

# RESURGENCE OF THE KONTSEVICH-ZAGIER SERIES

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ABSTRACT. The paper is concerned with the resurgence of the Kontsevich-Zagier series

$$f(q) = \sum_{n=0}^{\infty} (1-q) \dots (1-q^n)$$

We give an explicit formula for the Borel transform of the power series when  $q = e^{1/x}$  from which its analytic continuation, its singularities (all on the positive real axis) and the local monodromy can be manifestly determined. We also give two formulas (one involving the Dedekind eta function, and another involving the complex error function) for the right, left and median summation of the Borel transform. We also prove that the limiting values of the median sum at rational multiples of  $1/(2\pi i)$  coincide with the values of  $f(q)$  at the corresponding complex roots of unity. Our resurgence theorem extends more generally to the power series of torus knots and Seifert fibered 3-manifolds associated by Quantum Topology.

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## 1. QUANTUM INVARIANTS OF KNOTTED OBJECTS AND THEIR PUZZLES

1.1. **Introduction.** The paper is concerned with the Kontsevich-Zagier formal power series

$$(1) \quad f(q) = \sum_{n=0}^{\infty} (1-q) \dots (1-q^n)$$

and its analytic properties. To begin with, we give an explicit formula for the Borel transform of the associated formal power series  $F(x) = e^{-1/(24x)} f(e^{-1/x})$  from which its analytic continuation, its singularities and their structure can be manifestly determined. This gives rise to right/left and median summation of the original power series. These sums, which are well-defined in the open right half-plane are expressed by an integral formula involving the Dedekind eta function. The median sum can also be expressed as a series involving the complex error functions. Moreover, it is shown using results of Zagier that the limiting values at  $-1/(2\pi i\alpha)$  for rational numbers  $\alpha$  coincide with  $F(-1/(2\pi i\alpha))$ . One motivation for studying the series  $f(q)$  is Quantum Topology, which assigns numerical invariants to knotted 3-dimensional objects. Our results encourage us to formulate a resurgence conjecture for the formal power series of knotted objects, which we prove in the case of the trefoil knot and the Poincaré homology sphere, and more generally for torus knots and Seifert fibered 3-manifolds. In a subsequent publication we will study resurgence for a class of geometrically interesting knotted 3-dimensional objects that include the simplest hyperbolic  $4_1$  knot.

1.2. **Numerical invariants of knotted 3-dimensional objects.** Perturbative quantum field theory assigns numerical invariants (such as formal power series invariants) to knotted objects. These formal power series, although they are given by explicit formulas, are typically factorially divergent, and somehow they are linked to numerical invariants of knotted objects, such as the Witten-Reshetikhin-Turaev invariants of 3-manifolds and the Kashaev invariants of knots.

These numerical invariants have poor analytic behavior, satisfy no known differential equations (linear or not) and the existence of asymptotic expansions is a difficult and interesting analytic problem.

In our paper, we formulate a resurgence conjecture for the formal power series invariants, and show how resurgence solves the numerous analytic problems, and implies the existence of asymptotic expansions, and even the presence of exponentially small corrections.

The bulk of our paper consists of a proof of our resurgence conjecture for the case of the simplest non-trivial knot, the trefoil  $(3_1)$ , and one of the simplest closed 3-manifolds, the Poincaré homology sphere. Our results extend without change to torus knots and Seifert fibered integer homology spheres.

En route, we explain the important notion of resurgence, due to Écalle, in a self-contained manner.

In a subsequent publication, we will show resurgence of power series associated to a class of geometrically interesting knotted objects, such as the simplest hyperbolic  $4_1$  knot; see [CG1, CG2]. For a detailed discussion of conjectures, see also [Ga].

1.3. **TQFT invariants of knotted objects.** Let us begin by recalling some of the numerical invariants of knotted objects. The reader who wishes to focus on the results, may skip this section, and go directly to Theorem 1 and Section 2.3.

*Topological Quantum Field Theory* (TQFT in short) assigns numerical invariants to knotted 3-dimensional objects. The invariants of knots/3-manifolds depend on some additional data, such as a *complex root of unity*  $\omega$ . We will denote the numerical invariants by  $\phi_{\mathcal{K}}(\omega)$  where  $\mathcal{K}$  denotes a *knotted object*, that is, a knot  $K$  in 3-space or an integer homology 3-sphere  $M$ . In other words, we have a map:

$$(2) \quad \phi : \text{Knotted objects} \longrightarrow \mathbb{C}^{\Omega}$$

where  $\Omega$  denotes the set of complex roots of unity. The invariant  $\phi_M$  is the *Witten-Reshetikhin-Turaev invariant* of the closed 3-manifold  $M$ ; see [RT, Tu1, Tu2, Wi]. The invariant  $\phi_K(e^{2\pi i/N})$  is the *Kashaev invariant* of a knot  $K$  in 3-space; see [Ka]. Murakami-Murakami showed that  $\phi_K(e^{2\pi i/N})$  is also equal to the value  $\langle K \rangle_N$  of the  $N$ -th colored Jones polynomial of  $K$  (normalized to be 1 at the unknot), evaluated at the  $N$ -th complex root of unity  $e^{2\pi i/N}$ ; see [MM].

The following problem was formulated by Witten (for closed 3-manifolds) and by Kashaev (for knots).

**Problem 1.** Show the existence of asymptotic expansions of the sequence  $(\phi_{\mathcal{K}}(e^{2\pi i/N}))$ , and identify the leading terms with known geometric invariants; see [Wi, Ka].

Unfortunately, the complex-valued function  $\phi_{\mathcal{K}}$ , defined on the set of complex roots of unity, does not seem to extend to a continuous function on the unit circle. Moreover, its asymptotic expansion around a complex root of unity is unknown, and seems to be a difficult analytic problem.

**1.4. Perturbative TQFT invariants of knotted objects.** There is an additional formal power series invariant of knotted objects:

$$(3) \quad F : \text{Knotted objects} \longrightarrow \mathbb{Q}[[1/x]]$$

which is usually thought of as a *perturbative expansion* of the quantum invariants  $\phi_{\mathcal{K}}$ . For a homology sphere  $M$ ,  $F_M(x)$  is the well-known Le-Murakami-Ohtsuki invariant (composed with the  $\mathfrak{sl}_2$  weight system); see for example [LMO] and [Le]. For a knot  $K$ ,  $F_K(x)$  is the Taylor series expansion at  $q = e^{-1/x}$  of a reformulation of the Kashaev invariant due to Huynh-Le, [HL]. In other words, we may write:

$$(4) \quad F_{\mathcal{K}}(x) = \sum_{n=0}^{\infty} a_{\mathcal{K},n} \frac{1}{x^n} \in \mathbb{Q}[[1/x]].$$

For every knotted object  $\mathcal{K}$ , the series  $F_{\mathcal{K}}(x)$  is known to be Gevrey-1 (see [GL]) and in general is expected to be divergent.

**Problem 2.** Show the existence of asymptotic expansions of the sequence  $(a_{\mathcal{K},n})$ , and identify the leading terms with known geometric invariants.

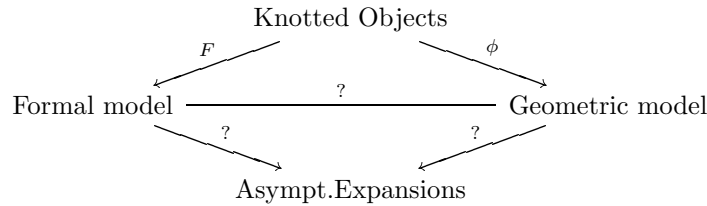
Thus, we have two types of invariants of a knotted object  $\mathcal{K}$ :

- (a) the function  $\phi_{\mathcal{K}} : \Omega \longrightarrow \mathbb{C}$ , and
- (b) the power series  $F_{\mathcal{K}}(x)$ .

Using suitable arithmetic completions, in [Ha1, Ha2] Habiro proves that either one of the following invariants:  $F_{\mathcal{K}}(x)$ ,  $\phi_{\mathcal{K}}$ , the sequence  $(\phi_{\mathcal{K}}(e^{2\pi i/n}))$ , the sequence  $(a_{\mathcal{K},n})$ , determines the other. We should point out that Habiro's proof is in a sense transcendental, of arithmetic nature. For example, finitely many terms of the sequence  $(a_{\mathcal{K},n})$  cannot determine  $\phi_{\mathcal{K}}(e^{2\pi i/3})$ .

**Problem 3.** Give an analytic proof of Habiro's result.

Summarizing, we have the following problems:



**1.5. A resurgence conjecture.** Despite the apparent analytic difficulties of the series (1) when  $q$  is inside or outside the unit circle, and the apparent factorial divergencies, there seems to be sufficient order and regularity. Our starting point is the formal power series  $F_{\mathcal{K}}(x)$ . Let us state the conjecture here, and explain the terms a little later. For further discussion, see also [Ga].

