CONCORDANCE AND 1-LOOP CLOVERS

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ABSTRACT. We show that surgery on a connected clover (or clasper) with at least one loop preserves the concordance class of a knot. Surgery on a slightly more special class of clovers preserves invertible concordance. We also show that the converse is false. Similar results hold for clovers with at least two loops vs. S-equivalence.

1. INTRODUCTION

1.1. History. M. Goussarov and K. Habiro have independently studied links and 3-manifolds from the point of view of surgery on objects called *Y-graphs, claspers or clovers*, respectively by [Gu, H] and [GGP]. Following the notation of [GGP], given a pair (M, K) consisting of a knot K in an integral homology 3-sphere M, and a clover $G \subset M - K$, surgery on the framed link associated to G produces a new pair $(M, K)_G$. Thus, by specifying a class of clovers $\mathfrak{c}$ we can define an equivalence relation (also denoted by $\mathfrak{c}$) on the set $\mathcal{KM}$ of knots in integral homology 3-spheres and sometimes on its subset $\mathcal{K}$ of knots in S^3.

It is often the case that for certain classes of clovers $\mathfrak{c}$, the equivalence relation is related to some natural topological equivalence relation. In this paper we will be particularly interested in *concordance* (in the smooth category) but will also discuss S-equivalence.

We begin by discussing some known facts. Using the terminology of [GGP], let $\mathfrak{c}^{\Delta\Delta}$ denote the class of clovers $G \subset S^3 - K$ of degree 1 (that is, the class of Y-graphs) whose leaves form a 0-framed unlink which bounds disks disjoint from G that intersect K geometrically twice and algebraically zero times. Surgery on such clovers was called a *double* Δ*elta* move by Naik-Stanford, who showed that

Theorem 1. [NS] *$\mathfrak{c}^{\Delta\Delta}$ coincides with S-equivalence on $\mathcal{K}$.*

Relaxing the above condition, let $\mathfrak{c}^{\mathrm{loop}}$ denote the class of clovers $G \subset M - K$ whose leaves have zero linking number with K. Surgery on such clovers was called a *loop move* by G.-Rozansky who showed that

Theorem 2. [GR] *$\mathfrak{c}^{\mathrm{loop}}$ coincides with S-equivalence on $\mathcal{KM}$.*

Let us make the following definition. If G is a clover in $M - K$ and $\mathcal{L}$ a set of leaves of G, we say $\mathcal{L}$ is *simple* if the elements of $\mathcal{L}$ bound disks in M each of which intersects K exactly once but whose interiors otherwise are disjoint from K, G and each other. Consider now for every non-negative integer n, the class $\mathfrak{c}^n$ of clovers $G \subset S^3 - K$ whose entire set of leaves is simple, and such that each connected component of G is a graph with at least n loops (i.e., whose first betti number is at least n). Kricker and Murakami-Ohtsuki showed that

Theorem 3. [Kr, MO] *$\mathfrak{c}^2$ implies S-equivalence on $\mathcal{K}$.*

In fact, if we let $\mathfrak{c}^{iv}$ denote the class of clovers G such that each component of G has at least one internal trivalent vertex, and G has a simple set of leaves containing one leaf from each component, then it is not hard to check that $\mathfrak{c}^2 \subset \mathfrak{c}^{iv}$ and [Kr, MO] actually proved that $\mathfrak{c}^{iv}$ implies S-equivalence. Combining this with a recent result of Conant-Teichner [CT] we actually have:

Theorem 4. [CT] *$\mathfrak{c}^{iv}$ coincides with S-equivalence on $\mathcal{K}$.*

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1.2. **Statement of the results.** In the present paper we will prove the following results.

Theorem 5. $\mathfrak{c}^1$ *implies concordance on* $\mathcal{K}$.

An different proof of Theorem 5 has been obtained by Conant-Teichner [CT] relying on the notion of *grope cobordism*. This result was also announced by the first author in [Le2], where an analogous statement was proved, and our proof will follow the lines of that argument. The result was also known to Habiro, according to private communication.

