CONSTRUCTING 1-CUSPED ISOSPECTRAL NON-ISOMETRIC HYPERBOLIC 3-MANIFOLDS

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Abstract. We construct infinitely many examples of pairs of isospectral but non-isometric 1-cusped hyperbolic 3-manifolds. These examples have infinite discrete spectrum and the same Eisenstein series. Our constructions are based on an application of Sunada's method in the cusped setting, and so in addition our pairs are finite covers of the same degree of a 1-cusped hyperbolic 3-orbifold (indeed manifold) and also have the same complex length-spectra. Finally we prove that any finite volume hyperbolic 3-manifold isospectral to the figure-eight knot complement is homeomorphic to the figure-eight knot complement.

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1. Introduction

Since Kac [21] formulated the question: *Can you hear the shape of a drum?*, there has been a rich history in constructing isospectral but non-isometric manifolds in various settings. We will not describe this in any detail here, but simply refer the reader to [17] for a survey. The main purpose of this note is to prove the following result (see also Theorem 2.5 for a more detailed statement).

Theorem 1.1. *There are infinitely many pairs of finite volume orientable 1-cusped hyperbolic 3-manifolds that are isospectral but non-isometric.*

Since we are working with cusped hyperbolic 3-manifolds, the statement of the theorem requires some clarification. Indeed, one can reasonably ask, *what does isospectral mean for cusped hyperbolic 3-manifolds*. We address this in Section 2, where we indicate the differences with the closed case. Our examples appear to be the first examples of 1-cusped hyperbolic 3-manifolds that are known to be isospectral and non-isometric. On the other hand, there has been considerable interest in this for surfaces (both non-compact finite area, and infinite area convex cocompact, see [4] and the survey [16]). In fact in [16], they raise the problem (Problem 1.2 of [16]) of finding such examples in much more generality. Note that these papers use the terms *isoscattering* or *isopolar*, but we prefer to stick with isospectral.

Theorem 1.1 is well-known for closed hyperbolic 3-manifolds, either using the arithmetic methods of [36], or the method of Sunada ([32] and which we recall below), as in [27, p.225]. As in this latter setting, our construction also uses Sunada's method, but we need some additional control. In addition to proving the existence of infinitely many pairs of examples of isospectral 1-cusped hyperbolic 3-manifolds, we also give more concrete examples of isospectral manifolds arising as low degree covers of small volume 1-cusped hyperbolic 3-manifolds arising in the census of hyperbolic manifolds of `Snap` and `SnapPy` [14, 7].

One motivation for Theorem 1.1 was to investigate the nature of the discrete spectrum of 1-cusped hyperbolic 3-manifolds. There has been considerable interest in this for non-compact surfaces of finite area (see [23], [20] and [31] to name a few), but little seems known in dimension 3. We discuss this further in Section 8, and in particular we prove the following.

Theorem 1.2. *Let M denote the complement of the figure-eight knot in S^3. Suppose that N is a finite volume hyperbolic 3-manifold which is isospectral with M. Then N is homeomorphic to M.*

Indeed we also show that the first ten 1-cusped orientable finite volume hyperbolic 3-manifolds are determined by their spectral data (see Definition 2.1).

2. What does isospectral mean for cusped manifolds?

As remarked upon in the Introduction, since we are in the setting of cusped orientable hyperbolic 3-manifolds, some clarification about the statement of Theorem 1.1 is required, and in this section we explain what we mean by *isospectral* 1-*cusped hyperbolic 3-manifolds*. Throughout this paper we will restrict ourselves to only discussing the spectrum for 1-cusped hyperbolic 3-manifolds. This simplifies things, but much of what is described in this section holds more generally, and similar statements can be made in the presence of multiple cusps.

We refer the reader to Chapters 4 and 6 of [10] or [6] for a detailed discussion of the spectral theory of cusped hyperbolic 3-manifolds.

2.1. **The spectrum of the Laplacian in the cusped setting.** Let $M = \mathbf{H}^3/\Gamma$ be a 1-cusped, orientable finite volume hyperbolic 3-manifold. The spectrum of the Laplacian on the space $L^2(M)$ consists of *a discrete spectrum* (i.e. a collection of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \ldots$ where each λ_j has finite multiplicity), together with *a continuous spectrum* (a copy of the interval $[1, \infty)$). Moreover, the discrete spectrum consists of finitely many eigenvalues in $[0, 1)$, together with those eigenvalues embedded in the continuous spectrum. However, unlike the closed setting, in general, it is unknown as to whether the discrete spectrum is infinite (we address this point in Section 2.3).

The eigenfunctions associated to eigenvalues in the discrete spectrum form an orthonormal system and the closed subspace of $L^2(M)$ that they generate is denoted by $L^2_{disc}(M)$. The orthogonal complement of $L^2_{disc}(M)$ in $L^2(M)$ is denoted by $L^2_{cont}(M)$ and "corresponds" (in a way that we need not make precise here) to the continuous spectrum (see [10] Chapters 4 and 6).

In the closed case, the Weyl law provides a way to prove that the discrete spectrum is infinite (see [10] Chapter 5). The precise analogue of this in the cusped setting is not available, and this necessitates understanding a contribution from an Eisenstein series associated to the cusp of M. To describe this further, conjugate Γ so that a maximal peripheral subgroup $P < \Gamma$ fixes infinity. Fixing co-ordinates on $\mathbf{H}^3 = \{w = (z, y) : z \in \mathbf{C}, y \in \mathbf{R}_+\}$, we define the *Eisenstein series associated to the cusp at infinity* by:

$$E(w, s) = \sum_{\gamma \in P \backslash \Gamma} y(\gamma w)^s,$$

where $y(p)$ denotes the y co-ordinate of the point $p \in \mathbf{H}^3$. Now since $E(w, s)$ is P-invariant, an analysis of the Fourier expansion at ∞ reveals a constant term of the form:

$$y(w)^s + \phi(s) y(w)^{2-s},$$

where $\phi(s)$ is the so-called *scattering function*. This is defined for $\mathrm{Re}(s) > 2$ and has a meromorphic extension to the complex plane. The poles of $\phi(s)$ are also the poles of the Eisenstein series and all lie in the half-plane $\mathrm{Re}(s) < 1$, except for at most finitely many in the interval $(1, 2]$. Moreover, if $t \in (1, 2)$ is a pole, the residue $\psi = \mathrm{Res}_{s=t} E(w, s)$ is an eigenfunction with eigenvalue $t(2 - t)$ (see [6]). In addition, if there is a pole at $s = 2$, the residue will be an eigenfunction with eigenvalue 0 ([6]). This subset of the discrete spectrum arising from residues of poles of the Eisenstein series is called the *residual spectrum*. If t is a pole of $E(w, s)$ (equivalently $\phi(s)$) we define the *multiplictity* at t to be the order of the pole at t, plus the dimension of the eigenspace in the case when t contributes to the residual spectrum as described above.

The following definition is, in part, motivated by what spectral information is required to determine the geometry in the cusped setting; e.g. the role of the scattering function and its poles is natural in the analogue of the Weyl law for cusped manifolds(see [10] Theorem 6.5.4).

Definition 2.1. *Let M_1 and M_2 be 1-cusped orientable hyperbolic 3-manifolds of finite volume with associated scattering functions $\phi_1(s)$ and $\phi_2(s)$. Assume that the discrete spectrum of M_1 is infinite. Say that M_1 and M_2 are isospectral if:*

- *M_1 and M_2 have the same discrete spectrum, counting multiplicities;*
- *$\phi_1(s)$ and $\phi_2(s)$ have the same set of poles and multiplicities.*

Remark 2.2. *(1) For a 1-cusped orientable finite volume hyperbolic 3-manifold M, its discrete spectrum, counting multiplicities, together with the set of poles and multiplicities of the scattering function will be referred to as its spectral data.*

(2) For a multi-cusped orientable finite volume hyperbolic 3-manifold M, the scattering function is a matrix (the scattering matrix), and in this case one takes the determinant of the scattering matrix to obtain a function $\tau_M(s)$ that plays the role of $\phi(s)$ above.

(3) Continuing with the discussion of the role of the scattering function in determining the geometry from spectral data, it is shown in [23] that an analogue of Huber and McKean's results for compact surface holds. Namely, the spectral data in Definition 2.1 (in the context of a non-compact hyperbolic surface of finite area), determines the length spectrum of the surface and vice versa (see Section 8 for a discussion of this for 1-cusped hyperbolic 3-manifolds). Moroever, there are only finitely many hyperbolic surfaces with the given spectral data.