**Conjecture 1.** *For every knotted object  $\mathcal{K}$ ,*

- (a) *the series  $F_{\mathcal{K}}(x)$  has resurgent Borel transform,*
- (b) *the median sum  $S_{\mathcal{K}}^{\text{med}}$  of  $F_{\mathcal{K}}(x)$  is an analytic function defined on the right half-plane  $\Re(x) > 0$ , with radial limits at the points  $\frac{1}{2\pi i}\mathbb{Q}$  of its natural boundary.*
- (c) *Moreover, for  $\alpha \in \frac{1}{2\pi i}\mathbb{Q}$ ,  $\alpha \neq 0$ , we have:*

$$(5) \quad S_{\mathcal{K}}^{\text{med}}\left(-\frac{1}{\alpha}\right) = \phi_{\mathcal{K}}(\alpha).$$

Our next result shows how resurgence answers the three problems mentioned above. To state it, recall some standard notation from asymptotic analysis. For a function  $f(x)$  defined a right-hand plane  $\Re(x) > 0$ , the notation

$$(6) \quad f(x) = O\left(\frac{1}{x^N}\right)$$

means that there exist positive constants  $C$  and  $M$  so that  $|f(x)| < C/|x|^N$  for all  $x$  with  $\Re(x) > 0$ ,  $|x| > M$ . Furthermore, we say that  $f(x)$  is *asymptotic in the sense of Poincaré* to a formal power series  $\hat{f}(x) = \sum_{k=0}^{\infty} c_k/x^k$  (and write  $f(x) \sim \hat{f}(x)$ ) iff for every  $N \in \mathbb{N}$  we have:

$$(7) \quad f(x) - \sum_{k=0}^{N-1} \frac{c_k}{x^k} = O\left(\frac{1}{x^N}\right).$$

**Theorem 1.** *Assuming Conjecture 1, it follows that*

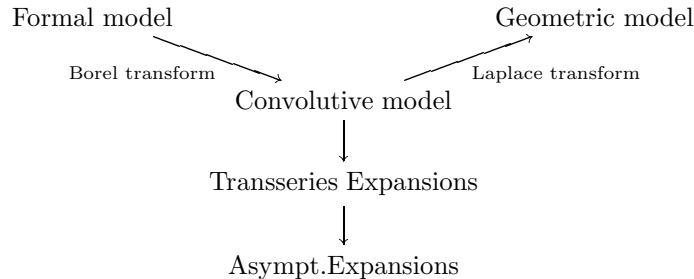
- (a) *In the interior  $\Re(x) > 0$ ,*

$$(8) \quad S_{\mathcal{K}}^{\text{med}}(x) \sim F_{\mathcal{K}}(x)$$

*for large  $x$ .*

- (b) *There exist transseries expansions for the sequence  $(a_{\mathcal{K},n})$  and for the sequence  $S_{\mathcal{K}}^{\text{med}}(n/(2\pi i))$ .*
- (c)  *$F_{\mathcal{K}}(x)$  determines  $\phi_{\mathcal{K}}$  and vice-versa.*

For a definition of a transseries and a proof, see Section 7. Schematically, Conjecture 1 implies the following:



Thus, our Resurgence Conjecture 1 solves at once Problems 1, 2 and 3 from Sections 1.3 and 1.4.

As a step towards Conjecture 1, in [GL] Le and the first author show that  $F_{\mathcal{K}}(x)$  is a Gevrey power series.

Aside from the applications in Quantum Topology, the conjectured resurgent series in Conjecture 1 seem to have a different origin than differential equations. Getting a little ahead of us, the resurgent function (16) below does not satisfy any linear or nonlinear differential equation with polynomial coefficients, as follows

from the structure of its singularities. Resurgence seems to come from the knotted objects themselves, their combinatorial encodings and the exact quantum field theory invariants. This will be investigated further in a subsequent publication.

2. TESTING CONJECTURE 1

2.1. **The Zagier-Kontsevich power series.** In the present paper we will verify the conjecture for the simplest non-trivial knot: the trefoil  $3_1$  (and also for the Poincaré Homology sphere; see A). Consider the *Kontsevich-Zagier* formal power series

$$(9) \quad f(q) = \sum_{n=0}^{\infty} (q)_n,$$

where the *q-factorial*  $(q)_n$  is defined by

$$(q)_n = (1 - q) \dots (1 - q^n)$$

for  $n > 0$  with  $(q)_0 = 1$ . Although  $f(q)$  is *not* an analytic function of  $q$  inside or outside the unit circle, it has a Taylor series for  $q = 1$ , as well as evaluations at complex roots of unity. With the notation of Section 1.4, we have:

$$F_{3_1}(x) = e^{-1/x} f(e^{-1/x}).$$

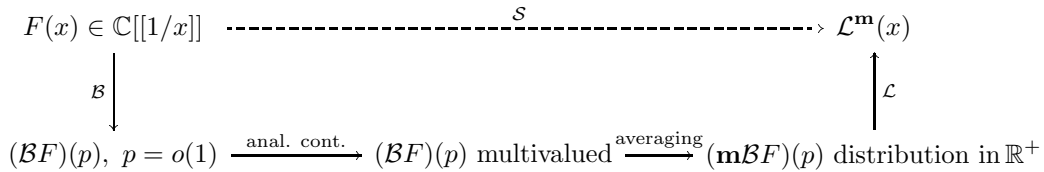
with  $f(q)$  given in (9). The power series  $f(q)$  appears in the beautiful paper of Zagier (see [Za]), and was also considered by Kontsevich in a talk at the Max-Planck-Institut für Mathematik in October 1997. Our basic object of study will be a modified version of  $F_{3_1}(x)$ , namely,

$$(10) \quad F(x) = e^{-1/(24x)} f(e^{-1/x}) \in \mathbb{Q}[[1/x]]$$

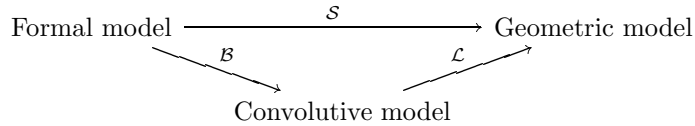
with  $f(q)$  given in (9).

2.2. **Three models of resurgence in a nutshell.** Before we proceed, we need to explain resurgence, the key aspect of Conjecture 1. Resurgence was introduced and studied by Écalle, see [Ec]. The input of resurgence are formal power series and the output are constructible analytic functions in suitable domains, which are asymptotic to the original formal power series. For an extended introduction to resurgence, the reader may also consult [DP, D].

The idea of resurgence is summarized in the following diagram:



and its shorthand version:



Let us explain the terminology here.

- The input is a *Gevrey-1* formal power series  $F(x) = \sum_{n=0}^{\infty} a_n x^{-n}$ . That is, a formal power series such that there exist constants  $C, C' > 0$  so that

$$(11) \quad |a_n| \leq C' C^n n!$$

for all  $n \in \mathbb{N}$ .

- The *Borel transform*  $\mathcal{B}$  is defined by

$$(12) \quad \mathcal{B} : \mathbb{C}[[\frac{1}{x}]] \longrightarrow \mathbb{C}[[p]], \quad \mathcal{B} \left( \sum_{n=0}^{\infty} a_n \frac{1}{x^n} \right) = \sum_{n=0}^{\infty} a_{n+1} \frac{p^n}{n!}$$

In other words,

$$\mathcal{B}(x^{-n-1}) = \frac{p^n}{n!}$$

If the series  $F(x)$  is Gevrey-1, it follows that  $(\mathcal{B}F)(p)$  is an analytic function in a neighborhood of  $p = 0$ .

- The two horizontal arrows *endlessly analytically continue*  $(\mathcal{B}F)(p)$  in the complex plane, minus a discrete set  $\mathcal{N}$  of singularities, as a multivalued function. In case the set  $\mathcal{N}$  of singularities of  $(\mathcal{B}F)(p)$  is a subset of the real line, one obtains a distribution  $(\mathbf{m}\mathcal{B}F)(p)$  on the positive real axis  $\mathbb{R}^+$  by means of an averaging  $\mathbf{m}$ . This is explained in detail in Section 4.2.
- The vertical arrow is the *Laplace transform* defined by

$$\mathcal{L}\mathbf{m}\mathcal{B}F : \{x \Re(x) > c\} \longrightarrow \mathbb{C}, \quad (\mathcal{L}\mathbf{m}\mathcal{B}F)(x) = \int_0^{\infty} e^{-xp} (\mathbf{m}\mathcal{B}F)(p) dp$$

under suitable hypothesis on the growth-rate of  $(\mathbf{m}\mathcal{B}F)(p)$  for large  $p$ .

- The final horizontal arrow is the *generalized Borel transform* which remembers the constant term of  $F(x)$  and is defined by

$$(13) \quad \mathcal{S}^{\mathbf{m}}(F)(x) = a_0 + (\mathcal{L}\mathbf{m}\mathcal{B})(F).$$

The result is an analytic function defined in a right half-plane.

**Definition 2.1.** When the above process can be completed, we say that

- the formal power series  $F(x) \in \mathbb{Q}[[1/x]]$  is *generalized Borel summable*, (and belongs to the *formal model*)
- its Borel transform  $G(p)$  is *resurgent*, (and belongs to the *convolutive model*)
- the resulting function  $\mathcal{S}^{\mathbf{m}}(F)(x)$  is *analyzable* (and belongs to the *geometric model*).

In what follows, given a generalized Borel summable series  $F(x)$ , we will denote by  $G(p)$  its Borel transform, and by  $\mathcal{S}^{\mathbf{m}}$  its summation with respect to  $\mathbf{m}$ .

Why is this a reasonable definition? The reader may find an answer in the following proposition. For a proof, see [Ec] and also the exposition in [DP, D, CNP1, CNP2, MI, Ra].

**Proposition 2.2.** (a) Generalized Borel summation  $\mathcal{S}^{\mathbf{m}}(x)$  coincides with  $F(x)$  in case  $F(x)$  is analytic in a neighborhood of  $x = \infty$ :

$$F(x) = \mathcal{S}^{\mathbf{m}}(x)$$

This follows from the following computation

$$x^{-n-1} = \int_0^{\infty} e^{-px} \frac{p^n}{n!} dp,$$

(valid for  $x \in \mathbb{C}$  with  $\Re(x) > 0$ , and  $n \in \mathbb{N}$ ) and the fact that if  $F(x)$  is analytic in a neighborhood of  $x = \infty$ , then its Borel transform  $G(p)$  is an entire function of exponential growth, thus the analytic continuation and the averaging steps do not change  $G(p)$ , and the Laplace transform reproduces  $F(x)$ .

