ptolemy assignment on Δ^3_n, and let c' be obtained from c by the diagonal action. By (5.7) c' is given by $d^i_j=\operatorname{diag}(d^i_{j0},\dots,d^i_{j,n-1})$. By (5.7) we have

$$c'_t=c_t\prod_{k=0}^{t_0}d_{0k}\prod_{k=0}^{t_1}d_{1k}\prod_{k=0}^{t_2}d_{2k}\prod_{k=0}^{t_3}d_{3k} \tag{7.15}$$

for diagonal matrices $d_i=\operatorname{diag}(d_{i0},\dots,d_{i,n-1})$. Letting log denote a logarithm, and $\widetilde{c}$ a lift of c, define a lift $\widetilde{c}'$ of c' by

$$\widetilde{c}'_t=\widetilde{c}_t+\sum_{k=0}^{t_0}\log(d_{0k})+\sum_{k=0}^{t_1}\log(d_{1k})+\sum_{k=0}^{t_2}\log(d_{2k})+\sum_{k=0}^{t_3}\log(d_{3k}). \tag{7.16}$$

Using this, one easily checks that $\lambda(c_\alpha)=\lambda(c'_\alpha)$ for each i and each $\alpha\in\Delta^3_{n-2}(\mathbb{Z})$. Applying this local argument to each simplex, the result follows from Proposition 7.7. ☐

Corollary 7.9. The extended Bloch group element of a peripherally well behaved boundary-unipotent representation ρ is independent of the decoration.

Proof. By performing a barycentric subdivision if necessary, we may assume that any decoration is generic. Since ρ is peripherally well behaved, the diagonal action is transitive on equivalence classes of decorations. Since equivalent decorations have the same fundamental class, the result follows. ☐

7.1. $p\,\mathrm{SL}(n,\mathbb{C})$**-decorations.** Let n be even. All results in this section have natural analogs for $p\,\mathrm{SL}(n,\mathbb{C})$. The proofs of these are obtained by replacing $2\pi\sqrt{-1}$ by $\pi\sqrt{-1}$, and logarithms by lifts of $\mathbb{C}\xrightarrow{\exp}\mathbb{C}^*/\langle\pm1\rangle$.

8. A cocycle formula for $\widehat{c}$

Let $i_*\colon H_3(\mathrm{SL}(n,\mathbb{C}))\to H_3(\mathrm{SL}(n,\mathbb{C}),N)$ denote the natural map. We wish to prove that the composition

$$H_3(\mathrm{SL}(n,\mathbb{C}))\xrightarrow{i_*}H_3(\mathrm{SL}(n,\mathbb{C}),N)\xrightarrow{\lambda}\widehat{\mathcal{B}}(\mathbb{C})\xrightarrow{R}\mathbb{C}/4\pi^2\mathbb{Z}\tag{8.1}$$

equals the Cheeger-Chern-Simons class $\widehat{c}$. Note that i_* is induced by the map $(g_0,\dots,g_3)\mapsto(g_0N,\dots,g_3N)$.

We shall make use of the canonical isomorphisms

$$H_3(\mathrm{SL}(n,\mathbb{C}))\cong H_3(\mathrm{SL}(3,\mathbb{C}))\cong H_3(\mathrm{SL}(2,\mathbb{C}))\oplus K_3^M(\mathbb{C}).\tag{8.2}$$

The first isomorphism is induced by stabilization (see Suslin [26]) and the second isomorphism is the $\pm$-eigenspace decomposition with respect to the transpose-inverse involution on $\mathrm{SL}(3,\mathbb{C})$ (see Sah [24]).

Lemma 8.1 (Suslin [26])**.** Let $H\subset\mathrm{SL}(3,\mathbb{C})$ be the subgroup of diagonal matrices. The $K_3^M(\mathbb{C})$ summand of $H_3(\mathrm{SL}(3,\mathbb{C}))$ is generated by the elements $B\rho_*([T])$, where $T=S^1\times S^1\times S^1$ is the 3-torus, and $\rho\colon\pi_1(T)\to H$ is a representation. □

Lemma 8.2. Let $T=S^1\times S^1\times S^1$ and let $\rho\colon\pi_1(T)\to H$ be a representation. The extended Bloch group element $[\rho]\in\widehat{\mathcal{B}}(\mathbb{C})$ of ρ is trivial.

Proof. We regard T as a cube C with opposite faces identified, and triangulate C as the cone on the triangulation on ∂C indicated in Figure 5 with cone point in the center. We order the vertices of each simplex by codimension, i.e. the 0-vertex is the cone point, the 1-vertex is a face point etc. Let $\rho\colon\pi_1(T)\to H$ be a representation, and pick a decoration of ρ by cosets in general position (the triangulation is such that this is always possible). Note that for every 3-simplex Δ of T, there is a unique opposite 3-simplex Δ^{opp}, such that the faces opposite the cone point are identified. We may assume that the cone point is decorated by the coset N. If a simplex Δ is decorated by the cosets (N,g_0N,g_1N,g_2N), the simplex Δ^{opp} must be decorated by the cosets (N,dg_0N,dg_1N,dg_2N), where d is the image of the generator of $\pi_1(T)$ pairing the faces of Δ and Δ^{opp}. It thus follows from (5.2) that the fundamental class is represented by a sum of terms of the form

$$(N,dg_0N,dg_1N,dg_2N)-(N,g_0N,g_1N,g_2N)\in C_3^{\mathrm{gen}}(\mathrm{SL}(n,\mathbb{C})/N).\tag{8.3}$$

Let c and c' be the Ptolemy assignments associated to the tuples (N,g_0N,g_1N,g_2N) and (N,dg_0N,dg_1N,dg_2N). Letting $d=\mathrm{diag}(d_1,\dots,d_n)$, we have $c'_t=c_t\prod_{i=t_0}^n d_i$. Fix a lift $\widetilde{c}$ of c, and consider the lift

$$\widetilde{c}'_t=\widetilde{c}_t+\sum_{i=t_0}^n\log(d_i)\tag{8.4}$$

of c'. One now checks that $\lambda(\widetilde{c}'_\alpha)=\lambda(\widetilde{c}_\alpha)$ for all $\alpha\in\dot{\Delta}_n^k(\mathbb{Z})$, so $\lambda(\widetilde{c})-\lambda(\widetilde{c}')=0$. This proves the result. □

Theorem 8.3. *The composition $R\circ\lambda\circ i_*$ equals $\widehat{c}$.*

Proof. Since λ commutes with stabilization, it follows from Goette-Zickert [17] that $R\circ\lambda\circ i_*=\widehat{c}$ on $H_3(\mathrm{SL}(2,\mathbb{C}))$. Since $\widehat{c}$ is zero on $K_3^M(\mathbb{C})$ (this follows from Lemma 8.1 and [6, Theorem 8.22]), the result follows from (8.2) and Lemma 8.2. □

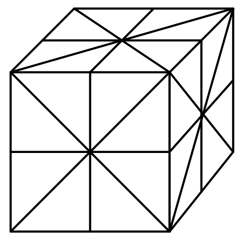

FIGURE 5. Triangulation of ∂C.

Remark 8.4. By defining $\widehat{c} = R \circ \lambda\colon H_3(\mathrm{SL}(n,\mathbb{C}),N) \to \mathbb{C}/4\pi^2\mathbb{Z}$, we have a natural extension of the Cheeger-Chern-Simons class to bundles with boundary-unipotent holonomy, and we can define the complex volume as in Definition 2.11.

Remark 8.5. The fact that the complex volume is independent of the choice of decoration can be seen as follows: We can regard $\widehat{c}$ as a map $P_n(\Delta^3) \to \mathbb{C}/4\pi^2\mathbb{Z}$, and a simple computation shows that the holomorphic 1-form $d\widehat{c}$ only involves coordinates on the boundary of Δ^3. Hence, for a closed 3-cycle K, $\widehat{c}\colon P_n(K) \to \mathbb{C}/4\pi^2\mathbb{Z}$ is locally constant. The result now follows from the fact that the space of decorations of a representation is path connected.

9. Recovering a representation from its Ptolemy coordinates

We now show that a Ptolemy assignment on K determines a generically decorated boundary-unipotent representation, which is given explicitly in terms of the Ptolemy coordinates. The idea is that a Ptolemy assignment canonically determines a (G,N)-cocycle on M.

9.1. The generic (G,N)-cocycle of a tuple.

Definition 9.1. An $n\times n$ matrix A is *counter diagonal* if the only non-zero entries of A are on the lower left to upper right diagonal, i.e. $A_{ij} = 0$ unless $j = n-i+1$. If $A_{ij} = 0$ for $j > n-i+1$ (resp. $j < n-i+1$), A is *upper* (resp. *lower*) *counter triangular*.

Given subsets I, J of $\{1,\dots,n\}$, let $A_{I,J}$ denote the submatrix of A whose rows and columns are indexed by I and J, respectively. If $|I| = |J|$, let $|A|_{I,J}$ denote the minor $\det(A_{I,J})$. Let I^c denote $\{1,\dots,n\}\setminus I$.

Recall that the adjugate $\mathrm{Adj}(A)$ of a matrix A is the matrix whose ijth entry is $(-1)^{i+j}|A|_{\{j\}^c,\{i\}^c}$. It is well known that $\mathrm{Adj}(A) = \det(A)A^{-1}$. The following result by Jacobi (see e.g. [1, Section 42]) expresses the minors of $\mathrm{Adj}(A)$ in terms of the minors of A.

Lemma 9.2. Let I, J be subsets of $\{1,\dots,n\}$ with $|I| = |J| = r$. We have

$$|\mathrm{Adj}(A)|_{I,J} = (-1)^{\sum(I,J)}\det(A)^{r-1}|A|_{J^c,I^c}, \tag{9.1}$$

where $\sum(I,J)$ is the sum of the indices occurring in I and J. $\square$

Definition 9.3. A matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is *generic* if $|A|_{\{k,\dots,n\},\{1,\dots,n-k+1\}} \neq 0$ for all $k \in \{1,\dots,n\}$.