(4) In general the scattering determinant is hard to compute explicitly. However, for arithmetic manifolds (and orbifolds) the scattering determinant is related to Dedekind zeta functions of number fields. For example, for $\mathrm{PSL}(2,\mathbb{Z})$ the poles of the scattering function are related to the zeroes of $\zeta(s)$ (see [20]), whilst for the Bianchi orbifolds with one cusp, the scattering function is expressed in terms of the zeta function $\zeta_K(s)$ attached to the quadratic imaginary number field K (see [9] or [10, Chpt.8.3]).

2.2. **Manifolds with the same Eisenstein series.** The following lemma will be useful in our construction. We fix some notation. Let $M = \mathbf{H}^3/\Gamma$ be a 1-cusped orientable finite volume hyperbolic 3-manifold with finite covers $M_i = \mathbf{H}^3/\Gamma_i$ $(i = 1, 2)$ both with one cusp and of same covering degree, n say. Conjugate Γ so that a maximal peripheral subgroup $P < \Gamma$ fixes ∞, and let $P_i = \Gamma_i \cap P$. Denote the Eisenstein series associated to M, M_1 and M_2 constructed in Section 2.1 by $E(w, s)$, $E_1(w, s)$ and $E_2(w, s)$ respectively.

Lemma 2.3. *Let M, M_1 and M_2 be as above. Then $E_1(w, s) = E_2(w, s)$. In particular M_1 and M_2 have the same scattering function.*

Proof. We begin the proof with a preliminary remark. Suppose that $N = \mathbf{H}^3/G$ is 1-cusped and is a finite covering of M. We claim that a set of distinct coset representatives for G in Γ can be chosen from elements of P. Briefly, since the preimage of the cusp of M is connected (i.e. is the single cusp of N), we have must equality of indices $[\Gamma : G] = [P : P \cap G]$. Thus a collection of coset representatives for $P \cap G$ in P also works as coset representatives G in Γ.

Given this, let $S = \{\delta_1, \ldots, \delta_n\} \subset P$ be a set of distinct (left) coset representatives for Γ_1 in Γ and $S' = \{\delta'_1, \ldots, \delta'_n\} \subset P$ be a set of distinct coset representatives for Γ_2 in Γ.

Now any term in $E(w,s)$ has the form $y(\gamma w)^s$ for $\gamma \in \Gamma$ not fixing ∞. Using the above decomposition of Γ as a union of cosets of both Γ_1 and Γ_2, there exists $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ and $\delta_j \in S$, $\delta'_k \in S'$ so that:

$$\delta_j \gamma_1 = \gamma = \delta'_k \gamma_2.$$

Since $\delta_j, \delta'_k \in P$ and $\gamma \notin P$, it follows that $\gamma_1 \notin P_1$ and $\gamma_2 \notin P_2$ (otherwise $\gamma \in P$, contrary to the definition of the Eisenstein series). Using the coset decomposition of Γ, it follows that $E(w,s)$ can be decomposed as a sum of terms of the form:

$$(*) \qquad \sum_{g \in P_1 \backslash \Gamma_1} y(\delta_j g w)^s \qquad \text{and} \qquad \sum_{h \in P_2 \backslash \Gamma_2} y(\delta'_k h w)^s.$$

Since $\delta_j, \delta'_k \in P$, they act by translation on $\mathbf{H}^3$, and in particular the y-cordinate is unchanged by this; i.e. $y(\delta_j g w) = y(gw)$ and $y(\delta'_k h w) = y(hw)$. Hence the terms in $(*)$ above reduce to

$$\sum_{g \in P_1 \backslash \Gamma_1} y(gw)^s \text{ and } \sum_{h \in P_2 \backslash \Gamma_2} y(hw)^s.$$

So, putting all of this together, we have the following:

$$nE_1(w,s) = E(w,s) = nE_2(w,s),$$

which proves the lemma. □

2.3. **Ensuring the discrete spectrum is infinite.** In this section we address the issue of ensuring that the discrete spectrum is infinite. In particular we state a result that can be proved using the methods of [34] (see also the comments in [10] at the end of Chapter 6.5). To state the result we need to recall the following.

A fundamental dichotomy of Margulis for a finite volume hyperbolic manifold $M = \mathbf{H}^3/\Gamma$, is whether M is arithmetic or not. In the commensurability class of a non-arithmetic manifold, there is a unique minimal element in the commensurability class. This minimal element arises as $\mathbf{H}^3/\mathrm{Comm}(\Gamma)$, where

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbf{H}^3) : g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}$$

is *the commensurator* of Γ.

The following can be proved following the methods in [34]. Note that in the statement of [34] Theorem 2, a certain matrix determinant is assumed to be non-vanishing. In our setting, since the manifold has one cusp, this matrix coincides with a function and can be shown to not be identically zero.

Theorem 2.4. *Let $M = \mathbf{H}^3/\Gamma$ be an orientable finite volume 1-cusped non-arithmetic hyperbolic 3-manifold that is not the minimal element in its commensurability class (i.e. $\Gamma \neq \mathrm{Comm}(\Gamma)$). Then the discrete spectrum of M is infinite.*

We will make some further comments on the nature of the discrete spectrum (when it is known to be infinite) in Section 8.

2.4. **The complex length spectrum.** Let $M = \mathbf{H}^3/\Gamma$ be an orientable finite volume hyperbolic 3-manifold. Given a loxodromic element $\gamma \in \Gamma$, the *complex translation length* of γ is the complex number $L_\gamma = \ell_\gamma + i\theta_\gamma$, where ℓ_γ is the translation length of γ and $\theta_\gamma \in [0, \pi)$ is the angle incurred in translating along the axis of γ by distance ℓ_γ. The *complex length spectrum* of M is defined to be the collection of all complex translation lengths counted with multiplicities.

Given the discussion of the previous subsections, we now give a more detailed statement of Theorem 1.1.

Theorem 2.5. *There are infinitely many pairs of finite volume orientable 1-cusped hyperbolic 3-manifolds that are isospectral but non-isometric. In addition, our pairs have the following properties:*

- *cover a 1-cusped hyperbolic 3-manifold of the same degree,*
- *have infinite discrete spectrum,*
- *have the same Eisenstein series,*
- *have the same complex length spectra.*

3. The Sunada construction in the 1-cusped setting

Let G be a finite group and H_1 and H_2 subgroups of G. We say that H_1 and H_2 are *almost conjugate* if they are not conjugate in G but for every conjugacy class $C \subset G$ we have:

$$|C \cap H_1| = |C \cap H_2|.$$

If the above condition is satisfied, we call (G, H_1, H_2) a *Sunada triple*, and (H_1, H_2) an almost conjugate pair in G. We prove the following using Sunada's method [32] (cf. [1, 2, 3, 26]).

Theorem 3.1. *Let $M = \mathbf{H}^3/\Gamma$ be a 1-cusped finite volume orientable hyperbolic 3-manifold that is non-arithmetic and the minimal element in its commensurability class. Let G be a finite group, (H_1, H_2) an almost conjugate pair in G, and assume that Γ admits a homomorphism onto G. Assume that the finite covers M_1 and M_2 associated to the pullback subgroups of H_1 and H_2 have 1 cusp. Then M_1 and M_2 are isospectral, have the same complex length spectra and are non-isometric.*

Proof. First, note that the manifolds M_1 and M_2 cannot be isometric, since if there exists $g \in \mathrm{Isom}(\mathbf{H}^3)$ with $g\Gamma_1 g^{-1} = \Gamma_2$, then this implies that $g \in \mathrm{Comm}(\Gamma)$. However, by assumption, $\mathrm{Comm}(\Gamma) = \Gamma$, and so projecting to the finite group G, we effect a conjugacy of the almost conjugate pair (H_1, H_2), a contradiction.

To prove isospectrality, there are two things that need to be established; that both M_1 and M_2 have the same infinite discrete spectrum with multiplicities, and that their scattering functions have the same poles with multiplicities. Since M_1 and M_2 are 1-cusped, and $[\Gamma : \Gamma_1] = [\Gamma : \Gamma_2]$, the latter follows immediately from the fact that their Eisenstein series are the same by Lemma 2.3.