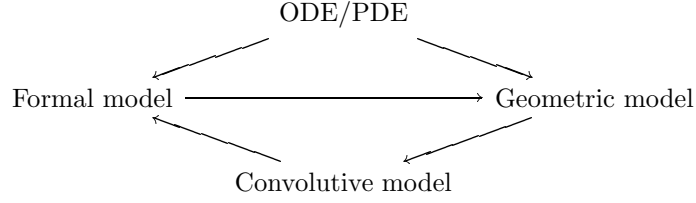
(b) If  $F(x) \in \mathbb{C}[[1/x]]$  is generalized Borel summable with  $\mathbf{m}$ -summation  $\mathcal{S}^{\mathbf{m}}(x)$ , then for large  $\Re(x)$  we have an asymptotic expansion:

$$\mathcal{S}^{\mathbf{m}}(x) \sim F(x).$$

(c) The set of generalized Borel summable is an algebra, closed under differentiation with respect to  $x$ . In particular, if  $F(x)$  is a formal solution of a differential or difference (linear or not) equation, then  $\mathcal{S}^{\mathbf{m}}(x)$  is an actual solution of the equation asymptotic to  $F(x)$ .

(d) Generalized Borel summability is a constructive approach, which has applications to the numerical approximation of analyzable functions which are asymptotic to divergent formal power series. See for example, the method of *truncation to least term* of factorially divergent series in [CK].

In other words, in analysis we have the following diagram:



**2.3. Statement of the results.** Let us postpone the remaining definitions to Section 4. Our main theorem is the following.

**Theorem 2.** (a) The formal power series  $F(x)$  of (10) has resurgent Borel transform.  
(b) The median summation  $S^{\text{med}}$  defined on  $\{x \in \mathbb{C} \mid \Re(x) > 0\}$  extends to the points  $\frac{1}{2\pi i}\mathbb{Q}$  of its natural boundary and for all  $\alpha \in \mathbb{Q}$ ,  $\alpha \neq 0$ , we have:

$$(14) \quad S^{\text{med}} \left( -\frac{1}{2\pi i\alpha} \right) = e^{\pi i\alpha/12} f(e^{2\pi i\alpha}).$$

The reader may compare Equation (10) that defines the formal power series  $F(x)$  with Equation (14) that evaluates the median summation  $S^{\text{m}}(x)$  of  $F(x)$ .

A side bonus is the following precise description of the Borel and Laplace transforms of  $F(x)$ . Among other things, it explains why we are using the median Laplace transform, and identifies the Laplace transforms in our paper with several functions considered by Zagier in [Za].

Let  $G(p)$  denote the formal Borel transform of the power series  $F(x)$  of (10). Recall the definition of the Dedekind eta function  $\eta$  and the modified eta function  $\tilde{\eta}$  from Section 6. Let  $\chi(\cdot)$  be the *unique primitive character of conductor 12*. In other words, we have:

$$(15) \quad \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \pmod{12} \\ -1 & \text{if } n \equiv 5, 7 \pmod{12} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3.** (a) The Borel transform  $G(p)$  of  $F(x)$  is given by:

$$(16) \quad G(p) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{\chi(n)n}{(-p + n^2\pi^2/6)^{5/2}}.$$

$G(p)$  is an analytic double-valued function on  $\mathbb{C} - \mathcal{N}$ , with singularities in the set  $\mathcal{N} \subset \mathbb{R}^+$ :

$$(17) \quad \mathcal{N} = \frac{\pi^2}{6} \left\{ n^2 \mid n \in \mathbb{N}, n \equiv 1, 5, 7, 11 \pmod{12} \right\}.$$

(b) The left and right summations  $S^{\text{mul}}$  and  $S^{\text{mur}}$  are given by:

$$(18) \quad S^{\text{mul}} : \{\arg(x) \in (-5\pi/2, \pi/2)\} \quad S^{\text{mul}}(x) = \sqrt{3}x^{3/2} \int_{\gamma_{\epsilon+\arg(x)}} \frac{\eta(2\pi iz)}{(x-z)^{3/2}} dz - 1$$

$$(19) \quad S^{\text{mur}} : \{\arg(x) \in (-\pi/2, 5\pi/2)\}, \quad S^{\text{mur}}(x) = \sqrt{3}x^{3/2} \int_{\gamma_{\epsilon-\arg(x)}} \frac{\eta(2\pi iz)}{(x-z)^{3/2}} dz - 1$$

where  $\gamma_{\theta}$  denotes the ray  $\{re^{i\theta} \mid r \geq 0\}$  in the complex plane from 0 to infinity.

(c) For every reality-preserving average  $\mathbf{m}$  (defined in Section 4.2), the summation  $S^{\text{m}}$  is independent of  $\mathbf{m}$ , agrees with the median summation and is given by:

$$S^{\text{med}} : \{x \in \mathbb{C} \mid \Re(x) > 0\} \longrightarrow \mathbb{C}$$

$$\begin{aligned} S^{\text{med}}(x) &= \frac{1}{2} (S^{\text{mur}}(x) + S^{\text{mul}}(x)) \\ &= \frac{\sqrt{3}x^{3/2}}{2} \left( \int_{\gamma_{\epsilon+\arg(x)}} \frac{\eta(2\pi iz)}{(x-z)^{3/2}} dz + \int_{\gamma_{\epsilon-\arg(x)}} \frac{\eta(2\pi iz)}{(x-z)^{3/2}} dz \right) - 1. \end{aligned}$$

Moreover,

$$\overline{S^{\text{med}}(x)} = S^{\text{med}}(\bar{x}).$$

(d) The associated Dirichlet series, defined by

$$(20) \quad \delta : \{\Re(x) > 0\} \longrightarrow \mathbb{C}, \quad \delta(x) = \frac{1}{2} (S^{\text{mur}}(x) - S^{\text{mul}}(x))$$

equals to:

$$(21) \quad \delta(x) = i\sqrt{2}(\pi x)^{3/2} \tilde{\eta}(2\pi i x)$$

where  $\tilde{\eta}$  is given in (56).  $\delta$  is a lacunary series, with natural boundary  $\Re(x) = 0$  and with well-defined radial limits at  $\frac{1}{2\pi i}\mathbb{Q}$ .

The above theorem gives a formula for the median Laplace transform of  $G(p)$  in terms of the modified  $\eta$ -function  $\tilde{\eta}$ . Our last theorem is an alternative formula for the median Laplace transform  $S^{\text{med}}$  in terms of the *complex error function*:

$$(22) \quad \text{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt$$

The complex error function is related to the better known *error function*

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

by

$$\text{Erfi}(x) = \frac{\text{Erf}(ix)}{i}.$$

Erfi is an entire odd function of  $x$ , with asymptotic expansion for large  $x$  with  $\arg(x) \in (0, \pi)$  of the form:

$$\text{Erfi}(x) \sim -i + \frac{e^{x^2}}{x\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2x^2)^k}$$

where  $(2k-1)!! = 1 \cdot 3 \cdot \dots \cdot (2k-1)$  and  $(-1)!! = 1$ . See for example, [O, Sec.2.2] or [Lb, Sec.2]. Consider the modified error function

$$(23) \quad \mathcal{E}(x) = e^{-x^2} x^3 \text{Erfi}(x) - \frac{x^2}{\sqrt{\pi}}.$$

Notice that

$$(24) \quad \mathcal{E}(x) = o(1)$$

for large  $x$  with  $\arg(x) \in (-\pi/4, \pi/4)$ .

**Theorem 4.** *The median Laplace transform of the series  $G(p)$  of Equation (26) is given by:*

$$(25) \quad S^{\text{med}}(x) = \frac{12\sqrt{3}}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} \mathcal{E}\left(n\pi\sqrt{\frac{x}{6}}\right) - 1.$$

Notice that Equation (24) implies that the series (25) is uniformly convergent in the open right half-plane.



**2.4. Plan of the proof.** Our goal is to motivate, introduce and use resurgence in a relatively self-contained fashion. In Section 3 we compute explicitly the Borel transform of the series (10) using as input the generating function of the Glaisher’s numbers, studied by Zagier. The trigonometric form of this generating function quickly leads, via a residue computation, to formula (16) for the Borel transform  $G(p)$  of Theorem 3. This formula is an example of what we call a “square root branched function”. In particular, this implies the existence of the analytic continuation of  $G(p)$  and locates its singularities.

In Section 4 we discuss at length the notion of averaging, following the work of Écalle, and give several examples of averages. Averaging leads to a Laplace transform, which in general depends on the averaging itself. Our key Proposition 4.2 shows that if  $G(p)$  is square root branched, then all reality-preserving averages coincide with the median average. Since our singularities are placed at the positive real numbers, and  $G(p)$  is square root branched, the difference between the left and right averages is a Dirichlet series, as we show in Proposition 4.2. We end this section by giving explicit formulas for the median Laplace transform in terms of the Dedekind  $\eta$ -function and in terms of the complex error function, proving Theorem 4 and part of Theorem 3.

In Section 5 we study the associated Dirichlet series of our problem, which turns out to be a modified Dedekind  $\tilde{\eta}$ -function. Zagier’s identity and modularity imply the existence of radial limits of our Dirichlet series. This concludes the proof of Theorems 2 and 3.

In Section 7 we explain how resurgence implies the existence of asymptotic (and more generally, transseries) expansions of sequences. In particular, we give a proof of Theorem 1.

Finally, in Section 2.5, we point out that our results apply without change to the power series  $F_{\mathcal{K}}(x) \in \mathbb{Q}[[1/x]]$  where  $\mathcal{K}$  is a  $(2, 2p)$  torus link or a Seifert fibered rational homology sphere.

**2.5. Extensions.** For simplicity, we state Theorems 2 and 3 for the power series  $F(x)$  of (10).

The proof of Theorem 2 works without change for the formal power series of  $F_{\mathcal{K}}(x)$  where  $\mathcal{K}$  is a torus link  $(2, 2p)$  or a Seifert-fibered homology sphere. In all those cases,

- the Borel transform is square root branched,
- the singularities of  $G(p)$  are a finite union of sets of the form

$$\mathcal{N} = \frac{\pi^2}{\beta} \left\{ n^2 \mid n \in \mathbb{N}, \chi(n) \neq 0 \right\}$$

for some quadratic character  $\chi$ .