Note that A is generic if and only if the Ptolemy coordinates of (N, AN) are non-zero. The following is a generalization of Zickert [31, Lemma 3.5].

Proposition 9.4. Let $A \in \mathrm{GL}_n(\mathbb{C})$ be generic. There exist unique $x \in N$ and $y \in N$ such that $q = x^{-1}Ay$ is counter diagonal. The entries of x, y and q are given by

$$q_{n,1} = A_{n,1}, \quad q_{n-j+1,j} = (-1)^{j-1}\frac{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}} \text{ for } 1 < j \leq n \tag{9.2}$$

$$x_{ij} = \frac{|A|_{\{i,j+1,\dots,n\},\{1,\dots,n-j+1\}}}{|A|_{\{j,\dots,n\},\{1,\dots,n-j+1\}}} \text{ for } j > i \tag{9.3}$$

$$y_{ij} = (-1)^{i+j}\frac{|A|_{\{n-j+2,\dots,n\},\{1,\dots,\widehat{i},\dots,j\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}} \text{ for } j > i. \tag{9.4}$$

Proof. It is enough to prove existence and uniqueness of x and y in N such that Ay and $x^{-1}A$ are upper and lower counter triangular, respectively. Suppose Ay is upper counter triangular. Then the vector $y_{\{1,\dots,j-1\},\{j\}}$ consisting of the part above the counter diagonal of the jth column vector of y must satisfy

$$A_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}y_{\{1,\dots,j-1\},\{j\}} + A_{\{n-j+2,\dots,n\},\{j\}} = 0. \tag{9.5}$$

The existence and uniqueness of y, as well as the given formula for the entries, now follow from Cramer's rule. Since $x^{-1}A$ is lower counter-triangular if and only if $A^{-1}x$ is upper counter-triangular, existence and uniqueness of x follows. The explicit formula for the entries follows from Jacobi's identity (9.1) and the formula for the entries of y. To obtain the formula for the entries of q, note that $q_{n-j+1,j} = (Ay)_{n-j+1,j}$. Hence, $q_{n,1} = A_{n,1}$, and for $1 < j \leq n$,

$$\begin{aligned} q_{n-j+1,j} &= \sum_{i=1}^{j-1} A_{n-j+1,i}y_{i,j} + A_{n-j+1,j} \\ &= \frac{\sum_{i=1}^{j}(-1)^{i+j}A_{n-j+1,i}|A|_{\{n-j+2,\dots,n\},\{1,\dots\widehat{i},\dots,j\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}} \\ &= (-1)^{j-1}\frac{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}}, \end{aligned}$$

where the second equality follows from (9.4). ☐

For a generic matrix A, let x_A, y_A and q_A be the unique matrices provided by Proposition 9.4. Given cosets g_iN, g_jN, g_kN, define

$$q_{ij} = q_{g_i^{-1}g_j}, \qquad \alpha^i_{jk} = (x_{g_i^{-1}g_j})^{-1}x_{g_i^{-1}g_k}. \tag{9.6}$$

Definition 9.5. The *generic cocycle* of a generic tuple $(g_0N, \dots, g_kN)$ is the (G, N)-cocycle on a truncated simplex $\overline{\Delta}$ defined by labeling the long edges by q_{ij} and the short edges by α^i_{jk} (see Figure 6).

Proposition 9.6. The diagonal left G-action on $C_k^{\mathrm{gen}}(G/N)$ is free when $k \geq 1$, and the chain complex $C_{*\geq 1}^{\mathrm{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ computes relative homology.

Proof. By Proposition 9.4, every generic tuple $(g_0N, \dots, g_kN)$ may be uniquely written as

$$g_0x_{g_0^{-1}g_1}(N, q_{01}N, \alpha^0_{12}q_{02}N, \dots, \alpha^0_{1k}q_{0k}N). \tag{9.7}$$

This proves that the G-action is free. Also note that for each generic tuple $(g_0N, \dots, g_kN)$, there exists a coset gN such that $(gN, g_0N, \dots, g_kN)$ is generic. Hence, $C_{*\geq 1}^{\mathrm{gen}}(G/N)$ is acyclic, and is thus a free resolution of $\mathrm{Ker}(C_0(G/N) \to \mathbb{Z})$. This proves the result (see e.g. Zickert [31, Theorem 2.1]). ☐

A generically decorated representation ρ thus determines a (G,N)-cocycle representing ρ. Let $\overline{B}_*^{\mathrm{gen}}(G,N)$ be the subcomplex of $\overline{B}_*(G,N)$ generated by generic cocycles on a standard simplex.

Corollary 9.7. We have a canonical isomorphism

$$\overline{B}_*^{\mathrm{gen}}(G,N)=C_*^{\mathrm{gen}}(G/N)\otimes_{\mathbb{Z}[G]}\mathbb{Z}, \tag{9.8}$$

end the fundamental class of a decorated representation is represented as in (4.4).

Proof. The first statement follows from Proposition 9.6 and the second from Theorem 4.8. □

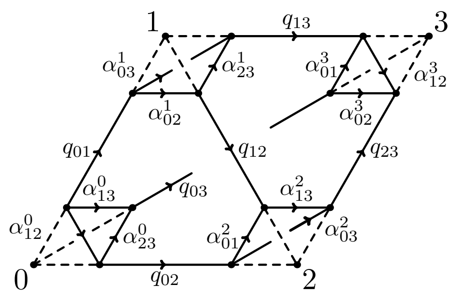

FIGURE 6. A (G,N)-cocycle on a truncated 3-simplex.

9.2. **Formulas for the long and short edges.** We wish to prove that a generic (G,N)-cocycle is uniquely determined by the Ptolemy coordinates.

Notation 9.8. Let $k\in\{1,\dots,n-1\}$.

(i) For $a_1,\dots,a_n\in\mathbb{C}^*$, let $q(a_1,\dots,a_n)$ be the counter-diagonal matrix whose entries on the counter-diagonal (from lower left to upper right) are $a_1,\dots,a_n$.
(ii) For $x\in\mathbb{C}$, let $x_k(x)$ be the elementary matrix whose $(k,k+1)$ entry is x.
(iii) For $b_1,\dots,b_k\in\mathbb{C}$, let $\pi_k(b_1,\dots,b_k)=x_1(b_1)x_2(b_2)\cdots x_k(b_k)$.

Proposition 9.9. The long edges of a generic (G,N)-cocycle are determined by the Ptolemy coordinates as follows:

$$q_{ij}=q(a_1,\dots,a_n),\qquad a_k=(-1)^{k-1}\frac{c_{(n-k)e_i+ke_j}}{c_{(n-k+1)e_i+(k-1)e_j}}. \tag{9.9}$$

Proof. Let $(g_0N,\dots,g_kN)$ be a generic tuple, and let $A=g_i^{-1}g_j$. Then $q_{ij}=q_A$. Since

$$|A|_{\{n-j+1,\dots,n\},\{1,j\}}=\det(\{g_i\}_{n-k},\{g_j\}_k)=c_{(n-k)e_i+ke_j}, \tag{9.10}$$

the result follows from (9.2). □

The corresponding formula for the short edges requires considerable more work, and is given in Proposition 9.14 below.

Lemma 9.10. Let A be generic, and let $L=x_A^{-1}A$. The entries $L_{i,n-i+2}$ right below the counter diagonal are given by

$$L_{i,n-i+2}=(-1)^{n-i}\frac{|A|_{\{i,\dots,n\},\{1,\dots,\widehat{n-i+1},n-i+2\}}}{|A|_{\{i+1,\dots,n\},\{1,\dots,n-i\}}}. \tag{9.11}$$

Proof. We proceed as in the proof of Proposition 9.4. Let $x = x_A^{-1}$. Since L is lower counter-triangular, we must have

$$x_{\{i\},\{i+1,\dots,n\}}A_{\{i+1,\dots,n\},\{1,\dots,n-i\}} + A_{\{i\},\{1,\dots,n-i\}} = 0, \tag{9.12}$$

so by Cramer's rule,

$$x_{ij} = (-1)^{i+j}\frac{|A|_{\{i,\dots,\widehat{j},\dots,n\},\{1,\dots,n-i\}}}{|A|_{\{i+1,\dots,n\},\{1,\dots,n-i\}}} \text{ for } j > i. \tag{9.13}$$

We thus have

$$\begin{aligned}|A|_{\{i+1,\dots,n\},\{1,\dots,n-i\}}L_{i,n-i+2} &= A_{i,n-i+2}|A|_{\{i+1,\dots,n\},\{1,\dots,n-i\}}\\ &\qquad + \sum_{k=i+i}^{n}(-1)^{i+k}|A|_{\{j,\dots,\widehat{k},\dots,n\},\{1,\dots,n-j\}}A_{k,n-i+2}\\ &= \sum_{k=j}^{n}(-1)^{i+k}|A|_{\{j,\dots,\widehat{k},\dots,n\},\{1,\dots,n-i\}}A_{k,n-i+2}\\ &= (-1)^{n-i}|A|_{\{i,\dots,n\},\{1,\dots,\widehat{n-i+1},\dots,n-i+2\}},\end{aligned}$$

which proves the result. □

Definition 9.11. Let $A, B \in \mathrm{GL}(n,\mathbb{C})$.

(i) A and B are related by a *type* 0 *move* if all but the last column of A and B are equal.
(ii) A and B are related by a *type* 1 *move* if all but the second last column of A and B are equal.
(iii) A and B are related by a *type* 2 *move* if for some $j < n-1$, B is obtained from A by switching columns j and $j+1$.

Proposition 9.12. Let A and B be generic, and let A_i and B_i denote the ith column of A, resp. B.