A slight refinement of the class $\mathfrak{c}^1$ relates to a classical refinement of concordance known as *invertible concordance*. Recall that a knot in S^3 is called *double-slice* if it can be exhibited as the intersection of a 3-dimensional hyperplane in $\mathbb{R}^4$ with an *unknotted* imbedding of S^2 in $\mathbb{R}^4$; see e.g. [Su]. Such knots are obviously slice, and it is shown in [Su] that, for any knot K, the connected sum $K\sharp(-K)$ is double-slice, where $-K$ denotes the mirror image of K. On the other hand the Stevedore knot is slice but not double-slice (see [Su]). More generally, following [Su], we say that K is *invertibly concordant* to K' if there is a concordance V from K to K' and a concordance W from K' to K so that if we stack W on top of V, the resulting concordance from K to itself is diffeomorphic to the product concordance $(I \times S^3, I \times K)$. If we write $K \leq K'$, then $\leq$ is transitive and reflexive and perhaps even a partial ordering. It is easy to see that $0 \leq K$, where 0 denotes the trivial knot, if and only if K is double-slice.

Let $\mathfrak{c}^{1,\mathrm{nf}}$ denote the subclass of $\mathfrak{c}^1$ consisting of clovers with *no forks*— a fork is a trivalent vertex two of whose incident edges contain a univalent vertex. Then, we will prove:

Theorem 6. *If* G *is a clover in the class* $\mathfrak{c}^{1,\mathrm{nf}}$ *and* K' *is obtained from* K *by surgery on* G *then* $K \leq K'$.

It is natural to ask whether the converses to Theorems 3, 5 and 6 are true. If that were the case, one could extract from the rational functions invariants of [GK] many concordance invariants of knots. It was a bit of a surprise for us to show that the converses are all false.

First of all, it will follow easily from a recent result of Livingston that:

Proposition 1.1. *There are* S*-equivalent knots which are not* $\mathfrak{c}^2$*-equivalent.*

Then we will generalize some techniques of Kricker to prove:

Theorem 7. *There are double-slice knots which are not* $\mathfrak{c}^1$*-equivalent to the unknot.*

Remark 1.2. The proofs of Proposition 1.1 and Theorem 7 allow one to easily construct specific knots with the desired properties. See [Li, Theorem 10.1] for knots that satisfy Proposition 1.1. For the $(5,2)$-torus knot $T_{5,2}$, we have that $T_{5,2}\sharp(-T_{5,2})$ is a knot that satisfies Theorem 7.

1.3. **Plan of the proof.** Theorems 5 and 6 follow from an analysis of the surgery link corresponding to a clover.

Proposition 1.1 follows easily from the fact (proven recently by Livingston [Li], using Casson-Gordon invariants) that S-equivalence does not imply concordance.

Theorem 7 follows from the fact that under surgery on $\mathfrak{c}^1$-clovers, the Alexander polynomial changes under a more restrictive way than under a concordance.

2. Proofs

2.1. **Proof of Theorem 5.** Suppose that G is a connected clover of class $\mathfrak{c}^1$ and L is its associated framed link, [Gu, H, GGP]. We want to show that the knot K' obtained from K by surgery on L is concordant to K. Note that the manifold M obtained from S^3 by surgery on L is diffeomorphic to S^3, see [Gu, H, GGP].

Lemma 2.1. *We can express* L *as a union of two sublinks* L' *and* L'' *such that:*

- L' *is a trivial* 0*-framed link in* $S^3 - K$,
- L'' *is a trivial* 0*-framed link in* S^3.

Assuming this lemma we can complete the proof of Theorem 5 as follows.

