

AN EXTENDED VERSION OF ADDITIVE K -THEORY

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ABSTRACT. There are two infinitesimal (i.e., additive) versions of the K -theory of a field F : one introduced by Cathelineau, which is an F -module, and another one introduced by Bloch-Esnault, which is an F^* -module. Both versions are equipped with a regulator map, when F is the field of complex numbers.

In our short paper we will introduce an extended version of Cathelineau's group, and a complex-valued regulator map given by the entropy. We will also give a comparison map between our extended version and Cathelineau's group.

Our results were motivated by two unrelated sources: Neumann's work on the extended Bloch group (which is isomorphic to indecomposable K_3 of the complex numbers), and the study of singularities of generating series of hypergeometric multisums.

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1. INTRODUCTION

1.1. **Two flavors of infinitesimal K -theory.** There are two infinitesimal (i.e., additive) versions of the K -theory of a field F :

- (a) $\beta_n(F)$, introduced by Cathelineau in [Ca2, Sec.3.1], which is an F -module, and
- (b) $TB_n(F)$, introduced by Bloch-Esnault in [BE], which is an F^* -module.

Both versions are equipped with a regulator map, when $F = \mathbb{C}$ is the field of complex numbers; see [Ca1] and [BE]. Let us recall the definition of $\beta_2(F)$ and $TB_2(F)$ for a field F of characteristic zero. Let

$$(1) \quad F^{**} = F \setminus \{0, 1\}.$$

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Definition 1.1. [BE, Thm.3.4] For a field F of characteristic zero consider the abelian group $\text{TB}_2(F)$ which is an F^* -module with generators $\langle a \rangle$ for $a \in F^{**}$, subject to the 4-term relation:

$$(2) \quad \langle a \rangle - \langle b \rangle + a \star \left\langle \frac{b}{a} \right\rangle + (1-a) \star \left\langle \frac{1-b}{1-a} \right\rangle = 0.$$

Here, \star denotes the action of F^* on $\text{TB}_2(F)$.

The next definition is taken from [Ca1, Sec.1.1]; see also [Ca2, Sec.4.2].

Definition 1.2. [Ca1] If F is a field of characteristic zero, consider the abelian group $\beta_2(F)$ which is an F -module with generators $\langle a \rangle$ for $a \in F^{**}$ subject to the 4-term relation:

$$(3) \quad \langle a \rangle - \langle b \rangle + a \left\langle \frac{b}{a} \right\rangle + (1-a) \left\langle \frac{1-b}{1-a} \right\rangle = 0.$$

The groups $\text{TB}_2(F)$ and $\beta_2(F)$ fit into a short exact sequence of abelian groups

$$(4) \quad 0 \longrightarrow F \longrightarrow \text{TB}_2(F) \longrightarrow \beta_2(F) \longrightarrow 0$$

where the first and last terms are F -modules and the middle is an F^* -module; see [BE, Eqn.(1.7)].

In our short paper we will introduce an extended version $\widehat{\beta}_2(\mathbb{C})$ of Cathelineau's group $\beta_2(\mathbb{C})$, and a complex-valued regulator map given by the entropy. We will also give a comparison map between our extended version and Cathelineau's group.

Our results were motivated by two unrelated sources:

- (a) Neumann's work on the *extended Bloch group* and
- (b) the study of singularities of generating series of hypergeometric multisums.

In a sense, the extended Bloch group is forced upon us by the functional properties (a 5-term relation) and the analytic continuation of a single special function: the *Rogers dilogarithm*. For a discussion on the extended Bloch group see [Ne, DZ, GZ], and for its relation to indecomposable K -theory $K_3^{\text{ind}}(\mathbb{C})$ see [Ga1].

In our paper, we will study the functional properties (a 4-term relation) and the analytic continuation of the entropy function. This will naturally lead us to introduce an extended version of Cathelineau's group.

The 4-term relation for the entropy function has already appeared in the context of additive K -theory and infinitesimal polylogarithms; see [Ca1, Ca2, BE] and also [E-VG, Defn.2.7].

From our point of view, the entropy function appears in the asymptotics of classical binomial coefficients, via the Stirling formula. The relation to singularities of generating series of hypergeometric multisums will be explained in a separate publication; see [Ga2].