(i) If A and B are related by a type 0 move, $x_B = x_A$.
(ii) If A and B are related by a type 1 move, $x_B = x_A x_1(x)$, where

$$x = -\frac{\det(A_1,\dots,A_{n-1},B_{n-1})\det(e_1,e_2,A_1,\dots,A_{n-2})}{\det(e_1,A_1,\dots,A_{n-1})\det(e_1,A_1,\dots,A_{n-2},B_{n-1})}. \tag{9.14}$$

(iii) If A and B are related by a type 2 move switching columns j and $j+1$, $x_B = x_A x_{n-j}(x)$, where

$$x = -\frac{\det(e_1,\dots,e_{n-j-1},A_1,\dots,A_{j+1})\det(e_1,\dots,e_{n-j+1},A_1,\dots,A_{j-1})}{\det(e_1,\dots,e_{n-j},A_1,\dots,A_j)\det(e_1,\dots,e_{n-j},A_1,\dots,A_{j-1},B_j)}. \tag{9.15}$$

Proof. The first statement follows from the fact that x_A is independent of the last column of A. Suppose A and B are related by a type 1 move. Using (9.3), one sees that $(x_A)_{ij} = (x_B)_{ij}$ except when $i = 1$ and $j = 2$. It thus follows that $x_B = x_A x_1(x)$, where $x = (x_B)_{12} - (x_A)_{12}$. Letting C be the matrix obtained from A by replacing the nth column by the $(n-1)$th column of B, one has

$$\begin{aligned}|A|_{\{1,3,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n,2}, & |B|_{\{1,3,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n-1,2},\\ |A|_{\{2,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n,1}, & |B|_{\{2,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n-1,1},\end{aligned}$$

and it follows from (9.3) that

$$x = (x_B)_{12} - (x_A)_{12} = \frac{\mathrm{Adj}(C)_{n-1,2}}{\mathrm{Adj}(C)_{n-1,1}} - \frac{\mathrm{Adj}(C)_{n,2}}{\mathrm{Adj}(C)_{n,1}}. \tag{9.16}$$

We then have

$$\begin{aligned} x\operatorname{Adj}(C)_{n,1}\operatorname{Adj}(C)_{n-1,1} &= \operatorname{Adj}(C)_{n-1,2}\operatorname{Adj}(C)_{n,1} - \operatorname{Adj}(C)_{n-1,1}\operatorname{Adj}(C)_{n,2} \\ &= -|\operatorname{Adj}(C)|_{\{n-1,n\},\{1,2\}} \\ &= -\det(C)|C|_{\{3,\dots,n\},\{1,\dots,n-2\}} \\ &= -\det(A_1,\dots,A_{n-1},B_{n-1})\det(e_1,e_2,A_1,\dots,A_{n-2}), \end{aligned}$$

where the third equality follows from Jacobi's identity (9.1). Since

$$\operatorname{Adj}(C)_{n,1}\operatorname{Adj}(C)_{n-1,1} = \det(e_1,A_1,\dots,A_{n-1})\det(e_1,A_1,\dots,A_{n-2},B_{n-1}),$$

this proves the second statement.

Now suppose A and B are related by a type 2 move. Let $E_{j,j+1}$ be the elementary matrix obtained from the identity matrix by switching the jth and $(j+1)$th columns. Then $B = AE_{j,j+1}$. Since $L = x_A^{-1}A$ is lower counter triangular, $x_{n-j}(-\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}})LE_{j,j+1}$ must also be lower counter triangular. We thus have

$$x_B = x_A x_{n-j}(-\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}})^{-1} = x_A x_{n-j}(\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}}). \tag{9.17}$$

By (9.11) and (9.2), we have

$$\begin{aligned} L_{n-j+1,j+1} &= (-1)^{j-1}\frac{|A|_{\{n-j+1,\dots,n\},\{1,\dots,\widehat{j},j+1\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}} \\ L_{n-j,j+1} &= (-1)^{j}\frac{|A|_{\{n-j,\dots,n\},\{1,\dots,j+1\}}}{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}}}. \end{aligned} \tag{9.18}$$

Hence,

$$\begin{aligned} \frac{L_{n-j,j+1}}{L_{n-j+1,j+1}} &= -\frac{|A|_{\{n-j,\dots,n\},\{1,\dots,j+1\}}|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}}{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}}|A|_{\{n-j+1,\dots,n\},\{1,\dots,\widehat{j},j+1\}}} \\ &= -\frac{\det(e_1,\dots,e_{n-j-1},A_1,\dots,A_{j+1})\det(e_1,\dots,e_{n-j+1},A_1,\dots,A_{j-1})}{\det(e_1,\dots,e_{n-j},A_1,\dots,A_j)\det(e_1,\dots,e_{n-j},A_1,\dots,A_{j-1},B_j)}, \end{aligned}$$

proving the third statement. ☐

Note that any two matrices $A, B \in \mathrm{GL}(n,\mathbb{C})$ are related by a sequence of moves of type 1, 2 and 0 as follows:

$$\begin{aligned} &A \xrightarrow{1} [A_1,\dots,A_{n-2},B_1,A_n] \xrightarrow{2} [A_1,\dots,A_{n-3},B_1,A_{n-2},A_n] \xrightarrow{2} \dots \xrightarrow{2} \\ &[B_1,A_1,\dots,A_{n-2},A_n] \xrightarrow{1} [B_1,A_1,\dots,A_{n-3},B_2,A_n] \xrightarrow{2} \dots \xrightarrow{2} \\ &[B_1,B_2,A_1,\dots,A_{n-3},A_n] \xrightarrow{1,2} \dots \xrightarrow{1,2} [B_1,\dots,B_{n-1},A_n] \xrightarrow{0} B. \end{aligned} \tag{9.19}$$

Consider the tilings of a face ijk, $i<j<k$, of Δ_n^2 by *diamonds* shown in Figure 7. We refer to the diamonds as being of type i, j and k, respectively.

Definition 9.13. The *diamond coordinate* $d_{r,s}^k$ of a diamond (r,s) of type k is the alternating product of the Ptolemy coordinates assigned to its vertices, see Figure 7.

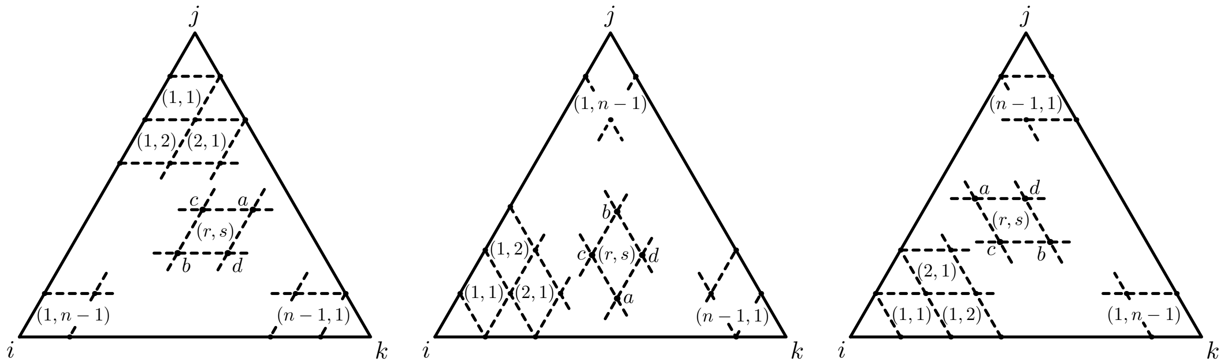

FIGURE 7. Diamonds of type i, j and k. The diamond coordinates are $d^i_{r,s} = d^k_{r,s} = \frac{-ab}{cd}$, and $d^j_{r,s} = \frac{ab}{cd}$, where a, b, c, and d are Ptolemy coordinates.

Proposition 9.14. The short edges α^i_{jk}, $j<k$, of a generic (G,N)-cocycle are determined by the Ptolemy coordinates as follows (π_* is defined in 9.8 (iii)):

$$\alpha^i_{jk} = \pi_{n-1}(d^i_{1,1},\dots,d^i_{1,n-1})\pi_{n-2}(d^i_{2,1},\dots,d^i_{2,n-2})\cdots\pi_1(d^i_{n-1,1}), \tag{9.20}$$

where the $d^i_{j,k}$'s are the type i diamond coordinates on the face ijk.

Proof. Let $(g_0N,\dots,g_lN)$ be a generic tuple, and let $A = g_i^{-1}g_j$ and $B = g_i^{-1}g_k$. We assume that $i<j<k$, the other cases being similar. Note that the Ptolemy coordinates on the ijk face are given by

$$c_{t_ie_i+t_je_k+t_ke_k} = \det(e_1,\dots,e_{t_i},A_1,\dots,A_{t_j},B_1,\dots,B_{t_k}). \tag{9.21}$$

Performing the sequence of moves in (9.19), the result follows from Proposition 9.12. □

Corollary 9.15. A generic tuple is determined up to the diagonal G-action by its Ptolemy coordinates. □

Example 9.16. Suppose Ptolemy assignments on Δ^2_n, $n\in\{2,3\}$, are given as in Figure 8. Using (9.9) and (9.20), we obtain that the corresponding (G,N)-cocycle is given by

$$\begin{gathered} q_{01} = q(a,-1/a), \quad q_{12} = q(b,-1/b), \quad q_{02} = q(c,-1/c), \\ \alpha^0_{12} = x_1\Big(\frac{-b}{ac}\Big), \quad \alpha^1_{02} = x_1\Big(\frac{c}{ab}\Big), \quad \alpha^2_{01} = x_1\Big(\frac{-a}{cb}\Big) \end{gathered} \tag{9.22}$$

when $n=2$, and

$$\begin{gathered} q_{01} = q(c,-a/c,1/a), \quad q_{12} - q(b,-e/b,1/e), \quad q_{02} = q(f,-g/f,1/g), \\ \alpha^1_{02} = x_1\Big(\frac{fa}{cd}\Big)x_2\Big(\frac{d}{ab}\Big)x_1\Big(\frac{gb}{de}\Big), \\ \alpha^0_{12} = x_1\Big(\frac{-bc}{ad}\Big)x_2\Big(\frac{-d}{cf}\Big)x_1\Big(\frac{-ef}{dg}\Big), \quad \alpha^2_{01} = x_1\Big(\frac{-cg}{fd}\Big)x_2\Big(\frac{-d}{ge}\Big)x_1\Big(\frac{-ae}{db}\Big) \end{gathered} \tag{9.23}$$

when $n=3$.