Regarding the former statement, Theorem 2.4 shows that the discrete spectrum is infinite for both M_1 and M_2, and we deal with remaining statement about the discrete spectra in a standard way following [32]. For completeness we sketch a proof of this.

Now it can be shown that to prove that M_1 and M_2 have the same discrete spectra with multiplicities, it suffices to show that $L^2_{disc}(M_1) \cong L^2_{disc}(M_2)$. To see this we find it convenient to follow [26] and we refer the reader to that paper for details. We need a lemma from [26] and this requires some notation. Let G be a finite group, and V is a G-module. Denote by V^G the submodule of V invariant under the G-action. The following is Lemma 1 of [26]:

Lemma 3.2. *Suppose G is a finite group, (H_1, H_2) an almost conjugate pair in G and suppose that G acts on the complex vector space V. Then there is an isomorphism $\iota : V^{H_1} \to V^{H_2}$, commuting with the action of any endomorphism Δ of V for which the following diagram commutes.*

$$\begin{array}{ccc} V^{H_1} & \xrightarrow{\iota} & V^{H_2} \\ \Delta\big\downarrow & & \big\downarrow\Delta \\ V^{H_1} & \xrightarrow{\iota} & V^{H_2} \end{array}$$

Now let M_0 be the cover of M corresponding to the kernel of the homomorphism to G. Taking V to be $L^2_{disc}(M_0)$ in Lemma 3.2, Δ to be the Laplacian, and noting that for $i = 1, 2$, $L^2_{disc}(M_i) = L^2_{disc}(M_0)^{H_i}$, it follows that $L^2_{disc}(M_1) \cong L^2_{disc}(M_2)$.

The proof that the manifolds have the same complex length spectra follows that given in [32]. $\square$

Remark 3.3. *As noted above, the method of Sunada* [32] *also produces pairs of finite volume hyperbolic 3-manifolds with the same complex length spectrum. More generally, in the case of closed hyperbolic 3-manifolds, the complex length spectrum is known to determine the spectrum of the Laplacian, see* [30, Thm.1.1]. *This also holds for cusped hyperbolic manifolds, as can be seen from* [22, Thm.2] *for example.*

Example 3.4. *For p a prime, we denote by $\mathbf{F}_p$ the finite field of p elements, and denote by* $\mathrm{PSL}(2,p)$ *the finite group* $\mathrm{PSL}(2,\mathbf{F}_p)$ *(which of course are simple for $p > 3$). It is known that (see* [19] *for example) for $p = 7, 11$ the groups* $\mathrm{PSL}(2,p)$ *contain almost conjugate pairs of subgroups of index* 7 *and* 11 *respectively.*

Remark 3.5. *In* [19]*, it is shown that there are no examples of almost conjugate (but not conjugate) subgroups of a finite group of index less than* 7*. Hence,* 7*-fold covers are the smallest index covers for which the Sunada construction can be performed.*

Given the previous set up, we can now prove the following straightforward proposition that is the key element in our construction. We require a preliminary definition. Following Riley [29] if $M = \mathbf{H}^3/\Gamma$ is an orientable finite volume 1-cusped hyperbolic 3-manifold, $P < \Gamma$ a fixed maximal peripheral subgroup and $\rho : \pi_1(M) \to \mathrm{PSL}(2,p)$ a representation, then ρ is called a *p-rep* if $\rho(P)$ is non-trivial and all non-trivial elements in $\rho(P)$ are parabolic elements of $\mathrm{PSL}(2,p)$. In which case, $\rho(P)$ is easily seen to have order p. More generally if M has more than 1-cusp we call ρ a p-rep of $\pi_1(M)$ if the image of all maximal peripheral subgroups satisfies the same condition as above.

Proposition 3.6. *Let $M = \mathbf{H}^3/\Gamma$ be an orientable non-arithmetic finite volume 1-cusped hyperbolic 3-manifold that is the minimal element in its commensurability class. Suppose that ρ is a p-rep of Γ onto $G =$* $\mathrm{PSL}(2,7)$ *or* $\mathrm{PSL}(2,11)$*. Then M has a pair of 1-cusped* *isospectral but non-isometric covers of degree* 7 *or* 11 *respectively. In addition this pair of manifolds have the same complex length spectra.*

Proof. Let $M_i = \mathbf{H}^3/\Gamma_i$ $(i = 1, 2)$, be the covers of M corresponding to the almost conjugate pair in Example 3.4 above in either of the cases $p = 7, 11$.

Once we establish that M_1 and M_2 both have 1 cusp, that M_1 and M_2 are isospectral and non-isometric follows from Theorem 3.1. This also shows that they have the same complex length spectra. We deal with the case of $p = 7$, the case of $p = 11$ is exactly the same.

Let P denote a fixed maximal peripheral subgroup of Γ. For $i = 1, 2$, let $P_i = \Gamma_i \cap P$. We claim that for $i = 1, 2$, $[P : P_i] = 7$. This implies that the covers M_1 and M_2 have one cusp, for then the degree of the cover on a cusp torus of M_i to the cusp of M is 7 to 1, ie M_i can have only one cusp.

To prove the claim, since the epimorphism ρ is a p-rep, the image of P consists of parabolic elements of $\mathrm{PSL}(2, p)$, and as remarked upon above, such subgroups have order 7. On the other hand, H_1 and H_2 have index 7 in $\mathrm{PSL}(2, 7)$, and since $\mathrm{PSL}(2, 7)$ has order 168, the subgroups H_1 and H_2 both have order 24, which is co-prime to 7. It follows from this that $\rho(P_i) = 1$, so that $[P : P_i] = 7$, and this completes the proof. □

We close this section by making the following observation. This will be helpful in computational aspects carried out in Section 7.

Suppose that M is a 1-cusped hyperbolic 3-manifold and $\rho : \pi_1(M) \to \mathrm{PSL}(2, p)$ a representation. We will say that ρ is a p-good-rep if ρ is an epimorphism and there exists a pair of non-conjugate p-index subgroups H_i of $\mathrm{PSL}(2, p)$ with the following property: if M_i is the cover of M obtained from H_i, then M_i is 1-cusped for $i = 1, 2$ and M_1 is not isometric to M_2. We are interested in $p = 7, 11$.

Lemma 3.7. *Fix* $p = 7, 11$. *If* H_1 *and* H_2 *are non-conjugate index* p *subgroups of* $\mathrm{PSL}(2, p)$, *then* (H_1, H_2) *is a Sunada pair in* $\mathrm{PSL}(2, p)$.

Proof. This can be done efficiently in `magma`, since a computation reveals that for $p = 7, 11$, the group $\mathrm{PSL}(2, p)$ has only two subgroups of index p, up to conjugation. Since $\mathrm{PSL}(2, p)$ has a Sunada pair, if follows that the above pair of subgroups is the unique Sunada pair, up to conjugation. Moreover, H_1 and H_2 are interchanged by the outer automorphism group $\mathrm{Out}(\mathrm{PSL}(2, p)) = \mathbb{Z}/2\mathbb{Z}$. □

Corollary 3.8. *Every* p*-good rep for* $p = 7, 11$ *is a* p*-rep.*

4. An example: covers of a knot complement in S^3

In the next section we will prove Theorem 1.1. It is instructive in this section to present an example of Proposition 3.6, as some of the methods used in this example will be employed below. We discuss the method in a more general framework in Section 6.

Let K be the knot $K11n116$ of the Hoste-Thistlethwaite table shown in Figure 1. K is known as $11n114$ in the `Snap` census [14], 11_{298} in the `LinkExteriors` table, $t12748$ in the `OrientableCuspedCensus` and $K8_{297}$ in the `CensusKnots`.

Using `Snap`, the manifold $M = S^3 \setminus K = \mathbf{H}^3/\Gamma$ has a decomposition into 8 ideal tetrahedra, has volume $7.754453760202655\ldots$ and invariant trace field $k = \mathbf{Q}(t)$ where $t = 0.0010656 -$

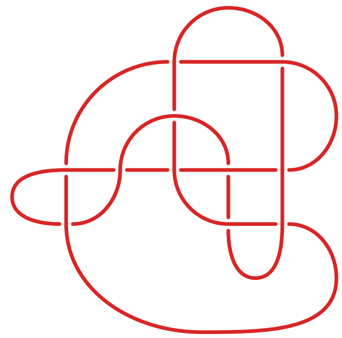

FIGURE 1. The knot $K11n116$.

