ON FINITE TYPE 3-MANIFOLD INVARIANTS V: RATIONAL HOMOLOGY 3-SPHERES

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Abstract. We introduce a notion of finite type invariants of oriented rational homology 3-spheres. We show that the map to finite type invariants of integral homology 3-spheres is one-to-one and deduce that the space of finite type invariants of rational homology 3-spheres is a filtered commutative algebra with finite dimensional nonzero graded quotients only in degrees divisible by 3. We show that the Casson-Walker invariant is of type 3. We mention an alternative B-finite-type notion of invariants of rational homology 3-spheres, with "better grading" properties, and show that type $3m$ invariants of rational homology 3-spheres are included in B-type m invariants. Finally, we prove a nonexistence theorem for finite type invariants of oriented, closed 3-manifolds.

Contents

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1. Introduction

1.1. Motivation. The present paper is a continuation of [Ga], [GL1], [GO], [GL2], [Oh1] and [Oh2] devoted to the study of finite type invariants of oriented integral homology 3-spheres. The main motivation is twofold:

- Chern-Simons theory predicts the existence of finite type invariants of rational homology 3-spheres.
- It is a natural question to try to extend the existing definition of finite type invariants of integral homology 3-spheres to the case of rational homology 3-spheres.

1.2. Statement of the results. We begin with an apology for the inconsistency of the notation between here and the previous papers. In the papers [Oh1], [Ga], [GL1], [GO] and [GL2], $\mathcal{M}$ denoted the rational vector space on the set of integral homology 3-spheres, but in the present paper $\mathcal{M}$ denotes the rational vector space on the set of rational homology 3-spheres, and $\mathcal{M}(\mathbb{Z})$ denotes the rational vector space on the set of integral homology 3-spheres.

All 3-manifolds to be considered will be oriented, and smooth unless otherwise mentioned. A link L in a 3-manifold M is called *algebraically split* (resp. *boundary*) if the linking numbers between its components vanish (resp. if each component bounds a Seifert surface, and the Seifert surfaces are all disjoint from each other). A *framing* of a link L in a rational homology 3-sphere M is a choice of (isotopy class) of essential simple closed curves in the boundary of tubular neighborhood of each of its components. If L is a *homologically trivial* link in a rational homology 3-sphere M (i.e. it represents the zero element in the first homology $H_1(M, \mathbb{Z})$), then a framing f on a r component link can be described in terms of a sequence $f = (f_1, \cdots, f_r)$, where $f_i \in \mathbb{Q} \cup \{1/0\}$, with the convention that $f_i = p_i/q_i$ is the isotopy class of the curve $p_i(meridian) + q_i(longitude)$. Note that a framing of a link does not require the choice of an orientation of it. An *integral framing* (resp. *rational framing*) f of a homologically trivial link in an integral homology 3-sphere one such that $f_i \in \mathbb{Z} - \{0\}$ (resp. $f_i \in \mathbb{Q}$ -$\{0\}$) for all i.

Remark 1.1. All the links considered in the present paper will be homologically trivial links in rational homology 3-spheres, unless otherwise explicitly mentioned. Note also that every rational homology 3-sphere can be obtained by surgery on a rationally framed algebraically split link in S^3.

If (L, f) is a framed link in a 3-manifold M, we denote by $M_{(L,f)}$ the 3-manifold obtained by doing Dehn surgery to each component of the framed link L. Let $\mathcal{M}$ (resp. $\mathcal{M}(\mathbb{Z})$) denote the set of of rational homology 3-sphere (resp. integral homology 3-spheres). For a rationally framed algebraically split link (L, f) in a rational homology 3-sphere M, let

$$[M, L, f] = \sum_{L' \subseteq L} (-1)^{|L'|} M_{L',f'} \in \mathcal{M} \tag{1}$$

where the sum is over all sublinks of L (including the empty one), $|L'|$ is the number of components of L' and f' is the restriction of the framing f to L'. Note that since (L, f) is a rationally framed algebraically split link, $M_{(L',f')}$ is a rational homology 3-sphere for every sublink L' of L.