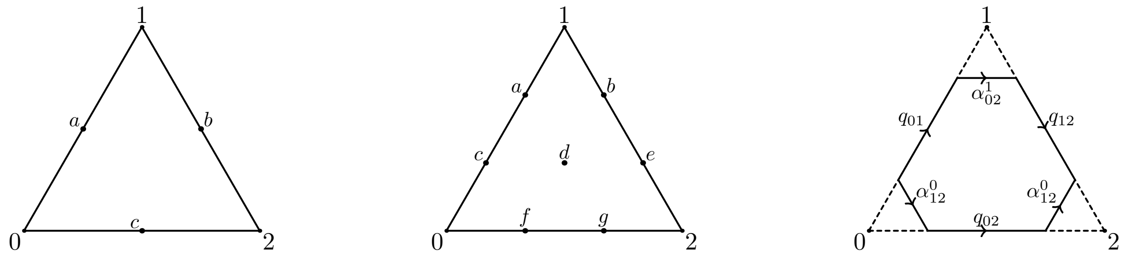

FIGURE 8. Ptolemy assignments and the corresponding cocycle for $n = 2$ and $n = 3$.

9.3. **From Ptolemy assignments to decorations.** Corollary 9.15 shows that here is at most one generic (G, N)-cocycle with a given collection of Ptolemy coordinates. We now prove that when $k \leq 3$ there is exactly one.

Lemma 9.17. Let $a_{i,j}$ and $b_{i,j}$ be non-zero complex numbers. The equality

$$\pi_{n-1}(a_{1,1}, \ldots, a_{1,n-1}) \cdots \pi_1(a_{n-1,1}) = \pi_{n-1}(b_{1,1}, \ldots, b_{1,n-1}) \cdots \pi_1(b_{n-1,1}) \tag{9.24}$$

holds if and only if $a_{i,j} = b_{i,j}$ for all i, j.

Proof. For any $c_{i,j}$, the nth column of $\pi_{n-1}(c_{1,1}, \ldots, c_{1,n-1}) \cdots \pi_1(c_{n-1,1})$ is equal to the nth column of $\pi_{n-1}(c_{1,1}, \ldots, c_{1,n-1})$, which equals

$$(\prod_{i=1}^{n-1} c_{1,i}, \prod_{i=2}^{n-1} c_{1,i}, \ldots, c_{1,n-1}).$$

This proves that $a_{1,j} = b_{1,j}$ for all j. The result now follows by induction. □

Proposition 9.18. For any assignment $c\colon \dot{\Delta}_n^2(\mathbb{Z}) \to \mathbb{C}^*$, there is a unique Ptolemy assignment $c \in Pt_2^n$ whose Ptolemy coordinates are c_t.

Proof. We prove that the Ptolemy coordinates c'_t of $(N, q_{01}N, \alpha_{12}^0 q_{02} N)$ equal c_t, where q_{01}, q_{02} and α_{12}^0 are given in terms of the c_t's by (9.9) and (9.20). First note that $c_t = c'_t$ if either t_1 or t_2 is 0, i.e. if t is on one of the edges of Δ_n^2 containing the 0th vertex. Each of the other integral points t is the upper right vertex of a unique diamond (r, s) of type 0. Let τ_k be the upper right vertex of the kth diamond D_k in the sequence

$$(1, n-1), (1, n-2), \ldots (1, 1), (2, n-2), \ldots, (2, 1), \ldots, (n-1, 1). \tag{9.25}$$

By Lemma 9.17, $d_{r,s}^{0\prime} = d_{r,s}^0$ for all diamonds (r, s) of type 0. It thus follows that if $c_t = c'_t$ for all but one of the vertices of a diamond D, then $c_t = c'_t$ for all vertices of D. In particular $c'_{\tau_1} = c_{\tau_1}$. Suppose by induction that $c'_{\tau_i} = c_{\tau_i}$ for all $i < k$. Then $c'_t = c_t$, for all vertices of D_k except τ_k. Hence, we also have $c'_{\tau_k} = c_{\tau_k}$, completing the induction. □

Proposition 9.19. For any assignment $c\colon \dot{\Delta}_n^3(\mathbb{Z}) \to \mathbb{C}^*$ satisfying the Ptolemy relations, there is a unique Ptolemy assignment $c \in Pt_3^n$ whose Ptolemy coordinates are c_t.

Proof. Let c'_t be the Ptolemy coordinates of the tuple $(N, q_{01}N, \alpha_{12}^0 q_{02} N, \alpha_{13}^0 q_{03} N)$ defined from the c_t's by (9.9) and (9.20). We wish to prove that $c'_t = c_t$ for all t. Note that if, for some subsimplex $\Delta^3(\alpha)$, $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ for all but one of the 6 α_{ij}'s, then $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ holds for all α_{ij}. This is a direct consequence of the Ptolemy relations. By Proposition 9.18, $c'_t = c_t$, when either t_2 or t_3 is zero. Hence, for each $\alpha = (a_0, a_1, a_2, a_3)$ with $a_2 = a_3 = 0$, $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ except possibly when $(i, j) = (2, 3)$.

As explained above, $c'_{\alpha_{23}} = c_{\alpha_{23}}$ as well. Now suppose by induction that $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ for all α with $a_2 + a_3 < k$. Then $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ holds except possibly when $(i,j) = (2,3)$. Again, $c'_{\alpha_{23}} = c_{\alpha_{23}}$ must also hold, completing the induction. $\square$

A (G,N)-cocycle on M obviously determines a decorated representation (up to conjugation). The main results of this section can thus be summarized by the diagram below.

$$\Big\{\text{Points in } P_n(K)\Big\} \longleftrightarrow \left\{\begin{matrix}\text{Generic } (G,N)\text{-cocycles}\\ \text{on } M\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Generically decorated}\\ (G,N)\text{-representations}\end{matrix}\right\} \tag{9.26}$$

Remark 9.20. We stress that the Ptolemy variety parametrizes decorated representations and *not* decorated representations up to equivalence. In particular, the dimension of $P_n(K)$ depends on the triangulation, and may be very large if K has many interior vertices.

9.4. **Obstruction cocycles and the $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy varieties.** Suppose n is even. The projection $G \to pG$ maps N isomorphically onto its image (also denoted by N), and by elementary obstruction theory (see e.g. Steenrod [25]), the obstruction to lifting a (pG,N)-representation ρ to a (G,N)-representation is a class in

$$H^2(M,\partial M;\mathbb{Z}/2\mathbb{Z}) = H^2(K;\mathbb{Z}/2\mathbb{Z}). \tag{9.27}$$

We can represent it by an explicit cocycle in $Z^2(K;\mathbb{Z}/2\mathbb{Z})$ as follows: Pick any $(p\,\mathrm{SL}(n,\mathbb{C}),N)$-cocycle $\bar\tau$ on M representing ρ and a lift τ of $\bar\tau$ to a (G,N)-cochain. Each 2-cell of K corresponds to a hexagonal 2-cell of M, and the 2-cocycle $\sigma \in Z^2(K;\mathbb{Z}/2\mathbb{Z})$ taking a 2-cell to the product of the τ-labelings along the corresponding hexagonal 2-cell of M represents the obstruction class.

Proposition 9.21. Suppose the interior of M is a 1-cusped hyperbolic 3-manifold with finite volume. The obstruction class in $H^2(K;\mathbb{Z}/2\mathbb{Z})$ to lifting the geometric representation is non-trivial.

Proof. By a result of Calegari [5, Corollary 2.4], any lift of the geometric representation takes a longitude to an element in $\mathrm{SL}(2,\mathbb{C})$ with trace -2. This shows that no lift is boundary-unipotent, so the obstruction class must be non-trivial. $\square$

Proposition 9.4 also holds in $p\,\mathrm{SL}(n,\mathbb{C})$, and we thus have a 1-1 correspondence between generically decorated representations and (pG,N)-cocycles on M.

Definition 9.22. Let $\sigma \in Z^2(K;\mathbb{Z}/2\mathbb{Z})$. A lifted (pG,N) cocycle on M with obstruction cocycle σ is a generic (G,N)-assignment on M lifting a (pG,N)-cocycle on M such that the 2-cocycle on K obtained by taking products along hexagonal faces of M equals σ.

A 1-cochain $\eta \in C^1(K;\mathbb{Z}/2\mathbb{Z})$ acts on a lifted (pG,N)-cocycle τ by multiplying a long edge e by $\eta(e)$. Note that if τ has obstruction cocycle σ, $\eta\tau$ has obstruction cocycle $\delta(\eta)\sigma$, where δ is the standard coboundary operator. Recall that there is a 1-1 correspondence between generic (G,N)-cocycles on M and points in the Ptolemy-variety. We shall prove a similar result for pG.

We wish to define a coboundary action on pG-Ptolemy assignments (see Definition 5.11). Let c be a pG-Ptolemy assignment on Δ, and let $\eta_{ij} \in C^1(\Delta;\mathbb{Z}/2\mathbb{Z})$ be the cochain taking the edge ij to -1 and all other edges to 1. Define

$$\eta_{ij}c\colon \dot\Delta^3_n(\mathbb{Z}) \to \mathbb{C}^*, \qquad (\eta_{ij}c)_t = (-1)^{t_it_j}c_t \tag{9.28}$$

and extend in the natural way to define ηc for a pG-Ptolemy assignment c on K and $\eta \in C^1(K;\mathbb{Z}/2\mathbb{Z})$. A priori ηc is only an assignment of complex numbers to the integral points of the simplices of K. However, we have:

Lemma 9.23. If c is a pG-Ptolemy assignment on K with obstruction cocycle σ, ηc is a pG-Ptolemy assignment on K with obstruction cocycle $\delta(\eta)\sigma$.