Consider $I \times K \subset I \times S^3$ and $\frac{1}{2} \times L \subset \frac{1}{2} \times (S^3 - K)$. Consider a union of disjoint disks D' in $\frac{1}{2} \times (S^3 - K)$ bounded by L' and push their interiors into $[0, \frac{1}{2}) \times (S^3 - K)$. Also consider a union of disjoint disks D'' in $\frac{1}{2} \times S^3$ bounded by L'' and push their interiors into $(\frac{1}{2}, 1] \times S^3$. Now let $X \subset I \times S^3$ be obtained from $[0, \frac{1}{2}] \times S^3$ by removing a tubular neighborhood of D' and adjoining a tubular neighborhood of D''. The boundary of X consists of $0 \times S^3$ and a copy of M, which is diffeomorphic to S^3. Thus X is diffeomorphic to $I \times S^3$ (indeed, add a D^4 to X along $0 \times S^3$ and observe that any two imbeddings of a 4-disk in a fixed 4-disk are isotopic). Moreover X contains $[0, \frac{1}{2}] \times K$, which is a concordance from $0 \times K \subset 0 \times S^3$ to $\frac{1}{2} \times K \subset M$, which is just K'. □

Proof of Lemma 2.1. This is a generalization of the argument used to prove Theorem 2 in [Le2]. Recall (eg. from [GGP, Section 2.3]) that surgery on a clover G with n edges corresponds to surgery on a link L of $2n$ components. Given an orientation of the edges of G, we can split L into the disjoint union of n-component sublinks L' and L'', where L' (resp. L'') consists of the sublink of L assigned to the tails of the edges of G (resp. of the heads of the edges of G, together with the leaves of G). As long as we avoid assigning all three of the components at a trivalent vertex to L' or L'', we will have the desired decomposition of L. The corresponding conditions imposed on the orientation of the edges of G are:

(1) No trivalent vertex is a source or a sink,
(2) Every edge with a univalent vertex is oriented toward the univalent vertex.

These are the same conditions as (i) and (ii) in the proof of Theorem 2 in [Le2] except that we now require no trivalent sinks also. But this will follow by the same argument as in [Le2] except that we need to choose the orientations of the cut edges more carefully. In particular we need to avoid choosing the orientation of two cut edges which share a trivalent vertex so that they both point into that vertex. But it is not hard to see that this can be done. □

The next two remarks are an addendum to Theorem 5.

Remark 2.2. Observe that the sublinks L' and L'' of L which are constructed from G have the same number of components, and that the linking matrix of L is hyperbolic. Lemma 2.1 is analogous to the case of a knot which bounds a Seifert surface with a metabolic Seifert surface. In that case, the knot is algebraically slice, and if a metabolizer can be chosen to be bands of the Seifert surface that form a slice link, then the knot is slice.

Remark 2.3. Suppose that a knot K' is obtained from the unknot K by surgery on a connected clover of class $\mathfrak{c}^1$. It follows from Theorem 5 that K' is slice. Using the calculus of clovers, one can show that K' is actually ribbon, as observed also by Kricker and Habiro.

2.2. **Proof of Theorem 6.** We need a refinement of Lemma 2.1. Consider a connected clover G of class $\mathfrak{c}^{1,\mathrm{nf}}$ and let L be its associated framed link.

Lemma 2.4. *There is a link $\bar{L}$ in $S^3 - K$, Kirby equivalent to L in $S^3 - K$, so that $\bar{L}$ is a union of two sublinks $\bar{L}', \bar{L}''$, each of which is trivial in $S^3 - K$.*

Assuming this lemma, we finish the proof following the lines of the argument following Lemma 2.1. The only difference is that we now use $\bar{L}$ instead of L and that $X' = \overline{I \times S^3 - X}$, which is also diffeomorphic to $I \times S^3$, now also contains $[\frac{1}{2}, 1] \times K$. Thus M splits the trivial concordance from K to itself. This, by definition, means $K \leq K'$. □

Proof of Lemma 2.4. For each univalent vertex of G, there is a corresponding part of L which looks like the left part of Figure 1.