$0.9101192i$ is a root of the irreducible polynomial

$$p(x) = x^8 - 3x^7 + 5x^6 - 3x^5 + 2x^4 + 2x^3 + 2x + 1.$$

Note that the discriminant of this polynomial is 156166337, a prime, and so this is the discriminant of k. Hence the ring of integers of k (denoted R_k) coincides with $\mathbb{Z}[t]$.

`Snap` shows that the geometric representation of Γ has traces, lying in R_k (see below). In [15] it is shown that $\Gamma = \mathrm{Comm}(\Gamma)$, and so we are in a position to apply Proposition 3.6.

4.0.1. *7-fold covers.* From above 7 is unramified in $k/\mathbf{Q}$ (since 7 does not divide the discriminant of k), and using Pari [33] for example, it can be shown that the ideal $(7) = 7R_k$ factors as a product $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ of prime ideals $\mathcal{P}_i$ for $i = 1, 2, 3$ of norm 7, 7^2 and 7^5 respectively. k has class number 1, so all ideals are principal, and in the above notation, the prime ideal $\mathcal{P}_1$ coincides with $(t-1)$.

We will use the prime ideal $\mathcal{P}_1$ (henceforth denoted simply by $\mathcal{P}$) to construct a p-rep as in Proposition 3.6. To that end, we need to identify a particular conjugate of Γ with matrix entries in R_k. `Snap` yields the following presentation of Γ:

$$\Gamma = \langle a, b, c \mid aaCbAccBB,\ aacbCbAAB \rangle$$

with peripheral structure

$$\mu = CbAcb, \qquad \lambda = AAbCCbacb\,,$$

where, as usual $A = a^{-1}$, $B = b^{-1}$ and $C = c^{-1}$. Using `Snap` it can shown that Γ can be taken to be a subgroup of $\mathrm{PSL}(2, R_k)$ represented by matrices as follows (note that from the irreducible polynomial of t we see that t is a unit):

$$a = \begin{pmatrix} -t^2+t-1 & t^7-3t^6+4t^5-t^4+t^2-t \\ -t^2+t-1 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} -t^7+2t^6-2t^5-3t^3+2t^2-3t-1 & t^6-2t^5+t^4+3t^3-2t^2+3t+2 \\ -t^7+3t^6-5t^5+4t^4-4t^3+2t^2-2t-1 & t^7-3t^6+5t^5-4t^4+4t^3-t^2+t+2 \end{pmatrix}$$

$$c = \begin{pmatrix} -t^6+4t^5-8t^4+7t^3-5t^2-t & -2t^7+7t^6-14t^5+15t^4-12t^3+t^2+3t-1 \\ t^5-3t^4+4t^3-3t^2+t & -t^7+4t^6-9t^5+11t^4-9t^3+3t^2+t-2 \end{pmatrix}$$

The meridian and longitude are given by

$$\mu = \begin{pmatrix} t^7 - 4t^6 + 8t^5 - 8t^4 + 5t^3 - 2t & -t^7 + 2t^6 - 3t^5 + t^4 - 2t^3 - 4t^2 - 2t - 1 \\ t^7 - 4t^6 + 9t^5 - 11t^4 + 10t^3 - 3t^2 + 3 & -t^7 + 4t^6 - 8t^5 + 8t^4 - 5t^3 + 2t - 2 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} -2t^7 + 6t^6 - 10t^5 + 7t^4 - 7t^3 + 3t^2 - 8t - 1 & 2t^7 - 9t^6 + 18t^5 - 19t^4 + 15t^3 - 11t^2 + 3t + 6 \\ 6t^7 - 20t^6 + 38t^5 - 35t^4 + 31t^3 - t^2 - t + 18 & 2t^7 - 6t^6 + 10t^5 - 7t^4 + 7t^3 - 3t^2 + 8t - 1 \end{pmatrix}$$

Now let $\rho_7 : \Gamma \to \mathrm{PSL}(2,7)$ denote the p-rep obtained by reducing entries of these matrices modulo $\mathcal{P}$. A computation gives:

$$\rho_7(a) = \begin{pmatrix} 6 & 1 \\ 6 & 0 \end{pmatrix} \qquad \rho_7(b) = \begin{pmatrix} 1 & 6 \\ 3 & 5 \end{pmatrix} \qquad \rho_7(c) = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$$

and

$$\rho_7(\mu) = \begin{pmatrix} 0 & 4 \\ 5 & 5 \end{pmatrix} \qquad \rho_7(\lambda) = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

We now check that ρ_7 is onto. To see this, note that $T = \rho_7(aB) = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$ and performing the conjugation $\rho_7(a)T\rho_7(A)$ gives the matrix $\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$.

Finally, after taking powers of these elements we see that $\rho_7(\Gamma)$ contains the elements $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These clearly generate $\mathrm{PSL}(2,7)$, and we are now in a position to apply Proposition 3.6 to complete the construction of examples.

4.0.2. *11-fold covers.* 11 is also unramified in $k/\mathbb{Q}$ and $(11) = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3$ where $\mathcal{Q}_i$ for $i = 1,2,3$ are prime ideals of norm 11, 11 and 11^6. Moreover, we can take $\mathcal{Q}_1 = (t+1)$ and $\mathcal{Q}_2 = (t^2 - t - 1)$.

Let $\rho'_{11}, \rho''_{11} : \Gamma \longrightarrow \mathrm{PSL}(2,11)$ denote the p-reps obtained by reducing entries of these matrices modulo $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. A computation gives:

$$\rho'_{11}(a) = \begin{pmatrix} 8 & 4 \\ 8 & 0 \end{pmatrix} \qquad \rho'_{11}(b) = \begin{pmatrix} 1 & 9 \\ 9 & 5 \end{pmatrix} \qquad \rho'_{11}(c) = \begin{pmatrix} 9 & 3 \\ 10 & 1 \end{pmatrix}$$

$$\rho'_{11}(\mu) = \begin{pmatrix} 9 & 6 \\ 9 & 0 \end{pmatrix} \qquad \rho'_{11}(\lambda) = \begin{pmatrix} 9 & 6 \\ 9 & 0 \end{pmatrix}$$

and

$$\rho''_{11}(a) = \begin{pmatrix} 9 & 6 \\ 9 & 0 \end{pmatrix} \qquad \rho''_{11}(b) = \begin{pmatrix} 4 & 36 \\ 12 & 12 \end{pmatrix} \qquad \rho''_{11}(c) = \begin{pmatrix} 32 & 12 \\ 28 & 4 \end{pmatrix}$$

$$\rho''_{11}(\mu) = \begin{pmatrix} 32 & 0 \\ 32 & 32 \end{pmatrix} \qquad \rho''_{11}(\lambda) = \begin{pmatrix} 32 & 0 \\ 28 & 32 \end{pmatrix}$$

Note that ρ'_{11} and ρ''_{11} are not intertwined by an automorphism of $\mathrm{PSL}(2,11)$ since $\rho'_{11}(\mu) = \rho''_{11}(\lambda)$ but $\rho'_{11}(\mu) \neq \rho''_{11}(\lambda)$.

Remark 4.1. *The construction of closed examples in [27] arise from Dehn surgery on the knot 9_{32} (a construction that we extend below). Proposition 3.6 can be applied to show that examples of isospectral 1-cusped manifolds arise as 11-fold covers of $S^3 \setminus 9_{32}$. The examples constructed above have much smaller volume and so are perhaps more interesting.*

5. Proof of Theorem 1.1: Infinitely many examples

In this section we complete the proof of Theorem 1.1 by exhibiting infinitely many examples. This builds on the ideas of [27, Sec.3] and Section 4.

5.1. **A lemma.** Using ideas from [27] together with Proposition 3.6, we will prove the following. This will complete the proof of Theorem 1.1, given the existence of a 2-cusped manifold as in Lemma 5.1 (which we exhibit in Subsection 5.2).