Proof. It is enough to prove this for a simplex Δ and for $\eta=\eta_{ij}$. Let $c'=\eta_{ij}c$. We assume for simplicity that $ij=01$; the other cases are proved similarly. For any $\alpha=(a_0,a_1,a_2,a_3)\in\Delta^k_{n-2}(\mathbb{Z})$, we then have

$$c'_{\alpha_{03}}c'_{\alpha_{12}}+c'_{\alpha_{01}}c'_{\alpha_{23}}-c'_{\alpha_{02}}c'_{\alpha_{13}}=(-1)^{a_0+a_1}(c_{\alpha_{03}}c_{\alpha_{12}}-c_{\alpha_{01}}c_{\alpha_{23}}-c_{\alpha_{02}}c_{\alpha_{13}}) \tag{9.29}$$

Let $\tau=\delta(\eta_{01})$. Since $\delta(\eta_{01})_2=\delta(\eta_{01})_3=-1$ and $\delta(\eta_{01})_0=1$, (9.29) implies that

$$\tau_2\tau_3c'_{\alpha_{03}}c'_{\alpha_{12}}+\tau_0\tau_3c'_{\alpha_{03}}c'_{\alpha_{01}}c'_{\alpha_{23}}=c'_{\alpha_{02}}c'_{\alpha_{13}}, \tag{9.30}$$

as desired. □

Definition 9.24. The diamond coordinates of a $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignment with obstruction cocycle σ are defined as in Definition 9.13, but multiplied by the sign (provided by σ) of the face.

Note that for $\eta\in C^1(K;\mathbb{Z}/2/\mathbb{Z})$, the diamond coordinates of c and ηc are identical.

Proposition 9.25. For any $\sigma\in Z^2(K;\mathbb{Z}/2\mathbb{Z})$, there is a 1-1 correspondence between $p\,\mathrm{SL}(n,\mathbb{C})$-Ptolemy assignments on K with obstruction cocycle σ, and lifted $(p\,\mathrm{SL}(n,\mathbb{C}),N)$-cocycles on M with obstruction cocycle σ. The correspondence preserves the coboundary actions.

Proof. It is enough to prove this for a simplex Δ. For a pG-Ptolemy assignment c on Δ with obstruction cocycle $\sigma\in Z^2(\Delta;\mathbb{Z}/2\mathbb{Z})$, define a cochain τ on $\overline{\Delta}$ by the formulas (9.9) and (9.20) using the σ-modified diamond coordinates (Definition 9.24). Let $\eta\in C^1(\Delta;\mathbb{Z}/2\mathbb{Z})$ be such that $\delta\eta=\sigma$, where δ is the standard coboundary map. By Lemma 9.23 ηc satisfies the $\mathrm{SL}(n,\mathbb{C})$ Ptolemy relations (5.4), and hence corresponds to an $(\mathrm{SL}(n,\mathbb{C}),N)$-cocycle τ'. Since the diamond coordinates of c and ηc are the same, the short edges of τ' agree with those of τ and the long edges differ from those of τ by η. This proves that τ is a lifted (pG,N)-cocycle with obstruction cocycle σ. The inductive arguments of Propositions 9.18 and 9.19 show that this is a 1-1 correspondence. The fact that the actions by coboundaries correspond is immediate from the construction. □

Corollary 9.26. Let $\sigma\in Z^2(K;\mathbb{Z}/2\mathbb{Z})$. There is an algebraic variety $P^\sigma_n(K)$ of generically decorated boundary-unipotent representations $\rho\colon\pi_1(M)\to p\,\mathrm{SL}(n,\mathbb{C})$ whose obstruction class to lifting to $\mathrm{SL}(n,\mathbb{C})$ is represented by σ. Up to canonical isomorphism, the variety $P^\sigma_n(K)$ only depends on the cohomology class of σ.

Proof. This follows immediately from Proposition 9.25. □

Note that the canonical isomorphisms in Corollary 9.26 respect the extended Bloch group element. This follows from the pG variant of Proposition 7.7. The analogue of (9.26) is

$$\Big\{\text{Points in } P^\sigma_n(K)\Big\}\longleftrightarrow\left\{\begin{matrix}\text{Lifted }(pG,N)\text{-cocycles on }M\\ \text{with obstruction cocycle }\sigma\end{matrix}\right\}\xrightarrow{k:1}\left\{\begin{matrix}\text{Generically decorated}\\ (pG,N)\text{-representations}\\ \text{with obstruction cocycle }\sigma\end{matrix}\right\}, \tag{9.31}$$

where k is the number of lifts, i.e. $k=|Z^1(K;\mathbb{Z}/2\mathbb{Z})|$.

9.5. **Proof of Theorems 1.3, 1.12, and 1.7.** Let $\mathcal{R}\colon P_n(K)\to R_{G,N}(M)$ be the composition of the map in (9.26) with the forgetful map ignoring the decoration. The fact that λ has image in $\widehat{\mathcal{B}}(\mathbb{C})$ follows from Proposition 6.12, and commutativity of (1.11) follows from Remark 8.4. The fact that $\mathcal{R}$ is surjective if K is sufficiently fine follows from Proposition 5.4. This concludes the proof of Theorem 1.3. The first part of Theorem 1.12 is proved similarly, and the last part follows from Theorem 11.7 below. The first statement of Theorem 1.7 follows from the definition of $\mathcal{R}$. The second statement follows from the fact that if ρ is boundary non-degenerate the only freedom in choosing a decoration is the diagonal action. Finally, the third statement is proved in Corollary 7.9.

10. Examples

In the examples below, all computations of Ptolemy varieties are exact, whereas the computations of complex volume are numerical with at least 50 digits precision.

Example 10.1 (The 5_2 knot complement)**.** Consider the 3-cycle K obtained from the simplices in Figure 9 by identifying the faces via the unique simplicial attaching maps preserving the arrows. The space obtained from K by removing the 0-cell is homeomorphic to the complement of the 5_2 knot, as can be verified by SnapPy [9].

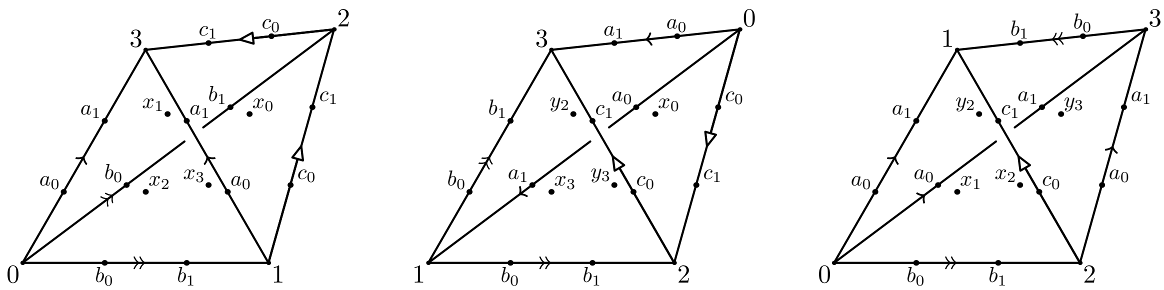

Figure 9. A 3-cycle structure on the 5_2 knot complement, and Ptolemy coordinates for $n = 3$.

Labeling the Ptolemy coordinates as in Figure 9, the Ptolemy variety for $n = 3$ is given by the equations

$$\begin{aligned} &a_0x_3 + b_0x_1 = b_0x_2, && a_0y_3 + a_0x_0 = c_0y_2, && a_0x_2 + b_0y_2 = a_0x_1 \\ &x_2c_0 + b_1x_0 = x_3a_0, && y_2b_0 + a_1x_3 = y_3b_0, && x_1a_0 + b_1y_3 = x_2c_0 \\ &x_1c_1 + x_3c_0 = b_1x_0, && x_0b_1 + y_3c_0 = c_1x_3, && y_2a_1 + x_2b_0 = a_1y_3 \\ &a_1x_0 + x_2c_1 = x_1a_1, && a_1x_3 + y_2c_1 = x_0b_1, && a_1y_3 + x_1b_1 = y_2c_1 \end{aligned} \tag{10.1}$$

together with an extra equation (involving an additional variable t)

$$a_0a_1b_0b_1c_0c_1x_0x_1x_2x_3y_2y_3t = 1, \tag{10.2}$$

making sure that all Ptolemy coordinates are non-zero. By (5.7) a diagonal matrix $\operatorname{diag}(x, y, z)$ acts by multiplying a Ptolemy coordinate on an edge by x^2y and a Ptolemy coordinate on a face by x^3. Since we are not interested in the particular decoration, we may thus assume e.g. that $a_0 = y_3 = 1$. Using Magma [3], one finds that the Ptolemy variety, after setting $a_0 = y_3 = 1$, has three zero-dimensional components with 3, 4 and 6 points respectively. One of these is given by

$$\begin{gathered} a_0 = a_1 = y_3 = 1, \quad x_1 = -1, \quad c_0 = c_1 = x_0^2 + 2x_0 + 1 \\ y_2 = x_0^2 + 2 = -x_2, \quad x_3 = -x_0^2 - x_0 - 1 \\ x_0^3 + x_0^2 + 2x_0 + 1 = 0 \end{gathered} \tag{10.3}$$

Thus, this component gives rise to 3 representations, one for each solution to $x_0^3 + x_0^2 + 2x_0 + 1 = 0$. Using the fact that $R(\lambda(c)) = i\operatorname{Vol}_{\mathbb{C}}(\rho)$, the complex volumes of these can be computed to be

$$0.0 - 4.453818209\ldots i \in \mathbb{C}/4\pi^2 i\mathbb{Z}, \qquad \pm 11.31248835\ldots + 12.09651350\ldots i \in \mathbb{C}/4\pi^2 i\mathbb{Z} \tag{10.4}$$

corresponding to the values $x_0 = -0.5698\dots$ and $x_0 = -0.2150 \mp 1.3071\dots i$, respectively.