- the associated Dirichlet series is nearly modular of weight  $1/2$ ,
- radial limits of the Dirichlet series exist, and Zagier’s identity and modularity holds.

For an example of the Poincaré homology sphere, see the Appendix. In forthcoming work [CG2] we will prove Conjecture 1 for a class of geometrically interesting knotted objects  $\mathcal{K}$  that include the simplest hyperbolic  $4_1$  knot.

**2.6. Acknowledgment.** An early version of this paper was presented by the second author in a conference in Columbia University, around the Volume Conjecture, in the fall of 2005 and spring of 2006. The second author wishes to thank the organizers of the conference for their wonderful hospitality.

### 3. THE BOREL TRANSFORM $G(p)$ OF $F(x)$

In this section we compute the formal Borel transform  $G(p)$  of the power series  $F(x)$  of (10).

**Theorem 5.** *We have:*

$$(26) \quad G(p) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{\chi(n)n}{(-p + n^2\pi^2/6)^{5/2}}.$$

Consequently,  $G(p)$  is resurgent, and its analytic continuation is double-valued in  $\mathbb{C} - \mathcal{N}$  with singularities in  $\mathcal{N}$ , defined as in Equation (17).

*Proof.* Let us define a sequence  $(a_n)$  by:

$$(27) \quad F(x) = e^{-1/(24x)} f(e^{-1/x}) = \sum_{n=0}^{\infty} \frac{a_n}{24^n} \frac{1}{x^n}.$$

Our sequence  $(a_n)$  coincides with Zagier's  $(T_n/n!)$  from [Za, Eqn.4], where  $(T_n)$  are the *Glaisher's T-numbers*. In [Za], Zagier proves that the *Glaisher's T-numbers* are given by the generating series

$$(28) \quad \sum_{n=0}^{\infty} \frac{a_n n!}{(2n+1)!} p^{2n+1} = \frac{\sin 2p}{2 \cos 3p} = \frac{\sin p}{1 - 4 \sin^2 p}$$

In the following calculations, it will be convenient to let  $H(p)$  denote the formal Borel transform of

$$\begin{aligned} F(x/24) &= e^{-1/x} f(e^{-24/x}) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{x^n}. \end{aligned}$$

It is easy to check that

$$G(p) = \frac{1}{24} H\left(\frac{p}{24}\right).$$

Thus, it suffices to show that

$$(29) \quad H(p) = 1296\sqrt{3}\pi \sum_{n=1}^{\infty} \frac{\chi(n)n}{(-144p + n^2\pi^2)^{5/2}}.$$

By the definition of  $H(p)$ , we have:

$$\begin{aligned} H(p) &= \mathcal{B}\left(1 + \sum_{n=0}^{\infty} a_{n+1} \frac{1}{x^{n+1}}\right) \\ &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} p^n \end{aligned}$$

On the other hand, Equation (28) implies that

$$p + \sum_{n=0}^{\infty} \frac{a_{n+1}(n+1)!}{(2n+3)!} p^{2n+3} = \frac{\sin p}{1 - 4 \sin^2 p}$$

thus

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} \frac{n!(n+1)!}{(2n+3)!} p^{2n} = \frac{1}{p^3} \left( \frac{\sin p}{1 - 4 \sin^2 p} - p \right).$$

Since

$$\sum_{n=0}^{\infty} \frac{(2n+3)!}{n!(n+1)!} p^n = 2 \sum_{n=0}^{\infty} (2n+3)(2n+1) \binom{2n}{n} p^n = \frac{6}{(1-4p)^{5/2}}$$

it follows that

$$H(p) = (f_1 \otimes f_2)(p)$$

where

$$\begin{aligned} f_1(p) &= \frac{1}{p^{3/2}} \left( \frac{\sin(p^{1/2})}{1 - 4 \sin^2(p^{1/2})} - p^{1/2} \right) \\ f_2(p) &= \frac{6}{(1-4p)^{5/2}} \end{aligned}$$

and  $\otimes$  denotes the *Hadamard product* of two formal power series at  $p = 0$ . The latter is the component-wise product defined by:

$$\left( \sum_{n=0}^{\infty} a_n p^n \right) \otimes \left( \sum_{n=0}^{\infty} b_n p^n \right) = \sum_{n=0}^{\infty} a_n b_n p^n.$$

It is easy to give a contour integral formula for the Hadamard product:

$$H(p) = \frac{1}{2\pi i} \int_{\gamma} f_1(s) f_2\left(\frac{p}{s}\right) \frac{ds}{s}$$

where  $\gamma$  is a small circle around 0. This will give an analytic continuation of the Hadamard product.

Observe that the set  $\mathcal{N}_1$  of singularities of  $f_1(p)$  is

$$(30) \quad \mathcal{N}_1 = \left\{ \left(2k + \frac{1}{6}\right)^2 \pi^2, \left(2k + \frac{5}{6}\right)^2 \pi^2, \left(2k + \frac{7}{6}\right)^2 \pi^2, \left(2k + \frac{11}{6}\right)^2 \pi^2 \mid k \in \mathbb{N} \right\}$$

Now, we enlarge the radius  $r$  of the circle  $\gamma := \gamma_r$ , and subtract the residues of the integrand at the singular points, applying Cauchy's theorem.

The integrand has single poles at the points  $\eta \in \mathcal{N}_1$ . By a straightforward calculation we get that the residue  $\psi_{\eta}(p)$  of the integrand at  $\eta \in \mathcal{N}_1$  is given by

$$(31) \quad \psi_{\eta}(p) = 1296\sqrt{3}\pi \begin{cases} -\frac{1+12k}{(-144p + (1+12k)^2\pi^2)^{5/2}} & \text{if } \eta = (1+12k)^2\pi^2/6 \\ +\frac{5+12k}{(-144p + (5+12k)^2\pi^2)^{5/2}} & \text{if } \eta = (5+12k)^2\pi^2/6 \\ +\frac{7+12k}{(-144p + (7+12k)^2\pi^2)^{5/2}} & \text{if } \eta = (7+12k)^2\pi^2/6 \\ -\frac{11+12k}{(-144p + (11+12k)^2\pi^2)^{5/2}} & \text{if } \eta = (11+12k)^2\pi^2/6 \end{cases}$$

for  $k \geq 0$ . The asymptotic behavior of  $\psi_{\eta}(p)$  for large  $\eta$  is  $O(1/\eta^2)$ , and thus the sum  $\sum_{\eta \in \mathcal{N}_1} \psi_{\eta}(p)$  converges.  $\psi_{\eta}(p)$  is double-valued with only one singularity  $\eta/4$  of the form:

$$\psi_{\eta}(p) = c_{\eta}(\eta/4 - p)^{-5/2}.$$

The definition of  $\chi$  and the above computation conclude the proof of Theorem 5.  $\square$

*Example 3.1.* As an numerical check, Theorem 5 implies that

$$G(p) = 54\sqrt{3} \frac{L(4, \chi)}{\pi^4} + O(p)$$

where

$$(32) \quad L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Since

$$(33) \quad L(2n+2, \chi) = \pi^{2n+2} \frac{(-4)^n}{\sqrt{3}(2n+1)!(n+1)} \left( B_{2n+2}\left(\frac{1}{12}\right) - B_{2n+2}\left(\frac{5}{12}\right) \right)$$

(see [Za, Eqn.(6)]) it follows that

$$(34) \quad G(p) = \frac{23}{24} + O(p).$$

On the other hand,

$$(35) \quad F(x) = 1 + \frac{23}{24} \frac{1}{x} + \frac{1681}{1152} \frac{1}{x^2} + \frac{257543}{82944} \frac{1}{x^3} + \frac{67637281}{7962624} \frac{1}{x^4} + O\left(\frac{1}{x^5}\right),$$

which confirms that the constant term of the Borel transform of  $F(x)$  is given by  $23/24$ , in accordance with (34).

*Exercise 3.2.* Using Theorem 5 and the special values of the L-series given in (33), show that

$$G(p) = \frac{23}{24} + \frac{1681}{1152}p + \frac{257543}{165888}p^2 + \frac{67637281}{47775744}p^3 + O(p^4)$$

in confirmation with the Borel transform of (35).

#### 4. THE LAPLACE TRANSFORM OF $G(p)$

In this section we compute the Laplace transform  $\mathcal{L}(x)$  of  $G(p)$ .

**4.1. Analytic continuation, averaging and Laplace transform.** Given a resurgent function  $G(p)$  with singularities in  $\mathbb{N}^+ \subset \mathbb{R}^+$ , there are three ways to average and take the Laplace transform.

- The first way is to use a uniformizing average  $\mathbf{m}$  of Écalle in order to get a single valued function  $\mathbf{m}G$  on  $\mathbb{R}^+$ . Unfortunately, this function is not integrable since  $\int_0^1 dp/(p-1)^{-5/2}$  does not exist. So, Écalle applies an acceleration operator to  $\mathbf{m}G$  and then takes the usual Laplace transform.
- Alternatively, Écalle applies a uniformizing average to the Laplace transform of the analytic continuation of  $G(p)$  along paths that avoid the singularities. The key property is that the set of such paths form a Riemann surface.
- The first author converts  $G(p)$  to a step-distribution on  $\mathbb{R}^+ \setminus \mathbb{N}^+$  and then applies an extended Borel transform  $\mathcal{B}_\alpha$ , followed by an extended Laplace transform. See [C, Sec.1.3].

Now, we arrive at a subtle point: there are many well-behaved uniformizing averages. In fact for every probability distribution  $f \in L^1(\mathbb{R})$  with  $\int_0^\infty |f(x)|dx = 1$ . Écalle-Menous construct a uniformizing average  $\mathbf{m}_f$ ; see [EM].