With these preliminaries in mind, let us introduce the following definition:

Definition 1.2. A map $v : \mathcal{M} \to \mathbb{Q}$ is called a type m invariant of rational homology 3-spheres if it satisfies the following Properties: (see however remark 1.3)

Property 0: For every (L, f) rationally framed algebraically split link (L, f) of $m + 1$ components in a integral homology 3-sphere M, we have that:

$$v([M, L, f]) = 0 \tag{2}$$

Property 1: If an algebraically split rationally framed link L in a integral homology 3-sphere M is the union of an integrally framed knot (L_1, n) and a rationally framed $m - 1$ component link (L_2, f), then

$$nv([M, L_1 \cup L_2, n \cup f]) = v([M, L_1 \cup L_2, +1 \cup f]) \tag{3}$$

Property 2: If an unknot with integer framing winds around 3 bands once as part of an m component algebraically split link, then we have an equality as in Figure 1.

Property 3: If an unknot with integer framing n winds m times around a band as part of an m component algebraically split link, then we have an equality as in figure 2.

Let $\mathcal{F}_{m+1}\mathcal{M}$ denote the subspace of $\mathcal{M}$ spanned by $[M, L, f]$ for all rationally framed algebraically split $m+1$ component links L in rational homology 3-spheres M together with the linear combination of the elements $[M, L', f']$ represented in Figures 1 and 2 where L' are m component links. Let $\mathcal{F}_m\mathcal{O}$ denote the space of type m invariants of rational homology 3-spheres, and let $\mathcal{O}$ denote their union $\cup_{m \geq 0} \mathcal{F}_m\mathcal{O}$. Similar to the case of integral homology 3-spheres, we have that $\mathcal{O}$ is a filtered commutative algebra under pointwise multiplication. Let $\mathcal{G}_\star\mathcal{O} = \oplus_{m \geq 0} \mathcal{F}_m\mathcal{O}/\mathcal{F}_{m+1}\mathcal{O}$ denote the associated graded algebra. More generally, let $\mathcal{G}_\star(obj)$ denote the associated graded object of the filtered object $\mathcal{F}_\star(obj)$.

A few remarks are in order:

Remark 1.3. Property 1 implies Property 2. Indeed, use Lemma 4.1 of [Oh2], or Lemma 3.4 of [GL1]. Property 2 implies Property 3. Indeed, use Lemma 3.7 of [Oh2]. Furthermore, the proof of theorem 1 (see remark 3.7) shows that Property 0 and property 3 imply property 1. At the moment, we are not sure about which one of Properties 1-3 is the most fundamental Property to choose. Using linking pairings of rational homology 3-spheres one can show that Properties 1,2,3 are *independent* of Property 0.

Remark 1.4. An invariant of 3-manifolds can be obtained from the universal Vassiliev-Kontsevich invariant such that the invariant of a 3-manifold is expressed as a power series of linear combinations of chord diagrams, see [LMO]. It is expected that the coefficients of the invariant should be finite type invariants.[1]

Remark 1.5. L. Rozansky [Rz2] mentioned invariants of rational homology 3-spheres that satisfy Property 0. We suspect that the vector space of invariants of rational homology 3-spheres that satisfy only Property 0 is *infinite* dimensional.

Remark 1.6. Note that the space of type 0 invariants is one dimensional, generated by the constant invariant. Note that the invariant $M \to |H_1(M,\mathbb{Z})|$ is *not* of type 0, or of finite type! If we chose another definition by changing the equation 1 with another formula such that the above invariant was of type 0, then the space of finite type invariants would either not be an algebra under pointwise multiplication, or else the space of type 0 invariants would be infinite dimensional. Note also that even though there are infinitely many linking pairings for rational homology 3-spheres, the space of type m invariants is finite dimensional for every m. This could make the suspicious reader appreciate the present definition a bit more. We hope to return to this question some time in the future.

Let $\mathcal{F}_m\mathcal{M}(\mathbb{Z}), \mathcal{F}_m\mathcal{O}(\mathbb{Z}), \mathcal{O}(\mathbb{Z}), \mathcal{G}_m\mathcal{O}(\mathbb{Z})$ denote the analogous definitions for finite type invariants of integral homology 3-spheres. We obviously have a map $I_{\mathbb{Q},\mathbb{Z},m} : \mathcal{F}_m\mathcal{O} \to \mathcal{F}_m\mathcal{O}(\mathbb{Z})$. An obvious problem is to determine the structure of $\mathcal{G}_m\mathcal{O}$ and to decide whether $I_{\mathbb{Q},\mathbb{Z},m}$ is an isomorphism.