Lemma 5.1. *Let $M = \mathbf{H}^3/\Gamma$ be an orientable non-arithmetic finite volume 2-cusped hyperbolic 3-manifold that is the minimal element in its commensurability class. Suppose that ρ is a p-rep of Γ onto $G = \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,11)$. Then there are infinitely many Dehn surgeries $r = p/q$ on one cusp of M so that the resultant manifolds $M(r)$ are hyperbolic and have 1-cusped covers that are isospectral but non-isometric.*

Proof. We will deal with the case of $G = \mathrm{PSL}(2,7)$, the other case is similar. Associated to the two cusps of M we fix two peripheral subgroups P_1 and P_2, and we will perform Dehn surgery on the cusp associated to P_2, thereby preserving parabolicity of the non-trivial elements of P_1 after Dehn surgery.

Fix a pair of generators μ and λ for P_2. By p/q-Dehn surgery on the cusp associated to P_2 we mean that the element $\mu^p\lambda^q$ is trivialized. We denote the result of p/q-Dehn surgery by $M(p/q)$. Note that for sufficiently large $|p| + |q|$, the resultant surgered manifolds will be 1-cusped hyperbolic manifolds and will still be the minimal elements in their commensurability class (see Theorem 3.2 of [27]).

Since ρ is a p-rep, $\rho(P_2)$ is non-trivial. Performing p/q-Dehn surgery on the cusp associated to P_2, if we can arrange that $\rho(\mu^p\lambda^q) = 1$, then the p-rep ρ will factor through $\pi_1(M(p/q))$, thereby inducing a p-rep of $\pi_1(M(p/q))$.

Now $\rho(P_2)$ is a cyclic subgroup $C = \langle x \rangle$ of order 7. Hence there are integers $s, t \in \{0, \pm 1, \pm 2, \pm 3\}$ (not both zero) so that $\rho(\mu) = x^s$ and $\rho(\lambda) = x^t$. Hence we need to find infinitely many co-prime pairs (p, q) which satisfy $ps + qt = 7d$ with s, t as above and for integers d. This is easily arranged by elementary number theory. For example, if exactly one of $\rho(\mu)$ or $\rho(\lambda)$ is trivial (say $\rho(\lambda)$), then we can choose integers $p = 7n$ and q coprime to $7n$ will suffice to prove the lemma in this case. If both $s, t \neq 0$, a simlar argument holds. For example suppose that $s = t = 2$. Then choosing $q = 1$ and p an integer of the form $7a - 1$ will work.

Thus we have constructed infinitely many 1-cusped hyperbolic 3-manifolds with a p-rep onto $\mathrm{PSL}(2,7)$ and so the proof is complete by an application of Proposition 3.6. □

5.2. **A 2-component link–9^2_{34}.** From [15] the 2-component link $L = 9^2_{34}$ of Rolfsen's table (which is the link $9a62$ in the `Snap` census and $L9a21$ in the Hoste-Thistlethwaite table) shown in Figure 2 has the property that $M = S^3 \setminus L$ is the minimal element in its commensurability class.

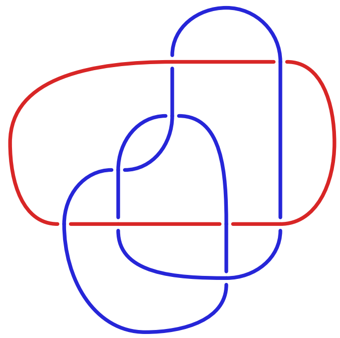

FIGURE 2. The link 9^2_{34}.

The link complement has volume approximately 11.942872449472... and invariant trace-field k generated by a root t of:

$$p(x) = x^{10} - x^9 - x^8 - x^7 + 6x^6 + x^5 - 3x^4 - 4x^3 + 2x^2 + 2x - 1.$$

As can be checked using Pari, $(7) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5$ where $\mathcal{P}_i$ for $i = 1, \dots, 5$ are prime ideals of norm 7, 7, 7^2, 7^3 and 7^3. Moreover, we can take $\mathcal{P}_1 = (t+1)$. The fundamental group has presentation

$$\Gamma = \langle a, b, c \mid aBACbccabCCBcabAcb,\ abAcbaCCBccABC \rangle$$

with peripheral structure

$$(\mu_1, \lambda_1) = (b, BBAcbaCC), \qquad (\mu_2, \lambda_2) = (BC, aBACbccaCCbccBACb)\,.$$

Following the ideas above it can be shown that the faithful discrete representation of $\pi_1(M)$ can be conjugated to lie in $\mathrm{PSL}(2, R_k)$ and that reducing modulo $\mathcal{P}_1$ provides a p-rep onto $\mathrm{PSL}(2, 7)$ given by

$$\rho(a) = \begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \qquad \rho(c) = \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}$$

$$\rho(\mu_1) = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \qquad \rho(\lambda_1) = \begin{pmatrix} 4 & 6 \\ 2 & 5 \end{pmatrix}, \qquad \rho(\mu_2) = \begin{pmatrix} 5 & 6 \\ 2 & 4 \end{pmatrix}, \qquad \rho(\lambda_2) = \begin{pmatrix} 3 & 6 \\ 2 & 2 \end{pmatrix}$$

Moreover, fixing a cusp, ρ can be conjugated to a representation such that the meridian and longitude pair of both map to $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ Choosing $p = -(7n + 1)$ (sufficiently large) and $q = 1$ provides explicit Dehn surgeries as given by Lemma 5.1.

6. TWO METHODS TO CONSTRUCT SUNADA PAIRS

We now discuss two methods for implementing Proposition 3.6. In Section 4, an example of a minimal knot complement was used to build examples (we will refer to this example as Example 1). The framework for this was to reduce the geometric representation, defined over a localization of the ring of integers of a number field, modulo a prime of norm 7 or 11. We shall call this *Method G*. A second method (which we refer to *Method R*), mentioned at the end of Section 3, is to compute all p-good reps for $p = 7, 11$. Each method has its own merits. Method R can be implemented efficiently by `magma` and `SnapPy` to search over lists of manifolds. Method G (which involves exact arithmetic computations) requires a combination of `Snap`, `SnaPy`, `pari` and `sage` and a lot of cutting and pasting, but produces infinitely many 1-cusped examples.

Let us describe Method G in more detail. We start with a cusped orientable hyperbolic 3-manifold M. Its geometric representation

$$\pi_1(M) \longrightarrow \mathrm{PSL}(2, R)$$

can be defined over a subring R of an extension of the invariant trace-field. In many cases, this is actually contained in the invariant trace-field k (e.g. for knots in integral homology 3-spheres). If we can find a prime ideal $\mathcal{P}$ in R of norm 7 or 11 which is not inverted in R, then we can reduce the geometric representation of M to get a representation $\rho : \pi_1(M) \longrightarrow \mathrm{PSL}(2, p)$ for $p = 7$ or $p = 11$. We can further check that ρ is a p-rep. If we can also compute the commensurator of $\pi_1(M)$, then we can apply Proposition 3.6.

Before we get into the details, let us recall that (Hoste-Thistlethwaite and Rolfsen) tables of hyperbolic knots are available from `SnapPy` [7] and from `Snap` [14]. A consistent conversion between these tables is provided by `SnapPy` [7].

7. More examples

7.1. **Example 1 via Method R.** Consider the knot $K = K11n116$ from Figure 1 of Section 4.

Setting $M = S^3 \setminus K$, `magma` computes that $\pi_1(M)$ has 4 epimorphisms in $\mathrm{PSL}(2, 7)$ and two of them are 7-good reps. (corresponding to those we found in Section 4). The corresponding pair M_1 and M_2 of index 7 covers are isospectral and non-isometric. We can also confirm that M_1 and M_2 are not isometric using the *isometry signature* (a complete invariant) of [11]. As shown by `magma` both have common homology $\mathbb{Z}/2 + \mathbb{Z}/110 + \mathbb{Z}$.

`SnapPy` computes that M has 42 11-fold covers. Of those, 8 have a total space with one cusp, and among those, we find 11-good covers: there is one pair of covers with homology $\mathbb{Z}/2 + \mathbb{Z}/210 + \mathbb{Z}$ and another pair with homology $\mathbb{Z}/2 + \mathbb{Z}/406 + \mathbb{Z}$ and non-isometric total spaces for either pair. These pairs (M_1', M_2') and (M_1'', M_2'') are built from the epimorphisms ρ_{11}' and ρ_{11}''. (M_1', M_2') and also (M_1'', M_2'') are isospectral.