1.2. The Stirling formula and the entropy function. In this section we will introduce the entropy function and state some of its elementary properties.

For $a > 0$, we can define $(an)! = \Gamma(an + 1)$, and for $a > b > 0$, we can define as usual $\binom{an}{bn} = (na)! / ((nb)!(n(a-b)!)$.

Definition 1.3. Consider the *entropy function* Φ , defined by:

$$(5) \quad \Phi(x) = -x \log(x) - (1-x) \log(1-x).$$

for $x \in (0, 1)$.

$\Phi(x)$ is a multivalued function on $\mathbb{C} \setminus \{0, 1\}$, and is given by the double integral of a rational function as follows from:

$$(6) \quad \Phi''(x) = -\frac{1}{x} - \frac{1}{1-x}.$$

The next lemma links the growth rate of the binomial coefficients with the entropy function.

Lemma 1.4. For $a > b > 0$, we have:

$$(7) \quad \binom{an}{bn} \sim e^{na\Phi(\frac{b}{a})} \sqrt{\frac{a}{2b(a-b)\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

The proof is an application of *Stirling's formula*, which computes the asymptotic expansion of $n!$ (see [O]):

$$(8) \quad \log n! \sim n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{n}\right)$$

The next lemma gives a 4-term relation for the entropy function.

Lemma 1.5. For $a, b, a+b \in (0, 1)$, Φ satisfies the 4-term relation:

$$(9) \quad \Phi(b) - \Phi(a) + (1-b)\Phi\left(\frac{a}{1-b}\right) - (1-a)\Phi\left(\frac{b}{1-a}\right) = 0.$$

The 4-term relation follows from the *associativity* of the multibinomial coefficients

$$(10) \quad \binom{(a+b+c)n}{an} \binom{(b+c)n}{bn} = \binom{(a+b+c)n}{bn} \binom{(a+c)n}{an} = \frac{((a+b+c)n)!}{(an)!(bn)!(cn)!}$$

and Lemma 1.4, and the specialization to $a+b+c=1$. In fact, the 4-term relation (9) uniquely determines Φ up to multiplication by a complex number. See for example, [Da] and [AD, Sec.5.4,p.66].

1.3. Analytic continuation of the entropy function. In this section we discuss in detail the analytic continuation of the entropy function, and the 4-term relation. This will be our motivation for introducing the extended Bloch-Cathelineau group in the next section. It is clear from the definition that the entropy function is a multivalued analytic function on the *doubly-punctured plane*

$$(11) \quad \mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}.$$

Let $\hat{\mathbb{C}}$ denote the *universal abelian cover* of \mathbb{C}^{**} . We can represent the Riemann surface $\hat{\mathbb{C}}$ as follows. Let \mathbb{C}_{cut} denote $\mathbb{C} \setminus \{0, 1\}$ cut open along each of the intervals $(-\infty, 0)$ and $(1, \infty)$ so that each real number r outside $[0, 1]$ occurs twice in \mathbb{C}_{cut} . Let us denote the two occurrences of r by $r+0i$ and $r-0i$ respectively. It is now easy to see that $\hat{\mathbb{C}}$ is isomorphic to the surface obtained from $\mathbb{C}_{\text{cut}} \times 2\mathbb{Z} \times 2\mathbb{Z}$ by the following identifications:

$$\begin{aligned} (x+0i; 2p, 2q) &\sim (x-0i; 2p+2, 2q) && \text{for } x \in (-\infty, 0) \\ (x+0i; 2p, 2q) &\sim (x-0i; 2p, 2q+2) && \text{for } x \in (1, \infty) \end{aligned}$$

This means that points in $\hat{\mathbb{C}}$ are of the form (z, p, q) with $z \in \mathbb{C}_{\text{cut}}$ and p, q even integers.

Definition 1.6. Let us define the entropy function:

$$(12) \quad \Phi: \hat{\mathbb{C}} \longrightarrow \mathbb{C}, \quad \Phi([z; p, q]) = \Phi(z) - \pi ipz + \pi iq(1-z).$$

Remark 1.7. Since $\hat{\mathbb{C}}$ is the Riemann surface of the analytic function $z \mapsto (\log(z), \log(1-z))$, it follows that a coordinate-free definition of Φ can be given by:

$$(13) \quad \Phi(\hat{z}) = -z \log(\hat{z}) - (1-z) \log(1-\hat{z})$$

where $\hat{z} \in \hat{\mathbb{C}}$ and z denotes its projection to \mathbb{C}^{**} .