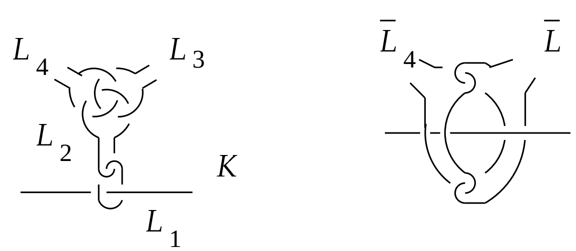


Figure 1. The associated link of a clover near a univalent vertex which is not a fork, before and after a Kirby move.

Now we can perform a Kirby move (see [Kr],[MO]) so that the four component link $\{L_1, \ldots, L_4\}$ in Figure 1 is replaced by two component link $\{\overline{L}_3, \overline{L}_4\}$. If we do this at every univalent vertex of G we obtain the link $\bar{L}$. Now consider the partition $L = L' \cup L''$ given by Lemma 2.1. The corresponding partition of $\overline{L}$ is given by $\bar{L}' = \{\bar{K} | K \in L' - \{L_1, L_2\}\}$ and $\bar{L}'' = \{\bar{K} | K \in L'' - \{L_1, L_2\}\}$. It is easy to see that both $\bar{L}'$ and $\bar{L}''$ are trivial in $S^3 - K$. This completes the proof. □

2.3. **Proof of Proposition 1.1.** Assume that S-equivalence implies $\mathfrak{c}^2$ on $\mathcal{K}$. Since $\mathfrak{c}^2$ implies $\mathfrak{c}^1$, and $\mathfrak{c}^1$ implies concordance (by Theorem 5), it follows that S-equivalence implies concordance. This is false. Livingston using Casson-Gordon invariants, shows that there are S-equivalent knots which are algebraically slice, but not slice, [Li, Theorem 0.4]. Since Livingston uses Casson-Gordon invariants, his examples have nontrivial Alexander module.

2.4. **Proof of Theorem 7.** We show that the Alexander polynomial Δ of a knot changes in a more restrictive way under $\mathfrak{c}^1$-equivalence than under concordance. Recall that if K and K' are concordant knots, then their Alexander polynomials satisfy $\Delta_{K'}(t)\theta'(t)\theta'(t^{-1}) = \Delta_K(t)\theta(t)\theta(t^{-1})$ for some $\theta(t), \theta'(t) \in \mathbb{Z}[t, t^{-1}]$ satisfying $\theta(1) = \theta'(1) = \pm 1$. Moreover, there are double-slice knots with Alexander polynomial $\theta(t)\theta(t^{-1})$ for any such θ. On the other hand,

Lemma 2.5. *Let K and K' be $\mathfrak{c}^1$-equivalent knots. Then,*

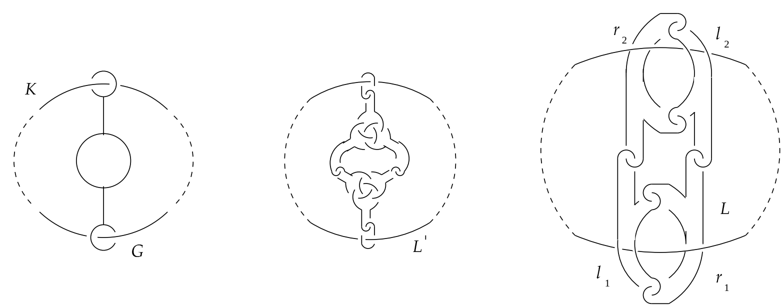
$$\Delta_{K'}(t)\theta'(t)\theta'(t^{-1}) = \Delta_K(t)\theta(t)\theta(t^{-1})$$

where $\theta(t)$ and $\theta'(t)$ are products of polynomials of the form $1 \pm t^k(t-1)^n$ for some integers k, n with $n > 0$.