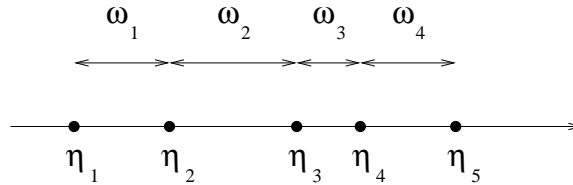
On the other hand, the first author extended Borel transforms  $\mathcal{B}_\alpha$  are parametrized by  $\alpha \in 1/2 + i\mathbb{R}$ . Of all those Borel transforms the most useful one is the balanced one  $\mathcal{B}_{1/2}$ , which satisfies the key property of approximation by summation to least term; see [CK].

In case a formal power series satisfies a generic differential equation (linear or not), all averages  $\mathbf{m}_f$  agree with the first author's balanced  $\mathcal{B}_{1/2}$ , as shown in [C]. This is also a consequence of Écalle's *bridge equation*; see [Ec, DP].

In our case, the series  $F(x)$  does not satisfy a differential equation. Nevertheless, Proposition 4.2 shows a universality, i.e., independence of averaging. Before we state the proposition, let us explain what averaging means.

**4.2. What is an averaging?** Averages were introduced and studied extensively by Écalle. Following Écalle-Menous (see [EM]), let us consider a multivalued function  $G(p)$  defined on  $\mathbb{C} - \mathcal{N}$ , with at most exponential growth at infinity, and with singularities on a discrete set  $\mathcal{N} = \{\eta_k \mid k \in \mathbb{N}\} \subset \mathbb{R}^+$ , where  $\eta_k < \eta_l$  for  $k < l$ .

Let us define the relative spacing  $\omega_k$   $k \in \mathbb{N}$  of the singularities by  $\omega_k = \eta_{k+1} - \eta_k$ . Thus, we have the picture:



With respect to the terminology of Écalle (cf. e.g. [EM]) we have that  $G \in \text{RAMIF}(\mathbb{R}^+)$  with ramification points  $\mathcal{N}$ . An averaging  $\mathbf{m}$  is a linear map

$$\mathbf{m} : \text{RAMIF}(\mathbb{R}^+) \longrightarrow \text{UNIF}(\mathbb{R}^+)$$

that maps multivalued functions with singularities at  $\mathcal{N}$  to single-valued distributions on  $\mathbb{R}^+$ . Averaging maps depend on a set of averaging weights. An *averaging weight*  $\mathbf{m}^\varpi$  is a collection

$$\{\mathbf{m}^\varpi = \mathbf{m}^{\varpi_1, \dots, \varpi_r} \mid r \in \mathbb{N}, \varpi_i = \left( \frac{\epsilon_i}{\omega_i} \right), \epsilon_i = \pm, \omega_i = \eta_{j+1} - \eta_j\}.$$

The tuple  $(\epsilon_1, \dots, \epsilon_r)$  is called an *address*. We always assume that for all  $r$ , we have:

$$(36) \quad \mathbf{m}^{\varpi_1, \dots, \varpi_r} = \mathbf{m}^{\varpi_1, \dots, \varpi_r, \left(\frac{+}{\varpi_{r+1}}\right)} + \mathbf{m}^{\varpi_1, \dots, \varpi_r, \left(\frac{-}{\varpi_{r+1}}\right)}$$

Recall that  $G(p)$  is a multivalued function. For fixed  $r \in \mathbb{N}$  and  $\varpi_1, \dots, \varpi_r$ , we now define a multivalued function  $G^{\varpi_1, \dots, \varpi_r}$  as follows. Let

$$(37) \quad G^{\varpi_1, \dots, \varpi_r} : (\eta_r, \eta_{r+1}) \longrightarrow \mathbb{C}$$

denote the analytic continuation of  $G$  along a path avoiding the singularity at  $\eta_i$  from above if  $\epsilon_i = +$  and from below otherwise. Then,  $\mathbf{m}G$  is defined by:

$$\mathbf{m}G(p) = \sum_{\epsilon_1 = \pm, \dots, \epsilon_r = \pm} \mathbf{m}^{\varpi_1, \dots, \varpi_r} G^{\varpi_1, \dots, \varpi_r}(p), \quad p \in (\eta_r, \eta_{r+1}).$$

There are several natural properties that are often required for averages  $\mathbf{m}$ . Three important properties are:

- (P1)  $\mathbf{m}$  preserves reality and has real-valued weights,
- (P2)  $\mathbf{m}$  preserves convolution,
- (P3)  $\mathbf{m}$  preserves lateral growth.

P1 is useful when the input is a power series with real coefficients and the needed output is an analytic function on the right half-plane  $\Re(x) > 0$  which takes real values for  $x > 0$ .

P2 is needed for commutation of generalized Borel summability with multiplication of power series.

P3 is necessary to be able to define Laplace transforms.

Let us give three rather trivial, but useful averages from [EM, p.85]:

$$\begin{aligned} \mathbf{mur}^{\varpi_1, \dots, \varpi_r} &= \begin{cases} 1 & \text{if } \epsilon_1 = \epsilon_2 = \dots = + \\ 0 & \text{otherwise,} \end{cases} \\ \mathbf{mul}^{\varpi_1, \dots, \varpi_r} &= \begin{cases} 1 & \text{if } \epsilon_1 = \epsilon_2 = \dots = - \\ 0 & \text{otherwise,} \end{cases} \\ \mathbf{med}^{\varpi_1, \dots, \varpi_r} &= \begin{cases} 1/2 & \text{if } \epsilon_1 = \epsilon_2 = \dots = + \\ -1/2 & \text{if } \epsilon_1 = \epsilon_2 = \dots = - \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$\mathbf{mul}$ ,  $\mathbf{mul}$ ,  $\mathbf{med}$  and  $\mathcal{B}_{1/2}$  satisfy P3.  $\mathcal{B}_{1/2}$  and  $\mathbf{med}$  satisfies P1.

**4.3. The Laplace transform of an averaged function.** Recall that the Laplace transform of a function  $G(p) \in L^1(\mathbb{R}^+, e^{-\nu x} dx)$  (for  $\nu > 0$ ) with at most exponential growth at infinity is defined by:

$$(38) \quad \mathcal{L}G : \{x \in \mathbb{C} | \Re(x) > 1/\nu\} \longrightarrow \mathbb{C}, \quad (\mathcal{L}G)(x) = \int_0^\infty e^{-px} G(p) dp.$$

If  $G(p)$  is defined in a sectorial neighborhood of 0 (i.e. in a set  $\{p \in \mathbb{C} | \arg(p) \in (-\epsilon, \epsilon')\}$ ) and is of exponential growth at infinity, then by moving the integration contour it follows that  $\mathcal{L}(x)$  is defined in an enlarged neighborhood  $\{x \in \mathbb{C} | |x| > 1/\nu, \arg(x) \in (-\epsilon' - \pi/2, \epsilon + \pi/2)\}$ .

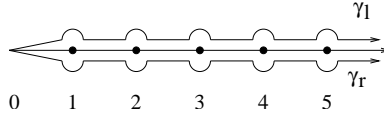
The definition of the Laplace transform makes sense in case  $G(p)$  is a *distribution* (e.g.  $G(p) = 1/(p-1)$ ), as was discussed by the first author in [C, Sec.2]. Likewise, we may define the Laplace transform of  $\mathbf{m}G(p)$ :

$$(39) \quad (\mathcal{L}^{\mathbf{m}}G)(x) = \int_0^\infty e^{-xp} \mathbf{m}G(p) dp$$

It turns out that  $(\mathcal{L}^{\mathbf{m}}G)(x)$  is an average of line integrals of  $G(p)$  along paths in  $\mathbb{C} - \mathcal{N}$  that start at 0 and end at  $\infty$ . For example, it is easy to see that

$$\begin{aligned} (\mathcal{L}^{\mathbf{mur}}G)(x) &= \int_{\gamma_r} e^{-px} G(p) dp \\ (\mathcal{L}^{\mathbf{mul}}G)(x) &= \int_{\gamma_l} e^{-px} G(p) dp \\ (\mathcal{L}^{\mathbf{med}}G)(x) &= \frac{1}{2} \left( \int_{\gamma_r} e^{-px} G(p) dp + \int_{\gamma_l} e^{-px} G(p) dp \right) \end{aligned}$$

where  $\gamma_r$  (resp.  $\gamma_l$ ) is a path in  $\mathbb{C} - \mathcal{N}$  from 0 to  $\infty$  that turns right (resp. left) at each singularity in  $\mathcal{N}$ :



Another useful average is  $\mathcal{B}_{1/2}$  of the first author; see [C, Eqn(1.20)].

In case the multivalued function  $G(p)$  is the Borel transform of a formal power series solution  $F(x)$  of a generic differential (or difference) equation, then it is known that the Laplace transforms  $\mathcal{L}^{\mathbf{m}}(x)$  for all averages that satisfy P1, P2, P3 agree. In our case,  $F(x)$  is not expected to satisfy a differential equation (linear or not) with polynomial coefficients, because the position of singularities (which is an analytic invariant) is qualitatively different from solutions to differential equations with polynomial coefficients. What is a natural average to consider? The next lemma states that for the singularities of  $G(p)$  in Equation (26), the Laplace transform is independent of the averaging.

To state the lemma, we need some notation. Motivated by Equation (26), let us introduce the following definition.

**Definition 4.1.** We will call a multivalued function  $G(p)$  *square root branched* if it is given by a (Mittag-Leffleg like) absolutely convergent sum:

$$(40) \quad G(p) = \sum_{\eta \in \mathcal{N}} G_{\eta}(p),$$

where  $\mathcal{N}$  is a discrete subset of  $\mathbb{R}^+$ ,

$$G_{\eta}(p) = c_{\eta}(\eta - p)^{-k_{\eta}/2},$$

and  $k_{\eta} \in \mathbb{N}^+$ . Thus, the support of  $c$ ,  $\{\eta \in \mathcal{N} \mid c_{\eta} \neq 0\}$  is the set of singularities of  $G(p)$ . A square root branched function  $G(p)$  has *weight*  $k$  when  $k_{\eta} = k \in \mathbb{N}^+$ .