In Zickert [31, Section 6], the complex volumes of the Galois conjugates of the geometric representation are computed to be

$$0.0 - 1.113454552\dots i \in \mathbb{C}/\pi^2 i\mathbb{Z}, \qquad \pm 2.828122088\dots + 3.024128376\dots i \in \mathbb{C}/\pi^2 i\mathbb{Z}. \tag{10.5}$$

Notice that (10.4) is (approximately) 4 times (10.5). It thus follows from Theorem 1.10 that the representations given by (10.3) are ϕ_3 composed with the geometric component of PSL$(2,\mathbb{C})$-representations and that the factor of 4 is exact.

Another component is given by

$$\begin{gathered} a_0 = a_1 = y_3 = 1, \quad x_1 = -1, \quad b_1 = -x_0 \\ b_0 = 1/4x_0^3 - 1/4x_0^2 + 3/4x_0 - 1/2 \\ c_0 = c_1 = 1/4x_0^3 - 1/4x_0^2 - 1/4x_0 + 1/2 \\ y_2 = -x_2 = 1/4x_0^3 + 3/4x_0^2 + 7/4x_0 + 3/2 \\ x_3 = -x_0^2 - x_0 - 1 \\ x_0^4 + x_0^3 + x_0^2 - 4x_0 - 4 = 0. \end{gathered} \tag{10.6}$$

In this case there are two distinct complex volumes given by:

$$0.0 + 2.631894506\dots i = \frac{4}{15}\pi^2 i \in \mathbb{C}/4\pi^2 i\mathbb{Z}, \quad 0.0 + 10.527578027\dots i = \frac{16}{15}\pi^2 i \in \mathbb{C}/4\pi^2 i\mathbb{Z}. \tag{10.7}$$

The third component has somewhat larger coefficients, but after introducing a variable u with $u^6 + 5u^4 + 8u^2 - 2u + 1 = 0$, the defining equations simplify to

$$\begin{gathered} a_0 = y_3 = 1, \quad a_1 = 1/4u^5 + 1/4u^4 + 5/4u^3 + 1/2u^2 + 2u - 3/4 \\ b_0 = b_1 = -1/4u^4 - 3/4u^2 - 1/4u - 3/4, \\ c_1 = -1/4u^5 - 3/4u^3 - 1/4u^2 - 3/4u, \\ c_0 = 1/2u^5 + 9/4u^3 + 1/4u^2 + 7/2u - 1/4, \\ y_2 = -8/17u^5 - 1/34u^4 - 79/34u^3 - 3/17u^2 - 105/34u + 26/17, \\ x_3 = 1/17u^5 - 1/17u^4 + 6/17u^3 - 6/17u^2 + 14/17u - 16/17, \\ x_2 = 9/34u^5 + 4/17u^4 + 37/34u^3 + 31/34u^2 + 75/34u + 13/17, \\ x_1 = 8/17u^5 + 1/34u^4 + 79/34u^3 + 3/17u^2 + 139/34u - 9/17, \\ x_0 = 15/34u^5 + 1/17u^4 + 73/34u^3 + 29/34u^2 + 125/34u - 1/17, \\ u^6 + 5u^4 + 8u^2 - 2u + 1 = 0. \end{gathered} \tag{10.8}$$

In this case, there are 3 distinct complex volumes:

$$0.0 + 1.241598704\dots i, \quad \pm 6.332666642\dots + 1.024134714\dots i \tag{10.9}$$

According to Conjecture 1.21, $6.33\dots + 1.02\dots i$ should (up to rational multiples of $\pi^2 i$) be an integral linear combination of complex volumes of hyperbolic manifolds. Using e.g. Snap [18], one checks that the complex volume of the manifold $m034$ is given by

$$3.166333321\dots + 2.157001424\dots i, \tag{10.10}$$

and we have

$$6.332666642\dots + 1.024134714\dots i = 2\,\mathrm{Vol}_{\mathbb{C}}(m034) - \frac{1}{3}\pi^2 i \in \mathbb{C}/4\pi^2 i\mathbb{Z}. \tag{10.11}$$

Example 10.2 (The figure 8 knot complement)**.** Let K be the 3-cycle in Figure 10. Then $M = M(K)$ is the figure 8 knot complement, and $H^2(K;\mathbb{Z}/2\mathbb{Z}) = H^2(M,\partial M;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

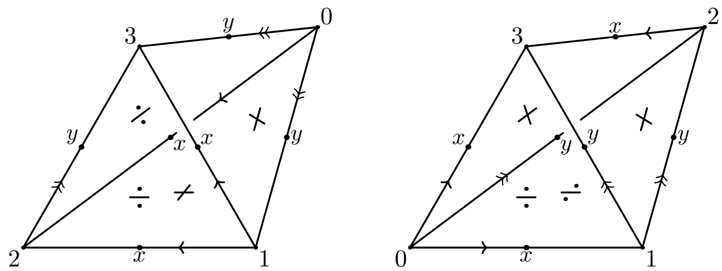

FIGURE 10. A 3-cycle structure on the figure 8 knot complement and Ptolemy coordinates for $n = 2$. The signs indicate the non-trivial second $\mathbb{Z}/2\mathbb{Z}$ cohomology class.

For the trivial obstruction class, the Ptolemy variety for $n = 2$ is given by

$$yx + y^2 = x^2, \qquad xy + x^2 = y^2, \tag{10.12}$$

and is thus empty since x and y are non-zero. In fact, the only boundary-unipotent representations in $\mathrm{SL}(2,\mathbb{C})$ are reducible, so this is not surprising. The non-trivial obstruction class can be represented by the cocycle indicated in Figure 10, and the Ptolemy variety is given by

$$yx - y^2 = x^2, \qquad xy - x^2 = y^2. \tag{10.13}$$

As in Example 10.1, we may assume $y = 1$. Hence, the Ptolemy variety detects two (complex conjugate) representations corresponding to the solutions to $x^2 - x + 1 = 0$. The extended Bloch group elements are

$$-(-\widetilde{x}, -2\widetilde{x}) + (\widetilde{x}, 2\widetilde{x}) \in \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}, \tag{10.14}$$

with complex volume

$$\pm\, 2.029883212\ldots + 0.0i. \tag{10.15}$$

We thus recover the well known complex volume of the figure 8 knot complement.

For $n = 3$, similar calculations as those in Example 10.1 show that the Ptolemy variety detects 3 zero-dimensional components, but the only one with non-zero volume is the one induced by the geometric representation. For $n = 4$, lots of new complex volumes emerge. For the trivial obstruction class, the non-zero complex volumes are

$$\pm\, 7.327724753\ldots + 0.0i = 2\operatorname{Vol}_{\mathbb{C}}(5_1^2) + \pi^2 i/4, \tag{10.16}$$

where the manifold 5_1^2 is the whitehead link complement. For the non-trivial obstruction class, the complex volumes are

$$\begin{gathered}
\pm 20.29883212\ldots + 0.0i = 10\operatorname{Vol}_{\mathbb{C}}(4_1) \in \mathbb{C}/\pi^2 i\mathbb{Z} \\
\pm 4.260549384\ldots \pm 0.136128165\ldots i \\
\pm 3.230859569\ldots + 0.0i \\
\pm 8.355502146\ldots + 2.428571615\ldots i = \operatorname{Vol}_{\mathbb{C}}(-9_{15}^3) + 2\pi^2 i/3 \\
\pm 3.276320849\ldots + 9.908433886\ldots i.
\end{gathered} \tag{10.17}$$

Example 10.3 ($S^1 \times S^2$)**.** Figure 11 shows a triangulation of $M = S^1 \times S^2$ taken from the Regina census [4]. Since $\pi_1(S^1 \times S^2) = \mathbb{Z}$, all representations in $\mathrm{PSL}(2,\mathbb{C})$ lift to $\mathrm{SL}(2,\mathbb{C})$, so we expect the Ptolemy variety for the non-trivial class in $H^2(M;\mathbb{Z}/2\mathbb{Z})$ to be zero. This class is represented by the cocycle shown in Figure 11, and the Ptolemy variety is given by

$$-zx + x^2 = y^2, \qquad x^2 + zx = y^2, \tag{10.18}$$

which indeed has no solutions in $\mathbb{C}^*$. For the trivial cohomology class, all signs are positive, and the two equations are equivalent. The extended Bloch group element is

$$(\widetilde{z} + \widetilde{x} - 2\widetilde{y}, 2\widetilde{x} - 2\widetilde{y}) - (\widetilde{z} + \widetilde{x} - 2\widetilde{y}, 2\widetilde{x} - 2\widetilde{y}) = 0 \in \widehat{\mathcal{B}}(\mathbb{C}). \tag{10.19}$$

In fact, the extended Bloch group element of a Ptolemy assignment is trivial for all n, as one easily verifies (the subsimplices cancel out in pairs).

We wish to find out which representations are detected by $P_2(K)$. A choice of fundamental domain F for K in L determines a presentation of $\pi_1(M)$ with a generator for each face pairing of F and a relation for each 1-cell of K (to see this consider the standard presentation for the dual triangulation of K). Letting F be the fundamental domain of $S^1 \times S^2$ given by gluing the bottom faces of the two simplices together, one easily checks that the generator of $\pi_1(M) = \mathbb{Z}$ is given by the self gluing of the first simplex taking the face opposite the third vertex to the face opposite the zeroth. For $\alpha \in \mathrm{SL}(2,\mathbb{C})$, the representation given by taking the generator to α has a decoration as in Figure 11. For $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, let $c(A) = c$, and note that $\det(e_1, Ae_1) = c(A)$. Letting x, y and z denote the Ptolemy coordinates, we have

$$x = c(\alpha), \quad y = c(\alpha^2) = x\mathrm{Tr}(\alpha), \quad z = c(\alpha^3) = x(\mathrm{Tr}(\alpha)^2 - 1), \tag{10.20}$$

and it follows that the Ptolemy variety detects all representations except those where $\mathrm{Tr}(\alpha) = \pm 1$.