7.2. **Example 2: covers of the manifold v2986 via Method G.** Let $M = \mathbf{H}^3/\Gamma$ denote the manifold from the `Snap` census $v2986$. `SnapPy` confirms that M is not a knot complement in S^3 (since it can be triangulated using 7 ideal tetrahedra and is not isometric to a manifold in `CensusKnots`, the complete list of hyperbolic knots with at most 8 tetrahedra) but it does have $H_1(M; \mathbb{Z}) = \mathbb{Z}$. The volume of M is approximately $6.165768948\ldots$ (which is less than the previous example). Again from [15], we have that $\Gamma = \mathrm{Comm}(\Gamma)$. `Snap` gives the the following presentation of the fundamental group Γ

$$\Gamma = \langle a, b, c \mid acbCBaBAc,\, abcbbAAC \rangle$$

with peripheral structure

$$\mu = C, \qquad \lambda = BCabAA\,.$$

From `Snap` we see that Γ has integral traces and has invariant trace-field generated by a root of the polynomial

$$p(x) = x^8 - 2x^7 - x^6 + 4x^5 - 3x^3 + x + 1$$

Using Pari, we get a decomposition $(7) = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ into prime ideals $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ of norm 7, 7^3 and 7^4. Moreover, we can take $\mathcal{P}_1 = (t^3 - t - 1)$. The geometric representation is still defined over R_k, and its reduction $\rho_7 : \Gamma \longrightarrow \mathrm{PSL}(2,7)$ modulo $\mathcal{P}_1$ is given by:

$$\rho_7(a) = \begin{pmatrix} 10 & 4 \\ 4 & 8 \end{pmatrix} \qquad \rho_7(b) = \begin{pmatrix} 0 & 8 \\ 6 & 12 \end{pmatrix} \qquad \rho_7(c) = \begin{pmatrix} 4 & 2 \\ 6 & 12 \end{pmatrix}$$

and

$$\rho_7(\mu) = \begin{pmatrix} 12 & 12 \\ 8 & 4 \end{pmatrix} \qquad \rho_7(\lambda) = \begin{pmatrix} 8 & 6 \\ 4 & 4 \end{pmatrix}$$

As before, one can check that ρ_7 is onto, and thereby construct isospectral covers with one cusp using Proposition 3.6.

7.3. **Example 3: covers of knot complements with at most 8 tetrahedra via Method R.** Of the 502 hyperbolic knots in `CensusKnots` with at most 8 ideal tetrahedra, `SnapPy` computes that the following 11 have trivial isometry group:

$$K8_{226}, K8_{252}, K8_{270}, K8_{277}, K8_{287}, K8_{290}, K8_{292}, K8_{293}, K8_{296}, K8_{297}, K8_{301}$$

Note that $K8_{297}$ is the knot of Example 1. `Snap` confirms that all of these knots have no hidden symmetries. Of the above 11 knots, `magma` finds that the following 8 have at least one 7-good-rep:

$$K8_{252}, K8_{270}, K8_{277}, K8_{290}, K8_{292}, K8_{293}, K8_{297}, K8_{301}$$

and all 11 have at least one 11-good-rep.

7.4. **Example 4: the list of 1-cusped manifolds of [15] via Method R.** [15] gave a list of 13486 hyperbolic manifolds with at least one cusp, along with their hidden symmetries. Of those with no hidden symmetries, 1252 have one cusp, 1544 have two cusps and 106 have four cusps.

There are 6 manifolds with one cusp and no hidden symmetries and with at most 7 ideal tetrahedra in the above list:

$$v2986, v3205, v3238, v3372, v3398, v3522$$

`magma` computes that all 7 of those manifolds have 7-good reps, and that the following 3

$$v3205, v3238, v3522$$

have 11-good reps.

Let $\mathcal{M}$ denote the list of 1252 one cusped manifolds with no hidden symmetries, and $\mathcal{M}_p$ the sublist of those with p-good reps for $p = 7, 11$. If $|X|$ denotes the number of elements of a set X, a computation shows that

$$(1)\quad |\mathcal{M}_7 \cap \mathcal{M}_{11}| = 809, \quad |\mathcal{M}_7 \setminus \mathcal{M}_{11}| = 165, \quad |\mathcal{M}_{11} \setminus \mathcal{M}_7| = 220, \quad |\mathcal{M} \setminus (\mathcal{M}_7 \cup \mathcal{M}_{11})| = 58\,.$$

The manifolds in $\mathcal{M}_7 \cap \mathcal{M}_{11}$ with at most 10 ideal tetrahedra are

$$v3205, v3238, v3522, K10n10, K11n27, K11n116, K12n318, K12n644.$$

The complete data (in `SnapPy` readable format) is available from [13].

8. Final comments

In this final section we discuss further the nature of the discrete spectrum for 1-cusped hyperbolic 3-manifolds. As described in Section 2, one issue in the cusped setting is whether there is any *interesting discrete spectrum*. Theorem 2.4 gives conditions when the discrete spectrum is infinite, and we will take this up here for 1-cusped hyperbolic 3-manifolds. In what follows $M = \mathbf{H}^3/\Gamma$ will denote a 1-cusped orientable finite volume hyperbolic 3-manifold with discrete spectrum $\lambda_1 \leq \lambda_2 \ldots$.

8.1. **Essentially cuspidal manifolds.** As was mentioned previously, there is no direct analogue of the Weyl law for cusped hyperbolic 3-manifolds, however the following asymptotic that takes account of a contribution from the continuous spectrum can be established using the Selberg trace formula (see [10] Chapter 6.5 and [31]). To state this, we introduce the following notation:

For $T > 0$ let $A(\Gamma, T) = |\{j : \lambda_j \leq T^2\}|$ and $M(\Gamma, T) = -\frac{1}{2\pi}\int_{-T}^{T} \frac{\phi'}{\phi}(1+it)dt$, then

$$A(\Gamma, T) + M(\Gamma, T) \approx \frac{1}{6\pi^2}\mathrm{vol}(M)T^3 \quad \text{as} \quad T \to \infty\,.$$

Therefore, getting good control on the growth of $M(\Gamma, T)$ implies a *Weyl law*

$$A(\Gamma, T) \approx \frac{1}{6\pi^2}\mathrm{vol}(M)T^3, \quad \text{as} \quad T \to \infty\,. \tag{$\dagger$}$$

In [31], Sarnak defines Γ or M to be *essentially cuspidal* if the Weyl law (†) holds. Thus the issue as to whether M is essentially cuspidal is, which of the terms $A(\Gamma, T)$ or $M(\Gamma, T)$ dominates in the expression (†) above. It is known that congruence subgroups of Bianchi groups are essentially cuspidal (see [28]); in this case $M(\Gamma, T) = O(T \log T)$. An example of a non-congruence subgroup of a Bianchi group that is also essentially cuspidal is given in [8].

In this regard, Sarnak [31] has conjectured, in a much broader context than discussed here, that *if M is essentially cuspidal then M is arithmetic.* In fact, in the case of surfaces, it is conjectured (see [20]) that *the generic* Γ in a given Teichmüller space is not essentially cuspidal, and indeed (apart from the case of the 1-punctured torus) *the generic* case should have only finitely many discrete eigenvalues. This is based on work of Philips and Sarnak [25] on how eigenvalues *dissolve* under deformation.

8.2. **Knot complements.** Even though Theorem 2.4 produces non-arithmetic 1-cusped hyperbolic 3-manifolds for which $A(\Gamma, T)$ is unbounded, the contribution from $M(\Gamma, T)$ is conjecturally enough to violate the Weyl law. Now there is no analogue of the Philips and Sarnak result in dimension 3, but it seems interesting to understand how the discrete spectrum behaves, for example for knot complements in S^3.

To that end, the *generic knot complement* will be the minimal element in its commensurability class, and so will likely have only finitely many discrete eigenvalues. In particular, we cannot apply Theorem 2.4 to deduce an infinite discrete spectrum.

The figure-eight knot complement is the only arithmetic knot complement, and it is also known to be a congruence manifold. Hence, the complement of the figure-eight knot is essentially cuspidal. Thus, given Sarnak's conjecture, the figure-eight knot should be the only knot complement that is essentially cuspidal. We cannot prove this at present, but we can prove Theorem 1.2 which we restate below for convenience.

Theorem 8.1. *Let M denote the complement of the figure-eight knot in S^3. Suppose that N is a finite volume hyperbolic 3-manifold which is isospectral with M. Then N is homeomorphic to M.*

Proof. Since N is isospectral with M, N cannot be closed since the poles of the scattering function are part of the spectral data. The result will follow once the following two claims are established.