In this direction we have the following results:

Theorem 1. *If (L,f) is a rationally framed algebraically split m-component link in an integral homology 3-sphere M, then*

$$[M,L,f] = \prod_{i=1}^{m} \frac{1}{f_i}[M,L,\{+1,\cdots,+1\}] \in \mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M} \tag{4}$$

This in turn proves the following Theorem:

Theorem 2. *The map $I_{\mathbb{Q},\mathbb{Z},m} : \mathcal{F}_m\mathcal{O} \to \mathcal{F}_m\mathcal{O}(\mathbb{Z})$ is one-to-one.*

[1] This was recently proven by T. Le [Le].

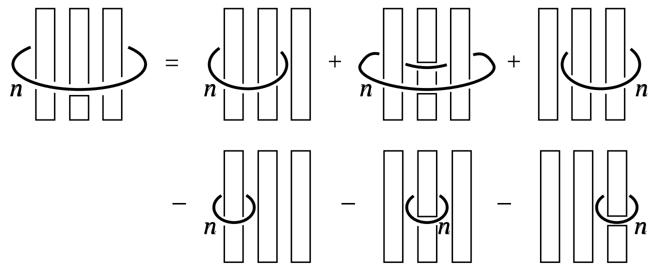

Figure 1. Shown here are parts of a link of m components, representing elements in $\mathcal{FM}_m$. A type m invariant of rational homology 3-spheres evaluated on these elements must satisfy the resulting identity.

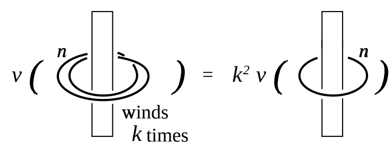

Figure 2. Same explanation as in Figure 1. v evaluated on the left hand side should equal to v evaluated on the right hand side.

Corollary 1.7. *Every type m invariant v of rational homology 3-spheres has an associated manifold weight system $W(v)$, defined in* [GO] *that determines it uniquely up to invariants of type $m-1$. Moreover, $W(v)$ is determined by the values of v on the set of integral homology 3-spheres. In particular, $\mathcal{F}_m\mathcal{O}$ is a finite dimensional space, and $\mathcal{F}_{3m}\mathcal{O} = \mathcal{F}_{3m+2}\mathcal{O}$.*

In small degrees m, the dimension of $\mathcal{G}_m\mathcal{O}$ is given by the following:

Proposition 1.8. *The Casson-Walker invariant is a type 3 invariant of rational homology 3-spheres. We therefore have the following table for the upper bounds of the dimensions of $\mathcal{G}_m\mathcal{O}$:*[2]

m	0	3	6	9	12
$dim\mathcal{G}_m\mathcal{O}$	1	1	$1 \leq \cdot \leq 2$	$1 \leq \cdot \leq 3$	$1 \leq \cdot \leq 5$

In case the reader considers the restriction to *nonzero* framing in definition 1.2 artificial, and the definition of finite type invariants of rational homology 3-spheres too restrictive, we have the following nonexistence Theorem for a possible definition of finite type invariants of oriented 3-manifolds:

[2]It follows essentially from the work of T. Le [Le] that the upper bounds are the exact dimensions of the graded vector spaces. T. Le considered only the case of integral homology 3-spheres, however the same argument applies (with the appropriate normalization) to the case of rational homology 3-spheres.

Theorem 3. *Let v be a (rationally valued) invariant of oriented closed 3-manifolds satisfying the following properties:*

- *There is a nonnegative integer m such that for every (L, f) algebraically split $m+1$-component link in a integral homology 3-sphere M, and framing $f_i \in \mathbb{Q}$, we have that*

$$v([M, L, f]) = 0 \tag{5}$$

- *The restriction of v to the set of rational homology 3-spheres is a type m invariant.*

Then, $v(M) = v(S^3)$ for every rational homology 3-sphere M.

Remark 1.9. Similarly, we can call a map $v : \mathcal{M} \to \mathbb{Q}$ a B-type m invariant of rational homology 3-spheres if for every (L, f) rationally framed boundary link (L, f) of $m+1$ components in a rational homology 3-sphere M, we have that $v([M, L, f]) = 0$, and v satisfies Properties 1,2, and 3 of definition 1.2. The same argument as in [GL2] shows that type $3m$ invariants of rational homology 3-spheres are included in B-type m invariants of rational homology 3-spheres. One advantage of using B-type m invariants is that it gives a "better grading" on the space of finite type invariants.