Proof. We prove this using a generalization of an argument of Kricker [Kr]. Consider a connected clover G of the class $\mathfrak{c}^1$. Suppose that K' is obtained from K by surgery on G. If G has at least one internal trivalent vertex, then K and K' are S-equivalent (see the discussion following Theorem 3); in particular $\Delta_K(t) = \Delta_{K'}(t)$. Otherwise, G must be a *wheel* with a certain number n of legs and with a total of $2n$ edges. Thus, the associated link L' in $S^3 - K$ has $4n$ components (see Figure below). Using the Kirby move in Figure 1 at every leaf of G we see that L' is Kirby-equivalent in $S^3 - K$ to a link L with $2n$ components, whose components can be numbered in pairs $l_1, r_1, \ldots, l_n, r_n$ so that:

(1) l_i (resp. r_i) bounds a disk d_i (resp. e_i) in $S^3 - K$,
(2) $d_i \cap e_i$, for $1 \leq i \leq n$, each consists of two oppositely oriented clasps,
(3) $e_i \cap d_{i+1}$, for $1 \leq i < n$ and $e_n \cap d_1$ each consists of a single clasp, and
(4) there are no other intersections among the disks.

An example for $n = 2$ is shown below:



We can now lift d_i and e_i to disks, $\widetilde{d}_i$ and $\widetilde{e}_i$, in the infinite cyclic cover $\widetilde{X}$ of $X = S^3 - K$. The lifts of l_i, r_i form a link $\widetilde{L}$ in $\widetilde{X}$ which has a linking matrix B with entries in $\mathbb{Z}[t, t^{-1}]$. To compute B note that we can choose the lifts $\widetilde{d}_i$ and $\widetilde{e}_i$ so that:

(1) $\widetilde{d}_i \cap \widetilde{e}_i$ consists of a single clasp, for every i,
(2) $\widetilde{d}_i \cap t(\widetilde{e}_i)$ consists of a single clasp, oriented opposite to that in (1), for every i,
(3) $\widetilde{e}_i \cap \widetilde{d}_{i+1}$, for $1 \leq i < n$, consists of a single clasp, and
(4) $\widetilde{e}_n \cap t^k(\widetilde{d}_1)$, for some integer k, consists of a single clasp.

In (4), k (up to sign) is just the linking number of K with the imbedded wheel of G.

Now it follows from this intersection data and the fact that L is 0-framed that we can orient L so that the linking matrix B is given by

$$B = \begin{pmatrix} 0 & D \\ D^\star & 0 \end{pmatrix} \qquad \text{where} \qquad D = \begin{pmatrix} t-1 & 1 & 0 & \dots & 0 \\ 0 & t-1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & t-1 & 1 \\ \pm t^k & 0 & \dots & 0 & t-1 \end{pmatrix}.$$

For any matrix A over $\mathbb{Z}[t, t^{-1}]$, $A^\star$ denotes the conjugate (under the involution $t \leftrightarrow t^{-1}$) transpose of A. The desired result $\Delta_{K'}(t) = \Delta_K(t)\theta(t)\theta(t^{-1})$ is now a consequence of the following lemma, which is proved by a standard argument going back to Kervaire-Milnor, generalized to covering spaces (see for example [Le1, p.140]). □

Suppose $K \subset S^3$ is a knot, L a framed link in $X = S^3 - K$, and $K' \subset S^3_L$ the knot produced from K by surgery on L. Assume that the components of L are null-homologous in X and the components of $\widetilde{L} \subset \widetilde{X}$, the lift of L into $\widetilde{X}$, are null-homologous. In this case we have well-defined linking numbers of the components of $\widetilde{L}$ which are organized into a matrix B with entries in $\mathbb{Z}[t, t^{-1}]$ in the usual way. Let $A(K) = H_1(\widetilde{X})$ and $A(K') = H_1(\widetilde{Y})$ denote the *Alexander modules* of K, K', where $Y = S^3_L - K'$.

Lemma 2.6. *There is an exact sequence of* $\mathbb{Z}[t, t^{-1}]$*-modules*

$$0 \to M \to A(K') \to A(K) \to 0$$

where M is a module with presentation matrix B. In particular, $\Delta_{K'} = \Delta_K \det(B)$.