Claims: *(1)* $\mathrm{Vol}(N) = \mathrm{Vol}(M)$*.*

(2) *N and M have the same set of lengths of closed geodesics (without counting multiplicities).*

Deferring discussion of these for now, we complete the proof. From Claim (1) and [5] the only possibility for N is the so-called sister of the figure-eight knot. However, as can be checked by `Snap` for example the shortest length of a closed geodesic in the sister is approximately $0.86255\ldots$ and for the figure-eight knot complement it is $1.08707\ldots$. In Section 8.3 we include a theoretical proof of the fact that the shortest geodesic in the sister has length $0.86255\ldots$ and that the figure-eight knot complement contains no closed geodesic of that length. $\square$

Note that both (1) and (2) are standard applications of the Weyl Law and trace formula in the setting of closed hyperbolic 3-manifolds (see for example [10] Chapter 5.3). This is the approach taken here, however, as we have already remarked upon, the cusped setting provides additional challenges. The proof of Claim (2) is given in subsection 8.4 and was kindly provided by Dubi Kelmer.

For Claim (1), the Weyl Law in the cusped setting takes the form (see [6] Chapter 7)

$$A(\Gamma, T) + M(\Gamma, T) = \frac{1}{6\pi^2}\mathrm{vol}(M)T^3 + O(T^2) + O(T\log\ T).$$

In the case at hand, for both M and N the left hand side is the same, and so it follow that we can read off the volume (on letting $T \to \infty$). A different proof of equality of volume is given in Section 8.4. $\square$

Using `Snap` and [12] we can prove the following by a similar method. We begin by recalling Theorem 7.4 of [12].

Theorem 8.2. *There are only ten finite volume orientable* 1*-cusped hyperbolic* 3*-manifolds with volume* ≤ 2.848*. These are (in the notation of the original* ***SnapPea*** *census):*

$$m003, m004, m006, m007, m009, m010, m011, m015, m016, m017.$$

Note that $m003$ and $m004$ are the sister of the figure-eight knot and the figure-eight knot respectively, $m006$ and $m007$ have the same volume (approximately $2.56897\ldots$), $m009$ and $m010$ have the same volume (approximately $2.66674\ldots$) and $m015$, $m016$, and $m017$ have the same volume (approximately $2.82812\ldots$).

Theorem 8.3. *Let M be any one the ten manifolds stated in Theorem 8.2. Then if N is an orientable finite volume hyperbolic 3-manifold isospectral with M than N is homeomorphic to M.*

Proof. As in the proof of Theorem 8.1, the manifold N must have cusps, and by [22, Thm.2] N must have 1 cusp. As before N also has the same volume as M. Note that all 10 manifolds in the above list have fundamental group that is 2-generator, and so the manifolds admit an orientation-preserving involution. Hence Theorem 2.4 applies to show that the discrete spectrum in all these cases is infinite. If N is isospectral to any one of the manifolds in the list then N has the same volume. Theorem 8.1 deals with $m004$, and also $m003$. Since $m011$ is the unique manifold of that volume, then this one is also accounted for. The only possibilities that remain to be distinguished are the pairs $(m006, m007)$, $(m009, m010)$ and the triple $(m015, m016, m017)$. This can be done using `snap` to compute the start of the length spectrum. To deal with $m006$ and $m007$, and $m009$ and $m010$ one can use the second shortest geodesic. To distinguish $m015$ from $m016$ and $m017$ one can use the second shortest geodesics, and $m016$ and $m017$ are distinguished by the shortest geodesic. $\square$

Note that $m015$ is the knot 5_2 in the standard tables and $m016$ is the $(-2, 3, 7)$-pretzel knot, and so these knots, like the figure-eight knot, have complements that are determined by their spectral data.

8.3. **Shortest length geodesics in the sister of the figure-eight knot.** Here we give a theoretical proof of the distinction in the lengths of the shortest closed geodesic in M (as above) and its sister manifold N. In what follows we let $M = \mathbf{H}^3/\Gamma_1$ and $N = \mathbf{H}^3/\Gamma_2$ As is well known $\Gamma_1, \Gamma_2 < \mathrm{PSL}(2, \mathbb{Z}[\omega])$ of index 12, and where $\omega^2 + \omega + 1 = 0$.

As can be easily shown (see for example [24, Thm.4.6]), the shortest translation length of a loxodromic element in $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ occurs for an element of trace ω or its complex conjugate $\overline{\omega}$ (up to sign) and is approximately $0.8625546276620610\ldots$; i.e. the length of the shortest closed geodesic in N.

Fix the following elements of trace ω and $\overline{\omega}$ (up to sign):

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & \omega \end{pmatrix}, \gamma_0' = \begin{pmatrix} 0 & -1 \\ 1 & \omega \end{pmatrix}$$

and

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & \overline{\omega} \end{pmatrix}, \gamma_1' = \begin{pmatrix} 0 & -1 \\ 1 & \overline{\omega} \end{pmatrix}.$$

As can be checked for $i = 0, 1$, γ_i and γ_i' are not conjugate in $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ (e.g. using reduction modulo the $\mathbb{Z}[\omega]$-ideal $< \sqrt{-3} >$).

Lemma 8.4. *For $i = 0, 1$, γ_i and γ'_i are representatives of all the* $\mathrm{PSL}(2, \mathbb{Z}[\omega])$*-conjugacy classes of elements of trace ω or $\overline{\omega}$ (up to sign).*

Proof. Suppose that $t + t^{-1} = \omega$ with $t = (\omega + \theta)/2$ where $\theta = \sqrt{\frac{-9-\sqrt{-3}}{2}}$ and let $k = \mathbb{Q}(\theta)$. It can be checked that k has discriminant 189 and has class number one. Using this and the formulae in Chapter III.5 of [35] one deduces that the number of conjugacy classes of elements of $\mathrm{PSL}(2, O_3)$ of trace ω is 2.

Since an element of trace $\overline{\omega}$ simply gives a conjugate of k given by $\mathbf{Q}(\bar{t})$, the same argument applies to also give two conjugacy classes in this case. □

The claim about the lengths will follow once we establish that none of the $\mathrm{PSL}(2, \mathbb{Z}[\omega])$-conjugacy classes of γ_i and γ'_i for $i = 0, 1$, meet Γ_1 and at least one meets Γ_2. This can be done efficiently using `magma` as we now describe. We begin with a preliminary observation.

Suppose that $M = \mathbf{H}^3/\Gamma \to Q = \mathbf{H}^3/\Gamma_0$ is a finite sheeted covering of finite-volume orientable hyperbolic 3-orbifolds. Denoting the covering degree by d, the action on cosets of Γ in Γ_0 determines a permutation representation $\rho : \Gamma_0 \to S_d$ with kernel K. Suppose that $[g_1], \ldots, [g_r]$ denote the conjugacy classes of loxodromic elements in Γ_0 of minimal translation length ℓ. Then M contains an element of length ℓ if and only if $\Gamma \cap [g_i] \neq \emptyset$ for some $i \in \{1, \ldots, r\}$, and this happens if and only $\rho(\Gamma) \cap [\rho(g_i)] \neq \emptyset$ for some $i \in \{1, \ldots, r\}$.