Consider the set

$$(14) \quad 4\mathbb{T} := \left\{ \left(y, x, \frac{x}{1-y}, \frac{y}{1-x} \right) \in (\mathbb{C}^{**})^4 \right.$$

of 4-tuples involved in the 4-term relation. Also, let

$$(15) \quad 4\mathbb{T}_0 := \{ (x_0, \dots, x_3) \in 4\mathbb{T} \mid 0 < x_0 < x_1 < x_0 + x_1 < 1 \}$$

and define $\widehat{4\mathbb{T}} \subset \widehat{\mathbb{C}}^4$ to be the component of the preimage of $4\mathbb{T}$ that contains all points $((x_0; 0, 0), \dots, (x_4; 0, 0))$ with $(x_0, \dots, x_4) \in 4\mathbb{T}_0$. For a comparison with the 5-term relation, see [DZ, Rem.2.1].

1.4. An extended version of $\beta_2(\mathbb{C})$. We have all the ingredients to introduce an extended version of Cathelineau's group $\beta_2(\mathbb{C})$.

Definition 1.8. The *extended group* $\widehat{\beta_2(\mathbb{C})}$ is the \mathbb{C} -module generated by the symbols $\langle z; p, q \rangle$ with $(z; p, q) \in \widehat{\mathbb{C}}$, subject to the *extended 4-term relation*:

$$(16) \quad \langle x_0; p_0, q_0 \rangle - \langle x_1; p_1, q_1 \rangle + (1 - x_0) \left\langle \frac{x_1}{1 - x_0}; p_2, q_2 \right\rangle - (1 - x_1) \left\langle \frac{x_0}{1 - x_1}; p_3, q_3 \right\rangle = 0$$

for $((x_0; p_0, q_0), \dots, (x_3; p_3, q_3)) \in \widehat{4\mathbb{T}}$, and the relations:

$$(17) \quad \langle x; p, q \rangle - \langle x; p, q' \rangle = \langle x; p, q - 2 \rangle - \langle x; p, q' - 2 \rangle$$

$$(18) \quad \langle x; p, q \rangle - \langle x; p', q \rangle = \langle x; p - 2, q \rangle - \langle x; p' - 2, q \rangle$$

for $x \in \mathbb{C}^{**}$, $p, q, p', q' \in 2\mathbb{Z}$.

It is easy to see that Φ satisfies the 4-term relations (17) and (18). Moreover, since $\widehat{4\mathbb{T}}$ is defined by analytic continuation and Φ satisfies the $4\mathbb{T}$ -relation of Equation 1.5, Φ satisfies the $\widehat{4\mathbb{T}}$ relation. Thus, the following definition makes sense.

Definition 1.9. Φ gives rise to a *regulator map*:

$$(19) \quad R : \widehat{\beta_2(\mathbb{C})} \longrightarrow \mathbb{C}$$

Our main theorem compares our extended group $\widehat{\beta_2(\mathbb{C})}$ with Cathelineau's group $\beta_2(\mathbb{C})$.

Theorem 1. (a) There is a well-defined map:

$$(20) \quad \pi : \widehat{\beta_2(\mathbb{C})} \longrightarrow \beta_2(\mathbb{C}), \quad \pi(\langle x; p, q \rangle) = \langle x \rangle.$$

(b) We have a short exact sequence of \mathbb{C} -modules:

$$(21) \quad 0 \longrightarrow \mathbb{C} \xrightarrow{\chi} \widehat{\beta_2(\mathbb{C})} \xrightarrow{\pi} \beta_2(\mathbb{C}) \longrightarrow 0$$

where

$$(22) \quad \chi(z) = z \cdot c$$

and $c = \langle x; 2, -2 \rangle - \langle x; 0, 0 \rangle$ is independent of $x \in \mathbb{C}^{**}$.

(c) The composition $R \circ \chi$ is given by:

$$(23) \quad (R \circ \chi)(z) = -2\pi iz.$$

In particular, the restriction of R on $\text{Ker}(\pi)$ is 1-1.