Proof. Observe that $\widetilde{Y} = \widetilde{X}_{\widetilde{L}}$. Consider the following diagram of exact sequences of $\mathbb{Z}[t, t^{-1}]$-modules.

$$\begin{array}{ccccccccc} & & & & H_2(\widetilde{Y}, \widetilde{X} - \widetilde{L}) & & & & \\ & & & & \downarrow{\scriptstyle \partial_*} & & & & \\ H_2(\widetilde{X}) & \longrightarrow & H_2(\widetilde{X}, \widetilde{X} - \widetilde{L}) & \longrightarrow & H_1(\widetilde{X} - \widetilde{L}) & \xrightarrow{i_*} & H_1(\widetilde{X}) & \longrightarrow & H_1(\widetilde{X}, \widetilde{X} - \widetilde{L}) \\ & & & & \downarrow & & & & \\ & & & & H_1(\widetilde{Y}) & & & & \\ & & & & \downarrow & & & & \\ & & & & H_1(\widetilde{Y}, \widetilde{X} - \widetilde{L}) & & & & \end{array}$$

Notice that $H_1(\widetilde{X}, \widetilde{X} - \widetilde{L}) = H_1(\widetilde{Y}, \widetilde{X} - \widetilde{L}) = 0$. Moreover, $H_2(\widetilde{X}, \widetilde{X} - \widetilde{L})$ is freely generated by the meridian disks of L, lifted to $\widetilde{X}$, and $H_2(\widetilde{Y}, \widetilde{X} - \widetilde{L})$ is freely generated by the disks attached by the surgeries. Thus, since the components of $\widetilde{L}$ are null-homologous in $\widetilde{X}$, $i_* \circ \partial_* = 0$. Also note that $H_2(\widetilde{X}) = 0$ and so we have a mapping

$$H_2(\widetilde{Y}, \widetilde{X} - \widetilde{L}) \to H_2(\widetilde{X}, \widetilde{X} - \widetilde{L})$$

induced by ∂_*, which can be interpreted as expressing the longitudes of $\widetilde{L}$ as linear combinations of the meridians of $\widetilde{L}$ in $H_1(\widetilde{X} - \widetilde{L})$. Therefore this map is given by the linking numbers of $\widetilde{L}$ and has B as a representative matrix. This completes the proof of Lemma 2.6 and, as a consequence, Lemma 2.5. □

To complete the proof of Theorem 7 we need the following lemma.

Lemma 2.7. *Let $f(t)$ be a polynomial of the form $1 \pm t^k(t-1)^n$, for any integers k, n with $n \neq 0$. Then any root of $f(t)$ which lies on the unit circle must be of the form $e^{\pm \pi i/3}$.*

Proof. If z is a root of $f(t)$ then $|z|^k|z-1|^n = 1$. Thus we have $|z| = |z-1| = 1$, from which the conclusion follows. □

Now choose some $\theta(t)$ with a root on the unit circle different from $e^{\pm\pi i/3}$ but with $\theta(1) = 1$—for example any cyclotomic polynomial of composite order not equal to 6. Let K be a double-slice knot with Alexander polynomial $\theta(t)\theta(t^{-1})$ (see [Su, Theorem 3.3]). Then it follows from Lemmas 2.5 and 2.7 that K is not $\mathfrak{c}^1$ equivalent to the trivial knot. □

We end with a remark concerning the inverse of surgery on a wheel.

Remark 2.8. Recall that if a knot K' is obtained from a knot K by surgery on a Y-graph G, then there exists a Y-graph G' such that K is obtained from K' by surgery on G', see [GGP, Theorem 3.2]. Recall also that surgery on a wheel is described in terms of surgery on a union of Y-graphs, as explained in [GGP, Section 2.3]; in particular the inverse of surgery on a wheel can be described in terms of surgery on a union of Y-graphs. One might guess that the inverse of surgery on a wheel can be described in terms of surgery on a wheel. This is false, since the proof of Lemma 2.5 implies that if K' is obtained from K by surgery on a wheel G, then Δ_K always divides (and it can happen that it is not equal to) $\Delta_{K'}$.

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