1.3. Questions.

Question 1. Is the map $I_{\mathbb{Q},\mathbb{Z},m} : \mathcal{F}_m\mathcal{O} \to \mathcal{F}_m\mathcal{O}(\mathbb{Z})$ an isomorphism?

The above question may be hard to answer (for large m). An advantage of dealing with finite type invariants of rational homology 3-spheres is that there is a complete set of moves (the two Kirby moves) that describe when Dehn surgery on two framed links in S^3 gives diffeomorphic rational homology 3-spheres. The analogous question for integral homology 3-spheres is not known (and is not likely to be known any time soon).

1.4. Plan of the proof. In section 2 we review an exceptional Dehn surgery isomorphism which allows us to prove a key Proposition 2.2. In section 3.1 we prove Theorems 1, 2, Corollary 1.7, and Proposition 1.8. In section 3.2 we prove the nonexistence Theorem 3. In section 1.5 we state the philosophical idea that underlines the present paper.

1.5. A philosophical comment. To justify the title "exceptional" in the title of the next section, we mention that for generic framings, Dehn surgery on a knot gives rise to irreducible 3-manifolds, [GoLu], however, for special knots and special framings, exceptional decompositions occur. It is a general philosophy in arithmetic, in algebraic geometry, in singularity theory, and in many other areas that exceptional phenomena are very important in the study of the generic situations. From this perspective, it is not at all a coincidence to observe the same behavior in topology.

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2. Exceptional Dehn surgeries

In this section we will use an exceptional Dehn surgery diffeomorphism that will be the key to the proof of Proposition 2.2. The proof will be given by appropriate drawings. In order to distinguish *in drawings* the difference between $[M, L, f]$ and $M_{(L,f)}$ we introduce the following notation:

Remark 2.1. For a *partially drawn* framed link (L, f) in a rational homology 3-sphere M, let L' denote the sublink consisting of all components with a black dot on them, as in Figure 3. Then the resulting drawing represents the element $[M_{(L',f')}, L-L', f-f']$ of $\mathcal{M}$. In, particular, if none of the components of a drawn link is marked with a black dot, then the resulting drawing represents the element $[M, L, f]$ of $\mathcal{M}$. See Figure 3.

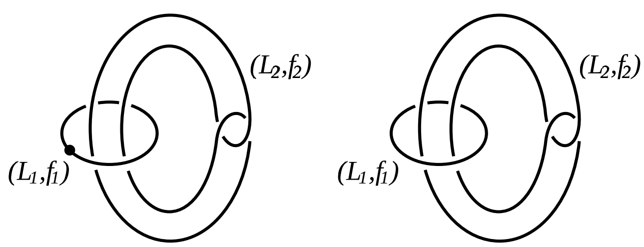

Figure 3. Each picture represents a linear combination of elements in $\mathcal{M}$. Note the difference between a black dot (on the left hand side) and not (on the right hand side). The left hand side represents the element $[M_{L_1,f_1}, L_2, f_2] = M_{L_1,f_1} - M_{L_1L_2,f_1f_2}$ of $\mathcal{M}$, whereas the right hand side represents the element $[M, L_1, L_2, f_1, f_2]$, which is a sum of four terms..

We can now state the following proposition:

Proposition 2.2. *Let p, q be coprime, nonzero integers.If a band is denoted as in Figure* 4*, then τ induces an orientation preserving diffeomorphism as in Figure* 5*. Note that $K_{p,q}$ is a knot such that the right hand side of Figure* 5 *is surgically equivalent* [GL1] *to Figure* 6*.*

Proof. Assume for the moment that both p, q are positive. Recall that Moser [Mo] constructed an orientation preserving diffeomorphism

$$\tau : S^3_{(O_1 \cup O_2, p/q \cup q/p)} \simeq S^3_{(T_{p,q}, pq)} \tag{6}$$

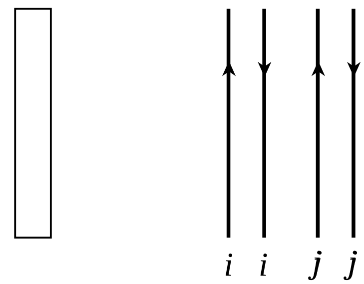

Figure 4. A convention of drawing bands. A band is a number of arcs of a (homologically trivial) algebraically split link in a rational homology 3-sphere $\mathcal{M}$. Note that the arcs are orianted and they are grouped in pairs, two for each component of the link, with opposite orientations. The subscripts i, j indicate the two (not necessarily distinct) components that the arcs of the band belong to.