An important ingredient for the proof of Theorem 1 is the following description of the $\widehat{4\mathbb{T}}$ relation.

Definition 1.10. Let $V \subset (\mathbb{Z} \times \mathbb{Z})^4$ be the subspace

$$(24) \quad V := \{ ((p_0, q_0), (p_1, q_1), (-q_0 + p_1, -q_0 + r), (p_0 - q_1, r - q_1)) \mid p_0, p_1, q_0, q_1, r \in 2\mathbb{Z} \}.$$

Let us define

$$(25) \quad 4\mathbb{T}^+ := \{(x_0, \dots, x_3) \in 4\mathbb{T} \mid \operatorname{Im}(x_i) > 0, i = 0, \dots, 3\}$$

and let $\widehat{4\mathbb{T}^+}$ denote the unique component of the inverse image of $4\mathbb{T}$ in $\widehat{\mathbb{C}^4}$ which includes the points $((x_0; 0, 0), \dots, (x_3; 0, 0))$ with $(x_0, \dots, x_3) \in \widehat{4\mathbb{T}^+}$.

Proposition 1.11. *We have:*

$$(26) \quad \widehat{4\mathbb{T}} = \widehat{4\mathbb{T}^+} + V = \{\mathbf{x} + \mathbf{v} \mid \mathbf{x} \in \widehat{4\mathbb{T}^+} \text{ and } \mathbf{v} \in V\},$$

where we are using addition to denote the action of $(\mathbb{Z} \times \mathbb{Z})^4$ by covering transformations on $\widehat{\mathbb{C}^4}$.

Corollary 1.12. *For $(y, x, x/(1-y), y/(1-x)) \in \widehat{4\mathbb{T}^+}$ and $p_0, q_0, p_1, q_1, r \in 2\mathbb{Z}$ we have:*

$$(27) \quad \langle y; p_0, q_0 \rangle - \langle x; p_1, q_1 \rangle + (1-y) \langle \frac{x}{1-y}; q_0 + p_1, -q_0 + r \rangle - (1-x) \langle \frac{y}{1-x}; p_0 + q_1, r - q_1 \rangle = 0.$$

1.5. A comparison of regulators. In [Ca2, Prop.7] Cathelineau defined purely real/imaginary regulators on the \mathbb{C} -modules $\beta_n(\mathbb{C})$ for all n :

$$(28) \quad d\tilde{\mathcal{D}}_n : \beta_n(\mathbb{C}) \longrightarrow \mathbb{R}(n-1)$$

where, we follow the notation of [Ca2] and, as usual, $A(n) = (2\pi i)^n A$. The regulators $d\tilde{\mathcal{D}}_n$ also known by the name of *infinitesimal polylogarithms* can be thought as an infinitesimal analogue of the *modified polylogarithms* \mathcal{D}_n :

$$(29) \quad \mathcal{D}_n : \mathbb{C} \longrightarrow \mathbb{R}(n-1), \quad \mathcal{D}_n(z) = \operatorname{Re}/\operatorname{Im}_n \left(\sum_{k=0}^{n-1} \frac{B_k 2^k}{k!} \log^k |z| \operatorname{Li}_{n-k}(z) \right)$$

where $\operatorname{Re}/\operatorname{Im}_n$ denotes the real (resp. i times the imaginary) part if n is odd (resp. even), B_k are the Bernoulli numbers, and $\operatorname{Li}_k(z) = \sum_k z^k/k^n$ is the classical polylogarithm. The modified polylogarithms were introduced by Zagier in [Za] (generalizing the $n = 2$ case of the $n = 2$ Bloch-Wigner polylogarithm) and have a motivic interpretation as a reality condition of Mixed Hodge Structures; see [BD].

The infinitesimal polylogarithms $d\tilde{\mathcal{D}}_n$ have also been studied by Elbaz-Vincent and Gangl; see [E-VG, Defn.2.7]. For $n = 2$, we have, up to a sign:

$$(30) \quad ir : \beta_2(\mathbb{C}) \longrightarrow i\mathbb{R}$$

where

$$(31) \quad r(x) = -x \log |x| - (1-x) \log |1-x|$$

is a modification of the entropy function. On the other hand, we have a complex-valued regulator map R on $\widehat{\beta_2(\mathbb{C})}$. We caution that $i\operatorname{Re} \circ R \neq ir \circ \pi$.