**Proposition 4.2.** (a) If  $G(p)$  is square root branched of odd weight  $k$  and  $\mathbf{m}$  is any Écalle average that preserves P1 (and may or may not preserve P2 or P3), then

$$(41) \quad \mathbf{m}G(p) = \sum_{\eta \in \mathcal{N}} \mathbf{m}G_{\eta}(p)$$

where

$$(42) \quad \mathbf{m}G_{\eta}(p) = \begin{cases} G_{\eta}(p) & \text{if } p < \eta \\ 0 & \text{if } p \geq \eta \end{cases}$$

does not depend on  $\mathbf{m}$ .

(b) For  $\Re(x) > 0$ , we have:

$$(43) \quad (\mathcal{L}^{\mathbf{med}}G)(x) = \frac{1}{2} \left( (\mathcal{L}^{\mathbf{mur}}G)(x) + (\mathcal{L}^{\mathbf{mul}}G)(x) \right).$$

where  $(\mathcal{L}^{\mathbf{mul}}G)$  and  $(\mathcal{L}^{\mathbf{mur}}G)$  are defined for  $x \in \mathbb{C}^*$  with  $\arg(x) \in (-5\pi/2, \pi/2)$  and  $\arg(x) \in (-\pi/2, 5\pi/2)$  respectively.

(c) In their common domain  $\arg(x) \in (-\pi/2, \pi/2)$ , the associated Dirichlet series, is defined by:

$$(44) \quad \delta(x) = \frac{1}{2} ((\mathcal{L}^{\mathbf{mur}}G)(x) - (\mathcal{L}^{\mathbf{mul}}G)(x))$$

(d) Consequently, we have:

$$(45) \quad (\mathcal{L}^{\mathbf{med}}G)(x) = (\mathcal{L}^{\mathbf{mul}}G)(x) + \delta(x)$$

$$(46) \quad = (\mathcal{L}^{\mathbf{mur}}G)(x) - \delta(x)$$

(e) If  $c_\eta \in \mathbb{R}$  for all  $\eta$ , then

$$\overline{(\mathcal{L}^{\mathbf{mul}}G)(x)} = (\mathcal{L}^{\mathbf{mur}}G)(\bar{x}), \quad \overline{(\mathcal{L}^{\mathbf{mur}}G)(x)} = (\mathcal{L}^{\mathbf{mul}}G)(\bar{x}), \quad \overline{(\mathcal{L}^{\mathbf{med}}G)(x)} = (\mathcal{L}^{\mathbf{med}}G)(\bar{x}).$$

(f) When  $k$  is odd, the Dirichlet series is given by:

$$(47) \quad \delta(x) = i \frac{2^{(k-1)/2} \sqrt{\pi} x^{k/2-1}}{(k-2)!!} \sum_{\eta \in \mathcal{N}} c_\eta e^{-\eta x}$$

where for a natural number  $n \in \mathbb{N}$  we denote  $(2n+1)!! = 1.3.5 \dots (2n+1)$ .

*Proof.* It suffices to consider the case

$$G(p) = c_\eta (\eta - p)^{-k_\eta/2}$$

where  $k_\eta$  is positive integer. Let us fix an average  $\mathbf{m}$  of [EM] which is symmetric (i.e., satisfies P2 of Section 4.2) and let us suppose that  $\eta = \eta_r$  for some  $r \in \mathbb{N}$ . Observe that  $G(p)$  is not singular for  $p \in [0, \eta_r)$ . Equation (36) implies that

$$G^{\varpi_1, \dots, \varpi_s}(p) = G(p), \quad p \in (\eta_s, \eta_{s+1})$$

for  $s < r$ . On the other hand, for  $p \in (\eta_r, \eta_{r+1})$ , the two analytic continuations of the square root differ only in sign; thus,

$$G^{\varpi_1, \dots, \varpi_r, \left(\frac{+}{\omega_{r+1}}\right)}(p) = -G^{\varpi_1, \dots, \varpi_r, \left(\frac{-}{\omega_{r+1}}\right)}(p),$$

which together with the symmetry condition P2 imply that  $\mathbf{m}G(p) = 0$  for  $p \in (\eta_r, \eta_{r+1})$ , and in fact for  $p > \eta_r = \eta$ . This proves (a).

Part (b) follows from Section 4.3.

Parts (d), (e) follow from (b) and (c).

The definition of the Dirichlet series implies that

$$\delta(x) = c_\eta \int_{C_\eta} (\eta - p)^{-k_\eta/2} e^{-px} dp$$

where  $C_\eta$  is a loop (Hankel contour) from  $+\infty$ ,  $\arg(p) = 0$  to  $+\infty$ ,  $\arg(p) = 2\pi$  which goes once around  $\eta$ , oriented counterclockwise. A residue calculation implies (f).  $\square$

**4.4. A formula for the Laplace transform of  $G(p)$ .** In this section we will prove of part (c) of Theorem 3 and Theorem 4. We will use the Dedekind  $\eta$  function as in (55). Recall the contours  $\gamma_\theta$  from Theorem 3. Recall also that  $S^{\mathbf{mul}}(x)$  denotes the Laplace transform of  $\mathbf{mul}G(p)$ .

**Theorem 6.** For  $x \in \mathbb{C}$ ,  $x \neq 0$ ,  $\arg(x) \in (-5\pi/2, \pi/2)$ , we have:

$$(48) \quad S^{\mathbf{mul}}(x) = \sqrt{3} x^{3/2} \int_{\gamma_{\epsilon+\arg(x)}} \eta(2\pi iz) \frac{dz}{(x-z)^{3/2}} - 1.$$

This proves Equation (18) of Theorem 3. (19) is completely analogous.

*Proof.* We have:

$$\begin{aligned}
S^{\text{mul}}(x) &= \int_{\gamma_l} e^{-px} G(p) dp \\
&= \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \chi(n)n \int_{\gamma_l} \frac{e^{-px}}{(-p + n^2\pi^2/6)^{5/2}} dp && \text{by Thm 5} \\
&= \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \chi(n)n \left( \int_{\gamma_l} \frac{2x}{3} \frac{e^{-px}}{(-p + n^2\pi^2/6)^{3/2}} dp - \frac{2 \cdot 6^{3/2}}{3n^3\pi^3} \right) && \text{by integration by parts} \\
&= \sqrt{3}x \sum_{n=1}^{\infty} \chi(n) \int_{\gamma_l} \frac{e^{-n^2\pi^2 qx/6}}{(-q+1)^{3/2}} dq + C && \text{by a change of variables } 6p = n^2\pi^2 q
\end{aligned}$$

where

$$C =: -6\sqrt{3} \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} = -6\sqrt{3} \frac{1}{\pi^2} L(2, \chi) = -1$$

where the last Equality follows from Equation (33). Since

$$\begin{aligned}
\sqrt{3}x \sum_{n=1}^{\infty} \chi(n) \int_{\gamma_l} \frac{e^{-n^2\pi^2 qx/6}}{(-q+1)^{3/2}} dq &= \sqrt{3} \sum_{n=1}^{\infty} \chi(n) \int_{\gamma_{\epsilon+\arg(x)}} \frac{e^{-n^2\pi^2 z/6}}{(-z/x+1)^{3/2}} dz && \text{by a change of variables } z = qx \\
&= \sqrt{3}x^{3/2} \int_{\gamma_{\epsilon+\arg(x)}} \eta(2\pi iz) \frac{dz}{(x-z)^{3/2}} && \text{by (55)}
\end{aligned}$$

the result follows.  $\square$

We now give a proof of Theorem 4.

*Proof.* (of Theorem 4) Recall the complex error function  $\text{Erfi}(x)$  and its modification  $\mathcal{E}(x)$  from Equations (22) and (23). The median Laplace transform is the average of the left and right Laplace transform. Moreover, a calculation shows that for  $x > 0$  we have:

$$(49) \quad \int_{\gamma_l} \frac{e^{-xp}}{(1-p)^{5/2}} dp = -\frac{2}{3} - \frac{4x}{3} - \frac{4}{3}i\sqrt{\pi}e^{-x}x^{3/2} + \frac{4}{3}\sqrt{\pi}e^{-x}x^{3/2}\text{Erfi}(\sqrt{x}).$$

Replacing  $\gamma_l$  by  $\gamma_r$  has the effect of replacing  $i$  by  $-i$  in the above equation. Thus, the median integral, which also coincides with the *principal value integral*, is given by:

$$(50) \quad \int_{\gamma_m} \frac{e^{-xp}}{(1-p)^{5/2}} dp = -\frac{2}{3} - \frac{4x}{3} + \frac{4}{3}\sqrt{\pi}e^{-x}x^{3/2}\text{Erfi}(\sqrt{x}) = -\frac{2}{3} + \frac{4}{3}\sqrt{\pi}\mathcal{E}(\sqrt{x})$$

where  $\gamma_m = 1/2(\gamma_l + \gamma_r)$ . On the other hand, the proof of Theorem 6 implies that

$$S^{\text{med}}(x) = \sqrt{3}x \sum_{n=1}^{\infty} \chi(n) \int_{\gamma_m} \frac{e^{-n^2\pi^2 qx/6}}{(1-q)^{3/2}} dq - 1.$$

Using Equation (50), the result follows.  $\square$

## 5. A DIRICHLET SERIES $\delta(x)$ ASSOCIATED TO $F(x)$

**5.1. A formula for a Dirichlet series  $\delta(x)$  associated to  $F(x)$ .** In this section we identify the associated Dirichlet series  $\delta(x)$  of the generalized Borel summable power series  $F(x)$  of (10) with the Eichler integral  $\tilde{\eta}$  of the Dedekind  $\eta$ -function given by (56). In particular, using Zagier's identity (see (57)) and a modular property of one of Zagier's functions (see (61)), allows us to prove the existence of radial limits at complex roots of unity and to finish the proof of Theorems 2 and 3.