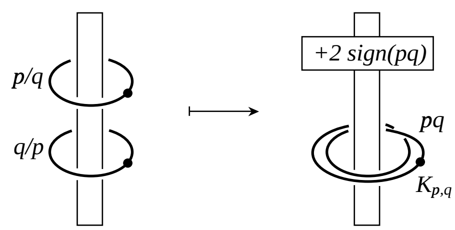

Figure 5. The statement of Proposition 2.2 in pictures.

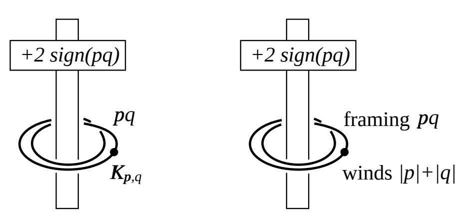

Figure 6. More on the statement of Proposition 2.2 in pictures.

where $O_1 \cup O_2$ is an unlink of two components in S^3, and $T_{p,q}$ is the (p,q) torus knot. Note that to make the notation clearer, we omit the p,q dependence on the diffeomorphism τ.

We will give a more explicit description of τ with the help of the figures. To analyze the map τ, recall that $S^3 = SD_1 \cup_T SD_2$ is the union of two solid tori SD_1, SD_2 along their common boundary $SD_1 \cap SD_2 = T$, and that the knot $T_{p,q}$ lies in T. Note that $T_{p,q}$ is a (p,q) torus knot with respect to SD_1, and a (q,p) torus knot with respect to SD_2. See Figure 7.

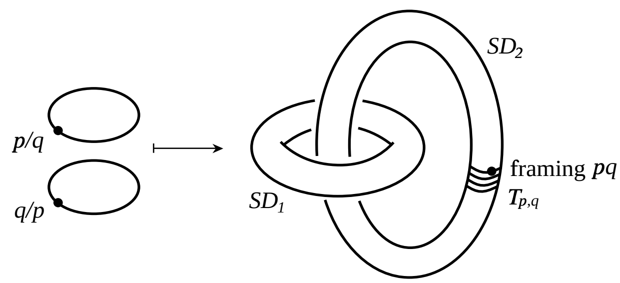

Figure 7. The diffeomorphism τ.

The images $\tau(m_i)$ of the meridians m_i (for $i = 1,2$) of the unknots O_i are the cores of the solid tori SD_i. The image $\tau(m_1 \sharp_b m_2)$ of their band sum is a band sum $\tau(m_1)\sharp_{\tau(b)}\tau(m_2)$. Therefore, if J is any satellite in the solid torus SD_0 (represented by tangles A, B) its image under τ is given by Figure 8.

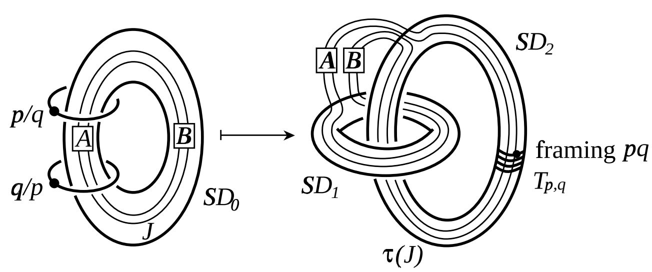

Figure 8. The image under τ of a satellite J.

Note that the link in the right hand side of Figure 8 is surgically equivalent to the link of Figure 5. This concludes the proof of Proposition 2.2 in the case of both p,q positive. In case they are of opposite signs, reverse the orientation of the two manifolds of the equation (6), use the fact that $\overline{S^3_{(L,f)}} = S^3_{(L^m,-f)}$ (where L^m is the mirror image of L, and $\overline{M}$ denotes the orientation reversed manifold of M), and proceed as in the case of both p,q are positive. In the case of both p,q are negative, reverse their signs, and proceed as in the case of both of them being positive. This concludes the proof of Proposition 2.2. $\square$

3. Proofs

3.1. Results for rational homology 3-spheres. In this section we prove Theorems 1, 2, as well as corollary 1.7 and Proposition 1.8. We begin by introducing some notation to be followed throughout this section.