Nevertheless, let $\widehat{\beta_2(\mathbb{R})}$ denote the \mathbb{R} -subspace of $\beta_2(\mathbb{C})$ generated by the real points in $\widehat{\mathbb{C}}$; i.e., the points of the form $(x; p, q)$ with $x \in \mathbb{R}^{**}$. The following corollary follows immediately from the definitions.

Corollary 1.13. *The following diagram commutes:*

$$\begin{array}{ccc} \widehat{\beta_2(\mathbb{R})} & \xrightarrow{R} & \mathbb{C} \\ \pi \downarrow & & \downarrow i\operatorname{Re} \\ \beta_2(\mathbb{R}) & \xrightarrow{ir} & i\mathbb{R} \end{array}$$

1.6. A comparison table. As we mentioned before, there is a close analogy between our paper and Neumann's work on the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$; see [Ne, DZ, GZ, Ga1]. Let us summarize this into a table, which enhances the table of [BE, Sec.5].

Theory	K -theory	Infinitesimal K -theory
Group	$\widehat{\mathcal{B}}(\mathbb{C})$	$\widehat{\beta}_2(\mathbb{C})$
Extended group	$\widehat{\mathcal{B}}(\mathbb{C})$	$\widehat{\beta}_2(\mathbb{C})$
Regulator	Rogers dilogarithm	Entropy
	$R: \widehat{\mathbb{C}} \rightarrow \mathbb{C}/\mathbb{Z}(2)$	$R: \widehat{\mathbb{C}} \rightarrow \mathbb{C}$
Functional Equation	5-term relation	4-term relation
Exact sequence	$0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \widehat{\mathcal{B}}(\mathbb{C}) \rightarrow \mathcal{B}(\mathbb{C}) \rightarrow 0$	$0 \rightarrow \mathbb{C} \rightarrow \widehat{\beta}_2(\mathbb{C}) \rightarrow \beta_2(\mathbb{C}) \rightarrow 0$

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2. PROOF OF PROPOSITION 1.11

In this section we will give a proof of Proposition 1.11, following the idea from the Appendix of [DZ]. Recall the definition of $4\mathbb{T}^+$ from Equation (25). Fix a point (x_0, \dots, x_3) and let

$$(32) \quad P = ((x_0; 0, 0), \dots, (x_3; 0, 0)) \in 4\mathbb{T}^+.$$

Let $x_0 = y$ and $x_1 = x$. Consider the curve in $\widehat{4\mathbb{T}}$ obtained by keeping x_1 fixed and letting x_0 move along a closed curve in $\mathbb{C} \setminus \{0, 1, 1 - x_1\}$. By an analysis of the 4-term relation we can examine how the values of the p_i 's, q_i 's change when x_0 moves around. This is recorded in Figure 1. It follows that if x_0 traverses a closed curve going $p_0/2$ times clockwise around 0, followed by $q_0/2$ times clockwise around 1, followed by $r/2$ times clockwise around $1 - x_1$, for p_0, q_0, r even, then the curve in $\widehat{4\mathbb{T}}$ ends in P' , where

$$P' = ((x_0; p_0, q_0), (x_1; 0, 0), (x_2, -q_0, q_0 + r), (x_3; p_0, r)).$$

Starting at P' , follow the curve in $\widehat{4\mathbb{T}}$ obtained by keeping x_0 fixed and letting x_1 move a curve going $p_1/2$ times clockwise around 0 followed by $q_1/2$ times clockwise around 1 for p_1, q_1 even. Then the curve in $\widehat{4\mathbb{T}}$ ends in P'' , where

$$((x_0; p_0, q_0), (x_1; p_1, q_1), (x_2; -q_0 + p_1, -q_0 + r), (x_3; p_0 + q_1, r - q_1)).$$

Since we can connect a point P in $\widehat{4\mathbb{T}}$ to a point in the preimage of $4\mathbb{T}_0$ by first sliding x_0 into the interval $(0, 1)$ and then sliding x_1 into the interval $(x_0, 1)$, Proposition 1.11 follows.