**Proposition 5.1.** (a) The Dirichlet series associated to  $F(x)$  is given by:

$$(51) \quad \delta : \{x \in \mathbb{C} | \Re(x) > 0\} \longrightarrow \mathbb{C}, \quad \delta(x) = i\sqrt{2}(\pi x)^{3/2} \tilde{\eta}(2\pi i x)$$

(b)  $\delta$  is a lacunary series with natural boundary the line  $\Re(x) = 0$ .

(c)  $\delta$  has radial limits at  $\frac{1}{2\pi i}\mathbb{Q}$  given by:

$$(52) \quad \delta\left(-\frac{1}{2\pi i\alpha}\right) = \zeta_{24}^3 \alpha^{-3/2} \phi(-1/\alpha)$$

for all  $\alpha \in \mathbb{Q}$ ,  $\alpha \neq 0$ , where  $\phi$  is a function of Zagier from (54) and  $\zeta_k = e^{2\pi i/k}$ .

*Proof.* Theorem 5 gives that

$$G(p) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{\chi(n)n}{(-p + n^2\pi^2/6)^{5/2}}.$$

Part (f) of Proposition 4.2 implies that the associated Dirichlet series is given by:

$$\begin{aligned} \delta(x) &= i\sqrt{2}(\pi x)^{3/2} \sum_{n=1}^{\infty} \chi(n) n e^{-\pi^2 n^2 x/6} \\ &= i\sqrt{2}(\pi x)^{3/2} \tilde{\eta}(2\pi i x) \end{aligned}$$

where the last equality follows from the definition of  $\tilde{\eta}$  in (56). This proves (a).

(b) follows from [Ma]. In other words,  $\delta(x)$  cannot be analytically continued beyond the line  $\Re(x) = 0$ . In general, lacunary series need not have radial limits at points of their natural boundary. Our series, however, has radial limits at rational multiples of  $1/(2\pi i)$ .

(c) follows from Equation (51) and Zagier's identity (57) below.  $\square$

**5.2. Proof of Theorem 2.** We are finally in a position to finish the proof of Theorem 2.

**Theorem 7.** *With the notation as in Theorem 2, for all  $\alpha \in \mathbb{Q}$ ,  $\alpha \neq 0$ , we have:*

$$S^{\text{med}}\left(-\frac{1}{2\pi i\alpha}\right) = \phi(\alpha).$$

*Proof.* Equations (18), (19) and the definition of Zagier's  $g$ -function of Equation (58) imply that for  $\alpha \in \mathbb{Q} - \{0\}$  we have:

$$(53) \quad g(\alpha) = \begin{cases} S^{\text{mul}}(-1/(2\pi i\alpha)) & \text{if } \alpha > 0 \\ S^{\text{mur}}(-1/(2\pi i\alpha)) & \text{if } \alpha < 0. \end{cases}$$

Let us assume  $\alpha \in \mathbb{Q}$ ,  $\alpha > 0$  (the other case is analogous). We have:

$$\begin{aligned} S^{\text{med}}\left(-\frac{1}{2\pi i\alpha}\right) &= S^{\text{mul}}\left(-\frac{1}{2\pi i\alpha}\right) + \delta\left(-\frac{1}{2\pi i\alpha}\right) && \text{by (46)} \\ &= S^{\text{mul}}\left(-\frac{1}{2\pi i\alpha}\right) + \zeta_{24}^3 \alpha^{-3/2} \phi\left(-\frac{1}{\alpha}\right) && \text{by Prop. 5.1 (c)} \\ &= g(\alpha) - (i\alpha)^{-3/2} \phi(-1/\alpha) && \text{by (53)} \\ &= \phi(\alpha) && \text{by (61)} \end{aligned}$$

$\square$

## 6. IDENTITIES FROM ZAGIER'S PAPER

In this section we collect several definitions, notations and results from Zagier's paper [Za], for the convenience of the reader. Zagier defines a function

$$(54) \quad \phi : \mathbb{Q} \longrightarrow \mathbb{C}, \quad \phi(\alpha) = e^{\pi i \alpha / 12} f(e^{2\pi i \alpha})$$

which evaluates at complex roots of unity the series  $f(q)$  of (9). Zagier considers the following formal power series in  $\mathbb{Z}[[q]]$ :

$$\begin{aligned} (q)_\infty &= \prod_{n=1}^{\infty} (1 - q^n) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= \sum_{n=1}^{\infty} \chi(n) q^{(n^2-1)/24} \\ H(q) &= \sum_{n=1}^{\infty} \chi(n) n q^{(n^2-1)/24} \end{aligned}$$

as well as the corresponding analytic functions for  $q = e^{2\pi i z}$ ,  $\Im(z) > 0$ :

$$(55) \quad \eta(z) = e^{\pi i z / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) = \sum_{n=1}^{\infty} \chi(n) e^{\pi i n^2 z / 12}$$

$$(56) \quad \tilde{\eta}(z) = \sum_{n=1}^{\infty} \chi(n) n e^{\pi i n^2 z / 12}$$

$\eta(z)$  is the famous Dedekind  $\eta$  function, a modular form of weight  $1/2$ , and  $\tilde{\eta}(z)$  is an *Eichler integral* of the Dedekind  $\eta$  function. Although  $\tilde{\eta}$  is not a modular form, Zagier proves that  $\tilde{\eta}$  has radial limits to the rational points  $z \in \mathbb{Q} \subset \mathbb{R}$  of its natural boundary.

Zagier's identity (coined "the strange identity" by Zagier himself) [Za, Eqn.7] identifies the radial limits of  $\tilde{\eta}$  with  $\phi$  for  $\alpha \in \mathbb{Q}$ :

$$(57) \quad \phi(\alpha) = -\frac{1}{2} \tilde{\eta}(\alpha).$$

At the last two pages of his seminal paper, Zagier introduces a  $C^\infty$  function

$$(58) \quad g : \mathbb{R} \longrightarrow \mathbb{C}, \quad g(x) = \int_0^\infty (z-x)^{-3/2} \eta(z) dz,$$

where  $\eta(z)$  is the *Dedekind*  $\eta$  function defined by (55). Zagier states that  $g(x)$  is real analytic everywhere except at  $x = 0$  and whose derivatives at 0 are given by

$$g^{(n)}(0) = (-\pi i / 12)^n n! a_n,$$

where

$$(59) \quad F(x) = e^{-1/(24x)} f(e^{-1/x}) = \sum_{n=0}^{\infty} \frac{a_n}{24^n} \frac{1}{x^n}.$$

Moreover, for  $\alpha \in \mathbb{Q}$ , we have:

$$(60) \quad g(\alpha) = (i\alpha)^{-3/2} g(-1/\alpha)$$

$$(61) \quad \phi(\alpha) + (i\alpha)^{-3/2} \phi(-1/\alpha) = g(\alpha) \text{ for } \alpha \in \mathbb{Q}$$

In other words, for  $h \rightarrow 0$  we have:

$$\begin{aligned}
 g(h) &\sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} h^n \\
 &= \sum_{n=0}^{\infty} \left(-\frac{\pi i}{12}\right)^n a_n h^n \\
 (62) \qquad &= e^{2\pi i h/24} f(e^{2\pi i h})
 \end{aligned}$$

where the last equality follows from Equation (27). In [Za, Eqn.6] Zagier gives a closed formula for the Taylor coefficients  $(a_n/24^n)$  of  $F(x)$ :

$$\begin{aligned}
 \frac{a_n}{24^n} &= 6 \frac{(-6)^n}{(n+1)!} \left( B_{2n+2} \left( \frac{1}{12} \right) - B_{2n+2} \left( \frac{5}{12} \right) \right) \\
 &= \frac{1}{2\sqrt{3}(\pi/6)^2 (2\pi^2/3)^n} \frac{(2n+1)!}{n!} L(2n+2, \chi)
 \end{aligned}$$

where

$$(63) \qquad L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Consider now the Borel transform

$$G(p) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{24^{n+1} n!} p^n$$

of  $F(x)$ . To simplify notation, let us write

$$(64) \qquad G(p) = \sum_{n=0}^{\infty} b_n p^n$$

instead. Then, we have:

$$(65) \qquad b_n = 6 \frac{(-6)^{n+1}}{(n+2)! n!} \left( B_{2n+4} \left( \frac{1}{12} \right) - B_{2n+4} \left( \frac{5}{12} \right) \right)$$

$$(66) \qquad = \frac{4\pi^2}{\sqrt{3}(2\pi^2/3)^n} \frac{(2n+3)!}{(n+1)! n!} L(2n+4, \chi)$$

Since

$$\frac{(2n+3)!}{(n+1)! n!} \sim 4^n n^{3/2} \left( \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \dots \right)$$

for computable constants  $\gamma_j$ , and since  $L(2n+4) = 1 + O(5^{-2n})$  for every  $M$ , it follows that the coefficients of the Borel transform have an asymptotic expansion of the form:

$$(67) \qquad b_n \sim \left( \frac{6}{\pi^2} \right)^n n^{3/2} \left( c_{1,0} + \frac{c_{1,1}}{n} + \frac{c_{1,2}}{n^2} + \dots \right)$$

for computable constants  $c_{1,l}$  for  $l \in \mathbb{N}^+$ . Disassembling the L-series into its monomial parts, Equations (66) reveals a *transseries expression* for the coefficients of the Borel transform:

$$(68) \qquad b_n \sim \left( \frac{6}{\pi^2} \right)^n n^{3/2} \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \frac{c_{k,l}}{n^l k^n}$$

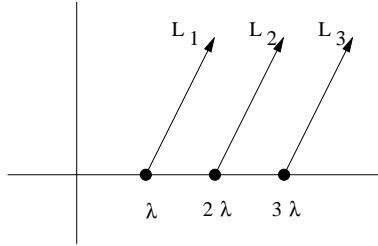
for a doubly indexed series of resurgence monomials  $(6/\pi^2)^n n^l k^n$ , and for computable constants  $c_{k,l}$ . Notice that the resurgence monomials form a well-order set of order type  $\omega^2$ .