Let (L, f_L) be an $m-1$ component algebraically split rationally framed link in S^3. Let C be an unknot that bounds a disc which intersects the link L as in Figure 9. We call (L, C) a *good pair*. Note that with the drawing conventions of figure 4 we have that $L \cup C$ is a homologically trivial algebraically split link. In order to simplify notation, for a rational framing f, we denote $\phi(f) = [M, L \cup C, f_L \cup f] \in \mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$.

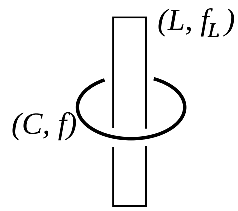

Figure 9. A good pair (C, L) together with its framing.

We begin with an elementary Lemma:

Lemma 3.1. *For an algebraically split rationally framed link (K, f) in a integral homology 3-sphere M, we have:*

$$M_{K,f} - M = \sum_{\emptyset \subsetneq K' \subseteq K} (-1)^{|K'|}[M, K', f'] \tag{7}$$

Proof. Immediate by definition. □

Lemma 3.2. *Let (L, C) be a good pair of m components, and n a natural number. Consider n-parallel copies of C with 0 framing; we denote them by $C_1, C_2, \cdots, C_n$. Let $\{f_i\}_{i=1}^n$ be a sequence of rational framings of $\{C_i\}$. Then we have the identity of Figure 10 in $\mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$.*

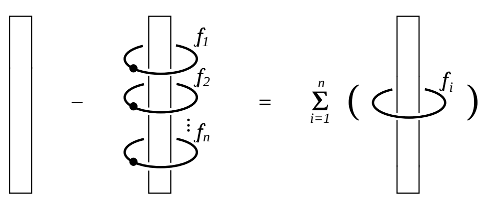

Figure 10. Lemma 3.2 in pictures. In the part of the figure before the equal sign is the link $L \cup C_1 \cdots \cup C_n$.

Proof. Using Lemma 3.1, we get the identity of Figure 11.

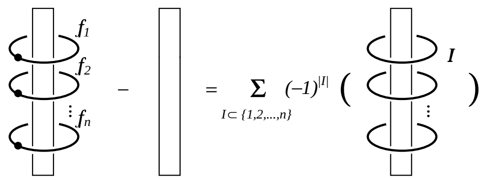

Figure 11. $|I|$ is the cardinality of I, and in the right hand side we have as many circles as elements of I, framed by I.

Since for $|I| > 1$ the right hand side of figure 11 lies in $\mathcal{F}_{m+1}\mathcal{M}$, the result follows. □

Corollary 3.3. *Let $\epsilon = -1, +1$. For p, q coprime such that $q(q + \epsilon p) \neq 0$, with the notation as in the beginning of this section, we have:*

$$\phi(\frac{p}{q + \epsilon p}) = \phi(\frac{p}{q}) + \epsilon\phi(1) \tag{8}$$

Proof. We begin by recalling in Figure 12 the second Kirby move for a rationally framed algebraically split m-component link, [Ro1], [Ro2]:

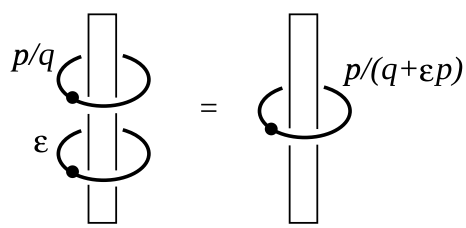

Figure 12. A Kirby move, representing diffeomorphic 3-manifolds. Here $\epsilon = -1, 1$.

The result now follows by Lemma 3.2 using the sequence of rational framings $(\frac{p}{q}, \epsilon)$, and the fact that $\phi(\epsilon) = \epsilon\phi(1)$, see Lemma 2.5 of [Oh2]. □

Proposition 3.4 (reciprocity law). *For p, q coprime, nonzero, we have the following:*

$$\phi(\frac{p}{q}) + \phi(\frac{q}{p}) = \frac{p^2 + q^2}{pq}\phi(1) \tag{9}$$

Proof. Let us first assume that $m \geq 2$. Using Proposition 2.2, and Lemma 3.1 (with the link K of Lemma 3.1 being the unlink of two components $O_1 \cup O_2$) end the fact that rational homology 3-spheres obtained by surgery on surcically equivalent

m component links have the same image in $\mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$, we obtain the following equality in $\mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$:

(10)
$$[M, L \cup O_1 \cup O_2, f \cup \frac{p}{q} \cup \frac{q}{p}] - \phi(\frac{p}{q}) - \phi(\frac{q}{p}) = -[M, \tilde{L} \cup K_{p,q}, f \cup pq] \in \mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$$

where $\tilde{L}$ differs by L by $sign(pq)2$ full twists. Since $[M, L \cup O_1 \cup O_2, f \cup \frac{p}{q} \cup \frac{q}{p}] \in \mathcal{F}_{m+1}\mathcal{M}$, and $pq[M, \tilde{L} \cup K_{p,q}, f \cup pq] = [M, \tilde{L} \cup K_{p,q}, f \cup +1]$ (by Property 1), we obtain that:

$$\phi(\frac{p}{q}) + \phi(\frac{q}{p}) = \frac{1}{pq}[M, \tilde{L} \cup K_{p,q}, f \cup +1] \tag{11}$$

Notice that up to now we deduced eqquation (11) *without* using any of the properties 1,2 or 3 that define the filtration $\mathcal{F}_m\mathcal{M}$. Now we will use property 1 to finish the proof of proposition 3.4.

Indeed, using the fact that $K_{p,q}$ winds around $\tilde{L}$ $(p+q)$ times, and Lemma 3.7 of [Oh2] and Property 1 we obtain that $[M, \tilde{L} \cup K_{p,q}, f \cup +1] = ((|p| + |q|)^2 - 2sign(pq))\phi(1)$ which concludes the proof of the Proposition in the case $m \geq 2$. In the case of $m = 1$, using the isomorphism τ of section 2 we obtain that: $\phi(\frac{p}{q}) + \phi(\frac{q}{p}) = \phi(pq)$, and proceeding as above we obtain the conclusion that $\phi(\frac{p}{q}) = \frac{q}{p}\phi(1) = 0$. The case of $m = 0$, is immediate. □

Remark 3.5. Note that alternatively, we could have deduced proposition 3.4 using Property 2 or 3 (since $K_{p,q}$ winds around $\tilde{L}$ $p+q$ times.

Corollary 3.6. *With the above notations, for coprime nonzero p, q we have that:*

$$\phi(\frac{p}{q}) = \frac{q}{p}\phi(1) \tag{12}$$

Proof. We claim that there is at most one function ϕ satisfying the equations (8) and (9). This is a standard argument from the Euclidean algorithm. Now the function $\phi(\frac{p}{q}) = \frac{q}{p}\phi(1)$ satisfies the equations (8) and (9). The result follows. □

Proof. [of Theorem 1] We first reduce Theorem 1 to the case of good pairs. This is a standard argument from [Oh2], [GL1], but for the convenience of the reader we repeat it here once again. Let (L, f) be a algebraically split rationally framed m-component link in a integral homology 3-sphere M. We are interested in the image of $[M, L, f] \in \mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$. Using the fundamental equation

$$[M, L \cup K, f \cup f_0] = [M, L, f] - [M_{(K, f_0)}, L, f] \tag{13}$$

and observing that the left hand side of the above equation lies in $\mathcal{F}_{m+1}\mathcal{M}$, we can replace M by M_{K,f_0} without affecting the image of $[M, L, f] \in \mathcal{F}_m\mathcal{M}/\mathcal{F}_{m+1}\mathcal{M}$.

Since every integral homology 3-sphere M can be obtained by a sequence of surgeries on a algebraically split link, we can assume that $M = S^3$. Furthermore, using the fundamental equation together with a Kirby move, we can replace L within its surgical equivalence class. Furthermore, using Property 1, we can replace L by a unit framed one, and therefore by a link obtained by a trivalent, vertex oriented tree as described in [Oh2], [GL2]. We can therefore assume that we are in the family of links described in Propositions 3.3 and 3.4. Now apply Corollary 3.6 to each of the components of the link L. □

Remark 3.7. The above proof of Theorem 1 and Remark 3.5 show that Property 3 and Property 0 of Definition 1.2. Indeed, Property 3 and 0 imply the equation (4) which of course implies Property 1.