3. PROOF OF THEOREM 1

For part (a) of Theorem 1, we need to show that the image under π of the extended 4-term relation of Corollary 1.12 is zero in $\beta_2(\mathbb{C})$. Let us replace x by $1 - x$. It suffices to show that for all $x, y \in \mathbb{C}^{**}$ with $x \neq y$ we have:

$$(33) \quad \langle y \rangle - \langle 1 - x \rangle + (1 - y) \left\langle \frac{1 - x}{1 - y} \right\rangle - x \left\langle \frac{y}{x} \right\rangle = 0.$$

In [Ca1, Sec.1.1] Cathelineau proves that the 4-term relation (2) implies the following relations:

$$\begin{aligned} \langle a \rangle &= \langle 1 - a \rangle \\ \langle 1/a \rangle &= -1/a \langle a \rangle \end{aligned}$$

It follows that

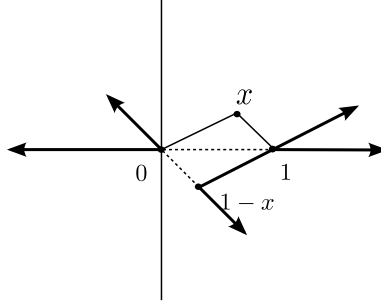


Figure 1. The 6 solid half-lines in the figure are the cuts of the function $z \mapsto (\text{Log}(z), \text{Log}(1-z))$ in the x_i -plane for $i = 0, 2, 3$, when $x = x_1$ is fixed. The relevant values of p_i and q_i increase by 2 whenever $y = x_0$ crosses the relevant line in the direction indicated by the arrows.

$$\begin{aligned}
 \langle y \rangle - \langle 1-x \rangle + (1-y) \left\langle \frac{1-x}{1-y} \right\rangle - x \left\langle \frac{y}{x} \right\rangle &= \langle y \rangle - \langle 1-x \rangle - (1-x) \left\langle \frac{1-y}{1-x} \right\rangle - x \left\langle \frac{y}{x} \right\rangle \\
 &= \langle y \rangle - \langle x \rangle - (1-x) \left\langle \frac{1-y}{1-x} \right\rangle - x \left\langle \frac{y}{x} \right\rangle \\
 &= 0
 \end{aligned}$$

by (3) for $x = a$, $y = b$.

In the remaining of the section we will prove the other parts of Theorem 1, following arguments similar to [Ne, Sec.7].

Lemma 3.1. For $x, y \in \mathbb{C}^{**}$, $p_0, q_0, p_1, q_1 \in 2\mathbb{Z}$ we have:

$$(34) \quad \langle y; p_0 - 2, q_0 + 2 \rangle - \langle y; p_0, q_0 \rangle = \langle x; p_1 - 2, q_1 + 2 \rangle - \langle x; p_1, q_1 \rangle$$

Proof. Replace (p_0, p_1, q_0, q_1, r) by $(p_0 - 2, p_1 - 2, q_0 + 2, q_1 + 2, r + 2)$ in the 4-term relation of Proposition 1.11 and subtract. Of the 8 terms, 4 cancel and the remaining 4 give the identity (34). \square

For every $x \in \mathbb{C}^{**}$, let us define

$$(35) \quad c_x = \langle x; 2, -2 \rangle - \langle x; 0, 0 \rangle \in \widehat{\beta_2(\mathbb{C})}.$$

Setting $(p_0, q_0, p_1, q_1) = (2, -2, 2, -2)$ in (34) and recalling (35) implies that

$$(36) \quad c := c_x$$

is independent of $x \in \mathbb{C}^{**}$. In addition, the left hand side of (34) is independent of p_1 and q_1 thus so is the right hand side. Thus,

$$(37) \quad \langle x; p_1 - 2, q_1 + 2 \rangle - \langle x; p_1, q_1 \rangle$$

is independent of p_1 and q_1 .