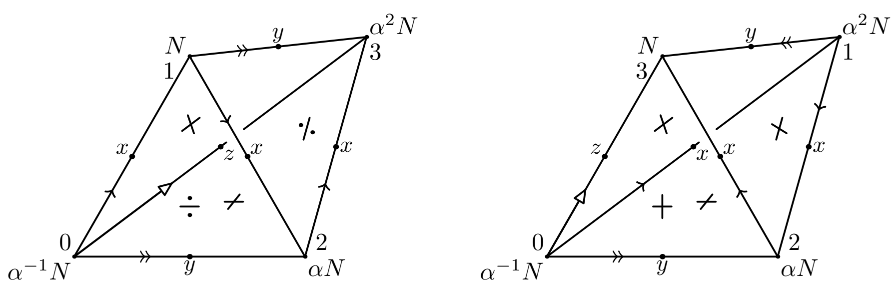

FIGURE 11. A triangulation of $S^1 \times S^2$. Both simplices have self gluings.

Remark 10.4. When $n = 2$, examples of Conjecture 1.21 are abundant. E.g. for the 10_{155} knot complement (10 simplices), the volumes of the representations detected by the Ptolemy variety are (numerically)

$$\mathrm{Vol}(m032(6,1)), \quad 2\,\mathrm{Vol}(4_1), \quad 3\,\mathrm{Vol}(10_{155}) - 4\,\mathrm{Vol}(v3461), \quad \mathrm{Vol}(10_{155}). \tag{10.21}$$

Remark 10.5. For the hyperbolic census manifolds, most of the components of the reduced Ptolemy varieties tend to be zero-dimensional. By a result of Menal-Ferrer and Porti [20], the composition of the geometric representation with ϕ_n is isolated among boundary-unipotent $p\,\mathrm{SL}(n,\mathbb{C})$-representations. Higher dimensional components also occur (rarely for $n = 2$, quite often for $n > 2$), but as mentioned earlier, the complex volume is constant on components.

Remark 10.6. If the face pairings do not respect the vertex orderings, one can still define a Ptolemy variety by introducing more signs. See Garoufalidis–Goerner–Zickert [15] for details.

Remark 10.7. The fact that the reduced Ptolemy varieties $P_n(K)_{\mathrm{red}}$ are given by setting some of the variables (chosen appropriately) equal to 1 is proved in [16].

11. The irreducible representations of $\mathrm{SL}(2,\mathbb{C})$

Let $\phi_n\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ denote the canonical irreducible representation. It is induced by the Lie algebra homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ given by

(11.1)
$$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \mapsto \mathrm{diag}^+(n-1,\dots,1), \quad \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \mapsto \mathrm{diag}^-(1,\dots,n-1), \quad \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \mapsto \mathrm{diag}(n-1,n-3,\dots,-n+1),$$

where $\mathrm{diag}^+(v)$ and $\mathrm{diag}^-(v)$ denote matrices whose first upper (resp. lower) diagonal is v and all other entries are zero. One has

$$\phi_n\left(\left[\begin{smallmatrix} 0 & -a^{-1} \\ a & 0 \end{smallmatrix}\right]\right) = q(a^{n-1}, -a^{n-3}, \dots, (-1)^{n-1}a^{-(n-1)}) \tag{11.2}$$

$$\phi_n\left(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\right) = \pi_{n-1}(x,\dots,x)\pi_{n-2}(x,\dots,x)\cdots\pi_1(x). \tag{11.3}$$

Proposition 11.1. Let c be a Ptolemy assignment on Δ^3_2, and let τ denote the corresponding cocycle. The Ptolemy assignment corresponding to $\phi_n(\tau)$ is given by

$$\phi_n(c)\colon \dot\Delta^3_n(\mathbb{Z}) \to \mathbb{C}^*, \qquad t \mapsto \phi_n(c)_t = \prod_{i<j} c_{ij}^{t_i t_j}. \tag{11.4}$$

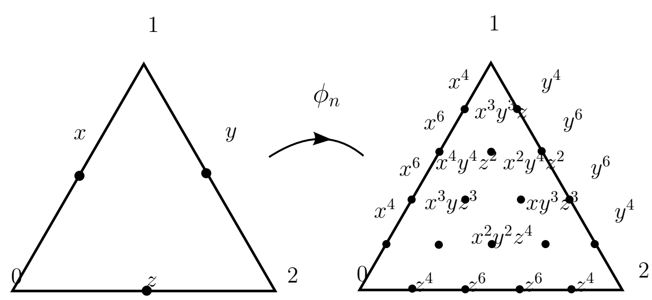

Figure 12. ϕ_n acting on Ptolemy assignments.

Proof. Let $\alpha = (a_0,\dots,a_3) \in \Delta^3_{n-2}(\mathbb{Z})$. Letting $k_\alpha = \prod_{i<j} c_{ij}^{a_i a_j}$, and $l_\alpha = \prod_{i<j} c_{ij}^{a_i + a_j}$, we have

(11.5)
$$\phi_n(c)_{\alpha_{03}}\phi_n(c)_{\alpha_{12}} = k_\alpha^2 l_\alpha c_{03}c_{12}, \quad \phi_n(c)_{\alpha_{01}}\phi_n(c)_{\alpha_{23}} = k_\alpha^2 l_\alpha c_{01}c_{23}, \quad \phi_n(c)_{\alpha_{02}}\phi_n(c)_{\alpha_{13}} = k_\alpha^2 l_\alpha c_{02}c_{13}.$$

Hence, the appropriate Ptolemy relations are satisfied. The long and short edges of the cocycle corresponding to $\phi_n(c)$ are given by (9.9) and (9.20), and we must prove that these agree with those of $\phi_n(\tau)$. For the long edges, this follows immediately from (11.2). For the short edges, an easy computation shows that all the diamond coordinates of a face are equal, and equal to the corresponding diamond coordinate of c. For example, the type 1 diamond coordinate on face 3 whose left vertex is $t = (t_0, t_1, t_2, 0)$ is given by

$$\begin{aligned} \frac{\phi_n(c)_{t+(0,-1,1,0)}\phi_n(c)_{t+(-1,1,0,0)}}{\phi_n(c)_t\phi_n(c)_{t+(-1,0,1,0)}} &= \frac{c_{01}^{t_0(t_1-1)}c_{02}^{t_0(t_2+1)}c_{12}^{(t_1-1)(t_2+1)}c_{01}^{(t_0-1)(t_1+1)}c_{02}^{(t_0-1)t_2}c_{12}^{(t_1+1)t_2}}{c_{01}^{t_0t_1}c_{02}^{t_0t_2}c_{12}^{t_1t_2}c_{01}^{(t_0-1)t_1}c_{02}^{(t_0-1)(t_2+1)}c_{12}^{t_1(t_2+1)}} \\ &= \frac{c_{02}}{c_{01}c_{12}}, \end{aligned} \tag{11.6}$$

which is a diamond coordinate for c. By (11.3) the short edges thus agree with those of $\phi_n(\tau)$, proving the result. ∎

Corollary 11.2. If a representation $\rho\colon \pi_1(M)\to \mathrm{PSL}(2,\mathbb{C})$ is detected by $P_2^\sigma(K)$ then $\phi_{2k+1}\circ\rho$ is detected by $P_{2k+1}(K)$ and $\phi_{2k}\circ\rho$ is detected by $P_{2k}^\sigma(K)$. □

Theorem 11.3. *Let ρ be a boundary-unipotent representation in* $\mathrm{SL}(2,\mathbb{C})$ *or* $\mathrm{PSL}(2,\mathbb{C})$*. The extended Bloch group element of $\phi_n\circ\rho$ is $\binom{n+1}{3}$ times that of ρ. In fact, the shapes of all subsimplices are equal.*

Proof. By refining the triangulation if necessary, we may represent ρ by a Ptolemy assignment c on K. Then $\phi=\phi_n(c)$ is a Ptolemy assignment representing $\phi_n\circ\rho$, and the extended Bloch group element of $\phi_n\circ\rho$ is given by

$$[\phi_n(\rho)]=\sum_i\epsilon_i\sum_{\alpha\in\Delta^3_{n-2}(\mathbb{Z})}(\widetilde{\phi}^i_{\alpha_{03}}+\widetilde{\phi}^i_{\alpha_{12}}-\widetilde{\phi}^i_{\alpha_{02}}-\widetilde{\phi}^i_{\alpha_{13}},\widetilde{\phi}^i_{\alpha_{01}}+\widetilde{\phi}^i_{\alpha_{23}}-\widetilde{\phi}^i_{\alpha_{02}}-\widetilde{\phi}^i_{\alpha_{13}}). \tag{11.7}$$

By Proposition 7.7, we may choose the logarithms independently as long as we use the same logarithm for identified points. Defining $\widetilde{\phi}^i_t=\sum_{j<k}t_jt_k\widetilde{c}^i_{jk}$, we see that

$$(\widetilde{\phi}^i_{\alpha_{03}}+\widetilde{\phi}^i_{\alpha_{12}}-\widetilde{\phi}^i_{\alpha_{02}}-\widetilde{\phi}^i_{\alpha_{13}},\widetilde{\phi}^i_{\alpha_{01}}+\widetilde{\phi}^i_{\alpha_{23}}-\widetilde{\phi}^i_{\alpha_{02}}-\widetilde{\phi}^i_{\alpha_{13}})=(\widetilde{c}_{03}+\widetilde{c}_{12}-\widetilde{c}_{02}-\widetilde{c}_{13},\widetilde{c}_{01}+\widetilde{c}_{23}-\widetilde{c}_{02}-\widetilde{c}_{13}), \tag{11.8}$$

which means that the flattenings assigned to each subsimplex of Δ^i_n are equal. By Lemma 5.6, $|\Delta^3_{n-2}(\mathbb{Z})|=\binom{n+1}{3}$, and the result follows. □

11.1. **Essential edges.**

Definition 11.4. An edge of K is *essential* if the lifts to L have distinct end points.

Note that an edge may be essential even though it is homotopically trivial in K. Let $L^{(0)}$ denote the zero skeleton of L.

Lemma 11.5. Let ρ be a representation in $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$. A decoration of ρ determines a ρ-equivariant map

$$D\colon L^{(0)}\to\partial\overline{\mathbb{H}}^3=\mathbb{C}\cup\{\infty\},\quad e_i\mapsto g_i\infty. \tag{11.9}$$

Every such map comes from a decoration, and the decoration is generic if and only if the vertices of each simplex of L map to distinct points in $\mathbb{C}\cup\{\infty\}$.