## 7. RESURGENCE IMPLIES TRANSSERIES EXPANSIONS

Let us examine more carefully the asymptotic equations from the last section. Although Equation (66) makes sense, the asymptotic series in (67) is factorially divergent. In view of this, one cannot naively make sense of Equation (68) since for example  $1/2^n$  is a monomial which is (exponentially) smaller than any of the monomials  $1/n^l$  for all  $l$ . In order to reach the monomial  $1/2^n$  we would have to subtract the infinite series of all previous monomials  $1/n^l$  for  $l \in \omega$ , and this series is factorially divergent. What we need is a way to subtract the whole series at once. It is at this point that resurgence is needed to make sense of the formal series in (16).

Recall that the singularities of  $G(p)$  are included in the set  $\lambda\mathbb{N}^+$  where  $\lambda = \pi^2/6$ .

Fix a small positive angle  $\theta$  and for every  $k \in \mathbb{N}^+$  draw the rays  $L_k = k\lambda\epsilon^{i\theta}\mathbb{R}^+$  from  $k\lambda$  to infinity along the direction of  $\theta$ . Assume that  $\theta$  is chosen so that the rays  $L_k$  are distinct:



The next proposition is a special case of a general result that will appear in subsequent work of the authors.

**Theorem 8.** (a) For every  $k \in \mathbb{N}^+$ , there exist analytic integrable functions  $R_k \in L^1[0, \infty)$  such that:

$$(69) \quad b_n = \lambda^{-n} n^{3/2} \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^{\infty} e^{-np} R_k(e^p) dp$$

(b) Moreover, for every  $k \in \mathbb{N}^+$ , we have an asymptotic expansion

$$(70) \quad \int_0^{\infty} e^{-np} R_k(p) dp \sim \sum_{l=0}^{\infty} \frac{c_{k,l}}{n^l} p^l.$$

(c) Thus,  $G(p)$  determines the transseries (68). Conversely,  $G(p)$  is uniquely determined by its transseries.

The functions  $R_k$  are constructed from the jump (i.e., variation) of the multivalued function  $G(p)$  at the rays  $L_k$ .

*Proof.* The proof is a well-known application of Cauchy's formula and a deformation of the contour; see for example [Ju]. For the benefit of the reader, we give the details. For a technical integrability reason we will work with the following variation  $g(p)$  of  $G(p)$ :

$$(71) \quad g(p) = \sum_{n=1}^{\infty} \frac{b_n}{n^2} p^n$$

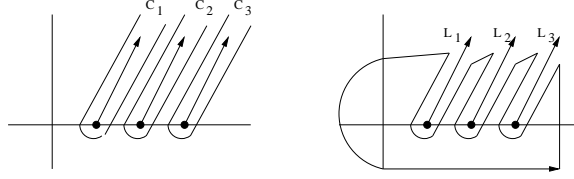
which of course satisfies

$$\left(p \frac{d}{dp}\right)^2 g(p) = G(p) - b_0.$$

Of course  $g$  and  $G$  have the same singularities. Since  $g(p)$  is analytic in a neighborhood of zero, Cauchy's formula implies that

$$\frac{b_n}{n^2} = \frac{1}{2\pi i} \int_{\gamma} \frac{g(p)}{p^n} \frac{dp}{p}.$$

Now, we will deform the contour  $\gamma$  in the following way. Choose (Hankel) contours  $C_k$  around each ray  $L_k$ , and choose a truncation  $C_k^r$  of them for  $r$  large. Join  $\cup_{k=1}^r C_k^r$  together as shown in Figure 1 for  $r = 3$ , and create a contour  $\gamma_r$ . For every  $r$ , there is a deformation of  $\gamma$  to  $\gamma_r$  which does not pass through the singularities of



**Figure 1.** On the left, Hankel contours  $C_k$  around each ray  $L_k$ , oriented counterclockwise. On the right, a truncated contour.

$g(p)$ . It follows that

$$\frac{b_n}{n^2} = \frac{1}{2\pi i} \sum_{k=1}^r \int_{C_k^r} \frac{g(p)}{p^n} \frac{dp}{p} + \frac{1}{2\pi i} \int_{\Gamma_r} \frac{g(p)}{p^n} \frac{dp}{p}$$

where  $\Gamma_r = \gamma_r - \cup_{k=1}^r C_k^r$  is the part of the contour  $\gamma_r$  that is not included in the truncated Hankel contours. Now, let  $r \rightarrow \infty$ . An estimate shows that

$$\lim_{r \rightarrow \infty} \int_{\Gamma_r} \frac{g(p)}{p^n} \frac{dp}{p} = 0.$$

Let  $H_k$  denote a Hankel contour around the ray  $L_k$ . For every  $k \in \mathbb{N}^+$ , we have

$$(72) \quad \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C_k^r} \frac{g(p)}{p^n} \frac{dp}{p} = \int_{H_k} \frac{g(p)}{p^n} \frac{dp}{p}.$$

Recall that  $g(p)$  is analytic in  $\mathbb{C} \setminus \cup_{k \in \mathbb{N}^+} L_k$ . For  $p \in L_k$ , we define the jump (i.e., the variation)  $g_k(p)$  of  $g(p)$  by

$$(73) \quad g_k(x) = \lim_{\epsilon \rightarrow 0^+} g(p + i\epsilon) - g(p - i\epsilon).$$

On the other hand, Theorem 3 implies that around  $p = k\lambda$ ,  $g$  has an expansion of the form

$$g(p) = \frac{S_k(p - k\lambda)}{(p - k\lambda)^{1/2}}$$

where  $S_k$  is analytic and integrable in  $[0, \infty)$ . It follows that for  $p \in L_k$  we have

$$g_k(p) = 2 \frac{S_k(p - k\lambda)}{(p - k\lambda)^{1/2}}$$

Thus, for  $t \in \mathbb{R}^+$  we can write

$$g_k(k\lambda e^t) = \frac{T_k(t)}{t^{1/2}}$$

where  $T_k(t)$  is analytic and integrable at  $[0, \infty)$ . A change of variables  $p = k\lambda e^{t+i\theta}$  in Equation (72) gives

$$\int_{H_k} \frac{g(p)}{p^n} \frac{dp}{p} = (k\lambda)^{-n} \int_0^\infty e^{-nt} \frac{T_k(t)}{t^{1/2}} dt.$$

Since

$$\int_0^\infty e^{-nt} t^c dt = \frac{\Gamma(c+1)}{n^{1+c}}$$

for all  $c \in \mathbb{C}$  with  $\Re(c) \geq -1/2$ , it follows that we can write

$$\int_0^\infty e^{-nt} \frac{T_k(t)}{t^{1/2}} dt = n^{-1/2} \int_0^\infty e^{-nt} R_k(t) dt$$

for  $R_k$  analytic and integrable at  $[0, \infty)$ . This proves part (a).

Part (b) follows from Watson's lemma; see [O].

Part (c) also follows from Watson's lemma.  $\square$

*Remark 7.1.* As is obvious from the statement and the proof, Theorem 8 holds for a wide class of resurgent functions  $G(p)$ , that includes all square root branched functions with singularities in a finite set of rays  $\lambda_1 \mathbb{N}^+ \cup \dots \cup \lambda_r \mathbb{N}^+$ .

Among other things, the above theorem makes clear the usefulness (and the necessity) of transseries versus asymptotic expansions. The asymptotic expansion (67) determines  $G(p)$  modulo *exponentially small corrections*. These corrections, beyond all orders in  $1/n$ , are precisely captured by the transseries. Theorem 8 gives a synthesis of  $G(p)$  by its transseries. In addition, Theorem 8 gives a proof of Theorem 1.

*Proof.* (of Theorem 1) Part (a) is a general statement about Laplace transforms, and follows from Watson's lemma.

Part (b) follows from Theorem 8 above, and from Equation (5).

For part (c),  $F_\mathcal{K}(x)$  determines (via object synthesis), the analytic function  $S_\mathcal{K}^{\text{med}}(x)$ , and its radial limits via (5). Conversely, the sequence  $(\phi_\mathcal{K}(e^{2\pi i/n}))$  determines its transseries, which in turn determines (via Theorem 8) the function  $S_\mathcal{K}^{\text{med}}(x)$ , which finally determines  $F_\mathcal{K}(x)$  by (8). This completes the proof of Theorem 1.  $\square$

#### APPENDIX A. RESURGENCE OF THE POWER SERIES OF THE POINCARÉ HOMOLOGY SPHERE

In this section, let  $M$  denote the *Poincaré homology sphere*, a closed 3-manifold. In [LZ], Lawrence-Zagier compute that

$$(74) \quad F_M(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \frac{1}{(120x)^n}$$

where

$$(75) \quad \sum_{n=0}^{\infty} \frac{a_n}{(2n)!} p^{2n} = \frac{\cos 5p \cos 9p}{\cos 15p}.$$

A computation analogous to the one in Section 3 shows that the Borel transform  $G_M(p)$  of  $F_M(x)$  is given by:

$$(76) \quad G_M(p) = c_1 \sum_{n=0}^{\infty} \frac{\chi_1(n)}{(-30p + n^2\pi^2)^{3/2}} + c_2 \sum_{n=0}^{\infty} \frac{\chi_2(n)}{(-30p + n^2\pi^2)^{3/2}}$$

where

$$(77) \quad c_1 = \frac{\sqrt{6(5 + \sqrt{5})}}{120}, \quad c_2 = \frac{\sqrt{6(5 - \sqrt{5})}}{120},$$

and  $\chi_1, \chi_2$  are periodic functions defined by the table:

$n \bmod 60$	7	13	17	23	37	43	47	53	other
$\chi_1(n)$	-1	-1	-1	-1	1	1	1	1	0

and

$n \bmod 60$	1	11	19	29	31	41	49	59	other
$\chi_2(n)$	-1	-1	-1	-1	1	1	1	1	0

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