Proof. [of Theorem 2] Recall the obvious inclusion map: $\mathcal{M}(\mathbb{Z}) \to \mathcal{M}$, compatible with the decreasing filtrations $\mathcal{F}_\star$. Theorem 2 implies that the associated graded map: $\mathcal{G}_m\mathcal{M}(\mathbb{Z}) \to \mathcal{G}_m\mathcal{M}$ is *onto*, and dually the map $I_{\mathbb{Q},\mathbb{Z},m} : \mathcal{O}_m \to \mathcal{O}_m(\mathbb{Z})$ is one-to-one. □

Proof. [of Corollary 1.7] It follows from Theorem 2, and the fact that finite type invariants of integral homology 3-spheres are determined (modulo invariants of lower type) by their associated manifold weight systems, [GO]. In particular, since $\mathcal{F}_{3m}\mathcal{O}(\mathbb{Z}) = \mathcal{F}_{3m+2}\mathcal{O}(\mathbb{Z})$, we obtain that $\mathcal{F}_{3m}\mathcal{O} = \mathcal{F}_{3m+2}\mathcal{O}$. □

Proof. [of Proposition 1.8] We show that the *Casson-Walker* invariant λ_{CW} is a type 3 invariant of rational homology 3-spheres. Indeed, let (L, f) be an algebraically split rationally framed m-component link in a integral homology 3-sphere M. For simplicity in notation, for a sublink L' of L, let $f_{L'} = \prod_{i\in L'} f_i$. Recall the following surgery formula for the Casson-Walker invariant from [Wa], [Lp]:

$$\lambda_{CW}(M_{(L,f)}) = \lambda_{CW}(M) + \sum_{\emptyset \not\subseteq L' \subseteq L} \frac{1}{f_{L'}}\phi(M, L') + \sum_{i=1}^{m} sign(p_i)s(q_i, p_i) \tag{14}$$

where $\phi(M, L')$ is the $|L'| + 1$ coefficient of the Conway polynomial of L' in M, and $s(q, p)$ is the Dedekind sum, [Ra]. Therefore, for $m \geq 2$ we obtain that:

$$\lambda_{CW}([M, L, f]) = \frac{1}{f_L}\phi(M, L) \tag{15}$$

Using Hoste's [Ho] observation that $\phi(M, L) = 0$ if $L \geq 4$, we get that $\lambda_{CW}[M, L, f] = 0$ for $m \geq 4$, and that λ_{CW} satisfies Properties 1, 2, and 3 and thus it is a type 3 invariant of rational homology 3-spheres. Note that the equation (15) follows by the equation (14) *without* using any arithmetic properties of the Dedekind sums. Of course, in the course of defining the Casson-Walker invariant explicit arithmetic properties of the Dedekind sums are used. □

3.2. Proof of the nonexistence Theorem 3. In this section we prove the nonexistence Theorem 3.

Proof. [of Theorem 3] Let v be an invariant of 3-manifolds with the asserted properties. We begin by recalling in Figure 13 an identity from Kirby calculus:

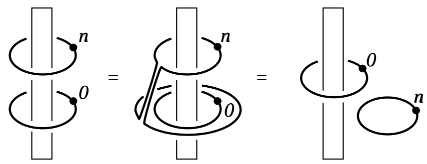

Figure 13. An identity of 0 framings. All 3 pictures represent the same (linear combination of) 3-manifolds. For a proof, add the 0 framed unknot to the n framed unknot with a band move, shown in the middle.

Using Lemma 3.2 we obtain the identity of Figure 14.

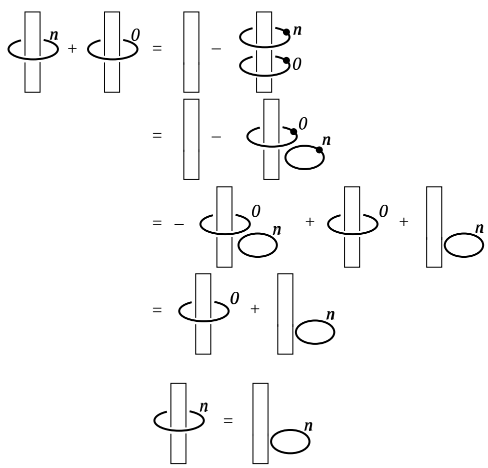

Figure 14. The proof of Theorem 3.

This implies, with the notation of section 3.1 that $\phi(n) = 0$, and the result follows. □

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