Proof. Equivariance of (11.9) follows from the definition of a decoration. A ρ-equivariant map $D\colon L^{(0)}\to\mathbb{C}\cup\{\infty\}$ is uniquely determined by its image of lifts $\widetilde{e}_i\in L$ of the zero cells e_i of K. Picking g_i such that $g_i\infty=D(\widetilde{e}_i)$, we define a decoration by assigning the coset g_iN to $\widetilde{e}_i$. The last statement follows from the fact that $\det(g_1e_1,g_2e_1)=0$ if and only if $g_1\infty=g_2\infty$. □

In the following we assume that the interior of M is a cusped hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ with finite volume.

Proposition 11.6. If all edges of K are essential, the geometric representation has a generic decoration.

Proof. We identify $\pi_1(M)$ with $\Gamma\subset\mathrm{PSL}(2,\mathbb{C})$. Each cusp of M determines a Γ-orbit of points in $\partial\overline{\mathbb{H}^3}$, and these orbits are distinct (if two orbits intersected, they would be identical, thus corresponding to the same cusp). Each vertex of L corresponds to either a cusp of M or an interior point of M. Accordingly, we have $L^{(0)}=L^{(0)}_{\mathrm{cusp}}\cup L^{(0)}_{\mathrm{int}}$. The stabilizer of a point in $L^{(0)}_{\mathrm{cusp}}$ is a parabolic subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and thus fixes a unique point in $\mathbb{C}\cup\{\infty\}$. We thus have an equivariant map $D\colon L^{(0)}_{\mathrm{cusp}}\to\mathbb{C}\cup\{\infty\}$ taking a vertex v to the fixed point in $\partial\mathbb{H}^3$ of $\mathrm{Stab}(v)\subset\mathrm{PSL}(2,\mathbb{C})$. Let e_1 and e_2 be points in $L^{(0)}_{\mathrm{cusp}}$ connected by an edge. Since all edges of K are essential, $e_1\neq e_2$. Since the Γ-orbits of different cusps are distinct, it follows that $D(e_1) \neq D(e_2)$ if e_1 and e_2 correspond to different cusps. If e_1 and e_2 correspond to the same cusp, there exists an element in Γ taking e_1 to e_2. Since only peripheral elements (i.e. cusp stabilizers) have fixed points in $\mathbb{C} \cup \{\infty\}$, it follows that $D(e_1) \neq D(e_2)$. We extend D to $L^{(0)}$ by choosing any equivariant map $L^{(0)}_{\text{int}} \to \mathbb{C} \cup \{\infty\}$. Since such map is uniquely determined by finitely many values (which may be chosen freely), we can pick the extension so that the vertices of each simplex map to distinct points. This proves the result. ☐

Theorem 11.7. *Suppose all edges of K are essential. The representation $\phi_n \circ \rho_{\text{geo}}$ is detected by $P_n(K)$ if n is odd, and by $P_n^{\sigma_{\text{geo}}}(K)$ if n is even.*

Proof. By Proposition 11.6, $P_2^{\sigma_{\text{geo}}}(K)$ detects ρ_{geo}. The result now follows from Corollary 11.2. ☐

Remark 11.8. The census triangulations all have essential edges.

12. Gluing equations and Ptolemy assignments

In this section we discuss the relation between Ptolemy assignments and solutions to the gluing equations. The latter were invented by Thurston [28] to explicitly compute the hyperbolic structure (and its deformations) of a triangulated hyperbolic manifold, and used effectively in [23, 18, 9]. The gluing equations make sense for any 3-cycle. They are defined by assigning a *cross-ratio* $z_i \in \mathbb{C} \backslash \{0, 1\}$ to each simplex Δ_i of K. Given these, we assign cross-ratio parameters to the edges of Δ_i as in Figure 13.

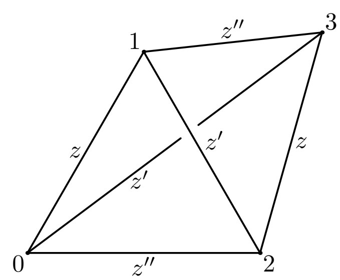

Figure 13. Assigning cross-ratio parameters to the edges of Δ_i. By definition, $z' = \frac{1}{1-z}$ and $z'' = 1 - \frac{1}{z}$.

There is a gluing equation for each edge E in K and each generator γ of the fundamental group of each boundary component of M. These are given by

$$\prod_{e \mapsto E} z(e)^{\epsilon_i(e)} = 1, \qquad \prod_{\gamma \text{ passes } e} z(e)^{\epsilon_i(e)} = 1. \tag{12.1}$$

Here $z(e)$ denotes the cross-ratio parameter assigned to e, and $\epsilon_i(e) = \epsilon_i$ if e is an edge of Δ_i. It follows that the set of assignments $\Delta_i \mapsto z_i \in \mathbb{C} \backslash \{0, 1\}$ satisfying the gluing equations (12.1) is an algebraic set $V(K)$.

Lemma 12.1. For every point $\{z_i\} \in V(K)$ there is a map $D\colon L^{(0)} \to \mathbb{C} \cup \{\infty\}$ such that if $\widetilde{\Delta}_i$ is a lift of Δ_i with vertices $e_1, \dots, e_3$ in L, the cross-ratio of the ideal simplex with vertices $D(e_1), \dots, D(e_3)$ is z_i. It is unique up to multiplication by an element in $\mathrm{PSL}(2, \mathbb{C})$. Moreover, there is a unique (up to conjugation) boundary-unipotent representation $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ such that D is ρ-equivariant.

Proof. Pick a fundamental domain F for K in L. Pick a simplex Δ in F and define D by mapping the first 3 vertices of Δ to 0, ∞ and 1. The map D is now uniquely determined by the cross-ratios. The fundamental group of M has a presentation with a generator for each face pairing of F. The second statement thus follows from the fact that $\mathrm{PSL}(2,\mathbb{C})$ is 3-transitive. We leave the details to the reader. □

Given a Ptolemy assignment on K, we assign the cross-ratio $z_i = \frac{c^i_{03}c^i_{12}}{c^i_{02}c^i_{13}}$ to Δ_i. Note that the Ptolemy relations imply that the cross-ratio parameters are given by

$$z_i = \frac{c^i_{03}c^i_{12}}{c^i_{02}c^i_{13}}, \qquad z'_i = \frac{c^i_{02}c^i_{13}}{c^i_{01}c^i_{23}}, \qquad z''_i = -\frac{c^i_{01}c^i_{23}}{c^i_{03}c^i_{12}}. \tag{12.2}$$

Theorem 12.2. *There is a surjective regular map*

$$\coprod_{\sigma\in H^2(K;\mathbb{Z}/2\mathbb{Z})} P^\sigma_2(K) \to V(K), \quad c \mapsto \{z_i = \frac{c^i_{03}c^i_{12}}{c^i_{02}c^i_{13}}\}. \tag{12.3}$$

The fibers are disjoint copies of $(\mathbb{C}^*)^h$, *where* h *is the number of zero-cells of* K.

Proof. By a simple cancellation argument (as in the proof of Zickert [31, Theorem 6.5]), the gluing equations would be satisfied if the formula (12.2) for z''_i did not have the minus sign. The minus sign appears whenever the edge is 02 or 13. As explained in the proof of Proposition 7.7, any curve passes these an even number of times. It thus follows that the cross-ratios satisfy the gluing equations. Surjectivity follows from Lemma 11.5, and the fact that fibers are $(\mathbb{C}^*)^h$ follows from the fact that $g_1\infty = g_2\infty$ if and only if $g_1N = g_2dN$ for a unique diagonal matrix d. □

Remark 12.3. Gluing equation varieties for $n > 2$ are studied in Garoufalidis-Goerner-Zickert [15].

13. Other fields

The Ptolemy varieties $P_n(K)$ and $P^\sigma_n(K)$ may be defined over an arbitrary field F, and as in Section 9, a Ptolemy assignment determines a boundary-unipotent representation in $\mathrm{SL}(n,F)$, respectively, $p\,\mathrm{SL}(n,F)$. If E is a primitive extension of F^* by $\mathbb{Z}$, there are maps

$$P_n(K)_F \to \widehat{\mathcal{B}}_E(F), \qquad P^\sigma_n(K)_F \to \widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}} \tag{13.1}$$

defined as in (5.10) using a set theoretic section of $E \to F^*$ instead of a logarithm. If F is infinite, the chain complex of Ptolemy assignments computes relative homology (see Proposition 9.6) and we have maps

$$H_3(\mathrm{SL}(n,F)) \to \widehat{\mathcal{B}}_E(F), \qquad H_3(p\,\mathrm{SL}(n,F)) \to \widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}}. \tag{13.2}$$

It thus follows that every boundary-unipotent representation has an extended Bloch group element $[\rho]$. If F is a number field, the extended Bloch groups are independent of the extension E.

Theorem 13.1. *Let F be a number field, and let $\rho\colon \pi_1(M) \to \mathrm{SL}(n,F)$ be a boundary-unipotent representation. If ρ is irreducible, $[\rho]$ lies in $\widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$.*

Proof. Let σ be an automorphism of F over $\mathrm{Tr}(\rho)$ and let $\tau\colon F \to \mathbb{C}$ be an embedding. Then ρ and $\sigma\circ\rho$ have the same traces, so $\tau\circ\rho$ and $\tau\circ\sigma\circ\rho$ are conjugate in $\mathrm{SL}(n,\mathbb{C})$, and thus have the same extended Bloch group element in $\widehat{\mathcal{B}}(\mathbb{C})$. By Corollary 3.6, it follows that $[\rho] = [\sigma\circ\rho] \in \widehat{\mathcal{B}}(F)$. Hence, $[\rho]$ is invariant under all automorphisms of F over $\mathrm{Tr}(\rho)$, so $[\rho] \in \widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$ by Galois descent. □

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