Lemma 3.2. For all $x \in \mathbb{C}^{**}$, $p, q \in 2\mathbb{Z}$ we have:

$$(38) \quad \langle x; p, q \rangle = \frac{1}{4} (pq \langle x; 2, 2 \rangle - p(q-2) \langle x; 2, 0 \rangle - q(p-2) \langle x; 0, 2 \rangle + (pq - 2p - 2q + 4) \langle x; 0, 0 \rangle).$$

Proof. Equation (17) implies that for all $p \in 2\mathbb{Z}$ we have:

$$(39) \quad \langle x; p, q \rangle = \frac{q}{2} \langle x; p, 2 \rangle - \frac{q-2}{2} \langle x; p, 0 \rangle.$$

Equation (18) implies that for all $q \in 2\mathbb{Z}$ we have:

$$(40) \quad \langle x; p, q \rangle = \frac{p}{2} \langle x; 2, q \rangle - \frac{p-2}{2} \langle x; 0, q \rangle.$$

Using (40), and telescoping, we can expand the right hand side of (39). The result follows. \square

Using (38), it follows that both sides of (34) are linear functions on p_0, q_0, p_1, q_1 . In particular, the coefficient of p_1 vanishes. This implies that for all $x \in \mathbb{C}^{**}$ we have:

$$(41) \quad \langle x; 2, 2 \rangle - \langle x; 2, 0 \rangle - \langle x; 0, 2 \rangle + \langle x; 0, 0 \rangle = 0.$$

Let $K = \text{Ker}(\pi)$ denote the kernel of $\pi : \widehat{\beta_2(\mathbb{C})} \rightarrow \beta_2(\mathbb{C})$. It follows that K is the \mathbb{C} -span of

$$\langle x; p+2, q \rangle - \langle x; p, q \rangle, \quad \langle x; p, q+2 \rangle - \langle x; p, q \rangle$$

for $x \in \mathbb{C}^{**}$, $p, q \in 2\mathbb{Z}$. Using Lemma 3.2, it follows that K is the \mathbb{C} -span of

$$\langle x; 2, 0 \rangle - \langle x; 0, 0 \rangle, \quad \langle x; 0, 2 \rangle - \langle x; 0, 0 \rangle, \quad \langle x; 2, 2 \rangle - \langle x; 0, 0 \rangle.$$

Using Equation (41), it follows that K is the \mathbb{C} -span of $\langle x; 0, 2 \rangle - \langle x; 0, 0 \rangle$ (for all $x \in \mathbb{C}^{**}$) and c .

For $x \in \mathbb{C}^{**}$, let us denote

$$(42) \quad \{x\} := -\frac{1}{1-x} (\langle x; 0, 2 \rangle - \langle x; 0, 0 \rangle).$$

Set $p_0 = q_0 = p_1 = q_1 = 0$ in the 4-term relation (27). It follows that

$$(43) \quad \langle y; 0, 0 \rangle - \langle x; 0, 0 \rangle + (1-y) \langle \frac{x}{1-y}; 0, r \rangle - (1-x) \langle \frac{y}{1-x}; 0, r \rangle = 0.$$

Replace r by $r+2$, subtract and divide by $1-x-y$. It follows that

$$(44) \quad \left\{ \frac{x}{1-y} \right\} = \left\{ \frac{y}{1-x} \right\}$$

which implies that for all $x, y \in \mathbb{C}^{**}$, we have:

$$(45) \quad \{x\} = \{y\}.$$

Thus, K is the \mathbb{C} -span of $\{x\}$ and c , where both elements are independent of x .

Going back to the 4-term relation (27), replace q_1 with q_1+2 and subtract. We get the 4-term relation:

$$\begin{aligned} \langle x; p_1, q_1 \rangle - \langle x; p_1, q_1+2 \rangle &= (1-x) \left(\left\langle \frac{y}{1-x}; p_0+q_1+2, r-q_1-2 \right\rangle - \left\langle \frac{y}{1-x}; p_0+q_1, r-q_1 \right\rangle \right) \\ &= (1-x) \left(\left\langle \frac{y}{1-x}; 2, -2 \right\rangle - \left\langle \frac{y}{1-x}; 0, 0 \right\rangle \right) \\ &= (1-x)c. \end{aligned}$$

Now set $p_1 = 0$ and $q_1 = 2$. It follows that $\{x\} = c$. Thus, K is the \mathbb{C} -span of $\{x\} = c$ which is independent of x . This implies that χ is well-defined and that $\text{Image}(\chi) = \text{Ker}(\pi)$. Since $(R \circ \chi)(z) = zR(c_x) = -2\pi iz$, it follows that $R \circ \chi$, and therefore χ , is 1-1. This concludes the proof of Theorem 1.

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