We apply this in the case that N is the Bianchi orbifold $Q = \mathbf{H}^3/\mathrm{PSL}(2, \mathbb{Z}[\omega])$ and M is either the figure-eight knot complement or its sister. In the former case, the permutation representation has kernel the congruence subgroup $\Gamma(4) < \mathrm{PSL}(2, \mathbb{Z}[\omega])$ (of index 1920) and in the latter case the permutation representation has kernel the congruence subgroup $\Gamma(2)$ (of index 60). To implement the magma routines we use the the presentation of $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ from [18], and express the subgroups Γ_1 and Γ_2 in terms of these generators. Setting

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \text{and } c = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned}
\mathrm{PSL}(2, \mathbb{Z}[\omega]) &= < a, b, c | b^2 = (ab)^3 = (acbC^2b)^2 = (acbCb)^3 = A^2CbcbCbCbcb = [a, c] = 1 > \\
\Gamma_1 &= < a, bcb > \\
\Gamma_2 &= < a^2, bcabaCbCb > .
\end{aligned}$$

The elements γ_i and γ'_i for $i = 0, 1$ are described in terms of these generators as:

$$\gamma_0 = bC, \gamma'_0 = bc, \gamma_1 = bac, \gamma'_1 = bAC.$$

Below we include the magma routine that executes the above computation showing no conjugates lie in Γ_1 but at least one does in Γ_2.

```
g<a,b,c>:=Group<a,b,c| b^2, (a*b)^3, (a*c*b*c^-2*b)^2,
(a*c*b*c^-1*b)^3, a^-2*c^-1*b*c*b*c^-1*b*c^-1*b*c*b, (a,c)>;
h1:= sub<g|a,b*c*b>;
h2:= sub<g|a^2, b*c*a*b*a*c^-1*b*c^-1*b>;
print AbelianQuotientInvariants(h1);
\\[0]
print AbelianQuotientInvariants(h2);
```

```
\\[ 5, 0 ]
x0:=g!b*c^-1;
x1:=g!b*c;
y0:=g!b*a*c;
y1:=g!b*a^-1*c^-1;
f1,i1,k1:=CosetAction(g,h1);
print Order(i1);
\\1920
f2,i2,k2:=CosetAction(g,h2);
print Order(i2);
\\60
l:=Class(i1,f1(x0)) meet Set(f1(h1));
print #l;
\\0
l:=Class(i1,f1(x1)) meet Set(f1(h1));
print #l;
\\0
l:=Class(i1,f1(y0)) meet Set(f1(h1));
print #l;
\\0
l:=Class(i1,f1(y1)) meet Set(f1(h1));
print #l;
\\0
k:=Class(i2,f2(x0)) meet Set(f2(h2));
print #k;
\\2
```

8.4. **Determining the length set.**

Proposition 8.5. *Let M_1 and M_2 be finite volume orientable 1-cusped hyperbolic 3-manifolds that are isospectral. Then they have the same set of lengths of closed geodesics (without counting multiplicities).*

Before commencing with the proof we recall the version of the trace formula given in [10] Chapter 6 Theorem 5.1. This needs some notation. Let $M = \mathbf{H}^3/\Gamma$ be 1-cusped finite volume orientable hyperbolic 3-manifold. Given a loxdromic element $\gamma \in \Gamma$ of complex length $\ell_\gamma + i\theta_\gamma$, we denote by γ_0 the unique primitive element such that $\gamma = \gamma_0^j$. For convenience, we denote the discrete spectrum by $\lambda_k = 1 + r_k^2$, and by $\phi(s)$ is (as before) the scattering function. The trace formula in this case then states ([10] Chapter 6 Theorem 5.1):

Theorem 8.6. *For any even compactly supported test function $g \in C_c^\infty(\mathbb{R})$ let $h(r)$ denote it's Fourier transform. Then*

$$\sum_k h(r_k) - \frac{1}{4\pi}\int_{\mathbb{R}} h(r)\frac{\phi'}{\phi}(ir)dr = \frac{\mathrm{vol}(M)}{4\pi^2}\int_{\mathbb{R}} h(t)r^2 dr$$
$$+ 4\pi \sum_{\gamma\in\Gamma_{\mathrm{lox}}} \frac{\ell_{\gamma_0} g(\ell_\gamma)}{\sqrt{2\sinh(\frac{\ell_\gamma + i\theta_\gamma}{2})}} + a_\Gamma g(0) + b_\Gamma h(0)$$
$$- \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$$

where the constants a_Γ and b_Γ are explicit constants depending only on Γ and the summation on the right-hand side is over Γ_{lox} which represents conjugacy classes of loxodromic elements in Γ.

Remark 8.7. *Our notation is slightly different from [10] and there are less terms due to our assumptions of only one cusp and no torsion. Also, the $\Gamma(s)$ in the last integral denotes the Γ-function and is not related to the Kleinian group Γ.*

We now prove Proposition 8.5.

Proof. For our purposes it will be helpful to rewrite the sum over the loxodromic classes and collect all the classes with the same ℓ_γ together. That is,

$$\sum_{\gamma\in\Gamma_{\mathrm{lox}}} \frac{\ell_{\gamma_0} g(\ell_\gamma)}{\sqrt{2\sinh(\frac{\ell_\gamma + i\theta_\gamma}{2})}} = \sum_\ell \left(\sum_{\substack{\gamma\in\Gamma_{\mathrm{lox}} \\ \ell_\gamma = \ell}} \frac{\ell_{\gamma_0}}{\sqrt{2\sinh(\frac{\ell_\gamma + i\theta_\gamma}{2})}} \right) g(\ell)$$
$$= \sum_\ell m_\Gamma(\ell) g(\ell)$$

where we defined the twisted multiplicities $m_\Gamma(\ell)$ by

$$m_\Gamma(\ell) = \sum_{\substack{\gamma\in\Gamma_{\mathrm{lox}} \\ \ell_\gamma = \ell}} \frac{\ell_{\gamma_0}}{\sqrt{2\sinh(\frac{\ell_\gamma + i\theta_\gamma}{2})}},$$

and the sum on the right is over the set of lengths of closed geodesics in M (in fact we can take the sum over all $\ell > 0$ since $m_\Gamma(\ell) = 0$ if ℓ is not a length of a closed geodesic).

We can thus rewrite the trace formula as

$$\sum_k h(r_k) - \frac{1}{4\pi}\int_{\mathbb{R}} h(r)\frac{\phi'}{\phi}(ir)dr = \frac{\mathrm{vol}(M)}{4\pi^2}\int_{\mathbb{R}} h(t)r^2 dr + 4\pi \sum_\ell m_\Gamma(\ell) g(\ell)$$
$$+ a_\Gamma g(0) + b_\Gamma h(0) - \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$$

Noting that $m_\Gamma(\ell) \neq 0$ if and only if ℓ is a length of a closed geodesic in M, the result will follow from the next proposition.

Proposition 8.8. *With M_1 and M_2 as in Proposition 8.4, then* $\mathrm{Vol}(M_1) = \mathrm{Vol}(M_2)$, $a_{\Gamma_1} = a_{\Gamma_2}$, $b_{\Gamma_1} = b_{\Gamma_2}$, *and* $m_{\Gamma_1}(\ell) = m_{\Gamma_2}(\ell)$ *for any* $\ell > 0$.

Proof. Let $\Delta V = \mathrm{Vol}(M_1) - \mathrm{Vol}(M_2)$, $\Delta a = a_{\Gamma_1} - a_{\Gamma_2}$, $\Delta b = b_{\Gamma_1} - b_{\Gamma_2}$, and $\Delta m(\ell) = m_{\Gamma_1}(\ell) - m_{\Gamma_2}(\ell)$. Taking the difference between the two trace formulas, the left hand side cancels and we get that for any even test function $g \in C_c^\infty(\mathbb{R})$

$$\frac{\Delta V}{4\pi^2}\int_{\mathbb{R}} h(t)r^2 dr + 4\pi \sum_{\ell} \Delta m(\ell) g(\ell) + \Delta a g(0) + \Delta b h(0) = 0$$

We can first take a test function g to be supported away from all the lengths in the length spectrum of both manifolds and from 0 (e.g., make it supported in the interval between zero and the shortest length), and satisfy that $h(0) = \int g(x)dx = 0$ but $\int_{\mathbb{R}} h(t)r^2 dr \neq 0$. Using such a test function we can deduce that $\Delta V = 0$ (which was already deduced from Weyl's law). The difference of the trace formula hence simplifies to

$$4\pi \sum_{\ell} \Delta m(\ell) g(\ell) + \Delta a g(0) + \Delta b h(0) = 0$$

Next, taking g supported away from all lengths and 0 but this time with $h(0) = 1$ we conclude that $\Delta b = 0$, and then taking g supported on a small neighborhood of 0 (smaller than the length of the shortest geodesic) we conclude that $\Delta a = 0$ as well. From this we get that for any test function

$$\sum_{\ell} \Delta m(\ell) g(\ell) = 0$$

Finally, for each $\ell > 0$ we can take g to be supported in a small enough neighborhood of ℓ, such that no other length in the length spectrum are in the support (except ℓ itself if it happens to be in the length spectrum of one of the manifolds). This implies that $\Delta m(\ell) = 0$ as well for any $\ell > 0$, thus concluding the proof. □

The proof of Proposition 8.5 is now complete. □

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