QUANTUM DILOGARITHMS OVER LOCAL FIELDS AND INVARIANTS OF 3-MANIFOLDS

STAVROS GAROUFALIDIS AND RINAT KASHAEV

ABSTRACT. To each local field (including the real or complex numbers) we associate a quantum dilogarithm and show that it satisfies a pentagon identity and some symmetries. Using an angled version of these quantum dilogarithms, we construct three generalized TQFTs in 2+1 dimensions, one given by a face state-integral and two given by edge state-integrals. Their partition functions rise to distributional invariants of 3-manifolds with torus boundary, conjecturally related to point counting of the A-polynomial curve. The partition function of one of these face generalized TQFTs for the case of the real numbers can be expressed either as a multidimensional Barnes-Mellin integral or as a period on a curve which is conjecturally the A-polynomial curve.

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1. Introduction

To each local field F (including the real and the complex numbers), we associate a quantum dilogarithm that satisfies a pentagon identities and some symmetries. Using an angled version of these quantum dilogarithms, we construct

(a) a face-type generalized TQFT whose states are elements of $\widehat{F^\times} \times F^\times$ associated to the faces of a triangulation, and
(b) a pair of edge-type generalized TQFTs whose states (in $F^\times$ and $\widehat{F^\times}$, respectively) are associated to the edges of the triangulation.

In all three cases, the partition function is a state-integral which is a distribution on the space of peripheral data.

These face and edge state-integrals are computable in terms of an ideal triangulation of a 3-manifold with torus boundary and are conjecturally expressed generically in terms of an F-point counting on the A-polynomial curve. This point counting is reminiscent to motivic integration developed by Igusa, Kontsevich, Denef, Loeser and others [18, 5, 7, 6], as well as to the counting of $\mathrm{PSL}_2(\mathbb{F}_p)$-representations and to the Bloch group of a finite field $\mathbb{F}_p$ developed by Karuo and Ohtsuki [20, 19, 28].

Although the above partition functions do not explicitly depend on a Planck's constant, the size of the residue field of F plays the role of Planck's constant as is evident from the point-count computations.

In the case of the real numbers, the $\widehat{\mathbb{R}^\times}$ edge state-integrals can be computed in two different ways: the first is expressed by Mellin–Barnes integrals of products of the beta function and of the cosine function, and the second is given by period integrals over a complex curve (the A-polynomial curve). The equality of the two computations follows from a Fourier transform formula and illustrates why some periods can be expressed in terms of Mellin–Barnes integrals, analogous to what has been observed in mirror symmetry of Calabi–Yau manifolds by Passare–Tsikh–Cheshel [29]. This dual presentation of the partition function fits well with another instance of face state-integrals, namely the meromorphic 3D-index of the authors [12] whose asymptotics were on the one hand expressed by beta integrals observed by Hodgson–Kricker–Siejakowski [17], and periods observed by Wheeler and the first author [13]. Our $\widehat{\mathbb{R}^\times}$ edge state-integral proves the equality of the two, and in particular of [13, Eqn.(18)]. As an example, for the case of the 4_1, we obtain an identity

$$\frac{1}{2\pi i}\int_{\epsilon-i\mathbb{R}} \frac{\mathrm{B}(z,z)^2}{\cos(\pi z)^2}\,\mathrm{d}z = 2\int_{-\infty}^{1} \frac{dx}{\sqrt{(1-x)(1-x+4x^2)}} = 5.60241216\ldots \tag{1}$$

between a beta-integral and a period of an elliptic curve defined over $\mathbb{Q}$.

As usual, all three generalized TQFTs give rise to representations of the Ptolemy groupoid, and in particular, unitary representations of the mapping class groups of punctured surfaces. A detailed discussion of such representations in the context of the Teichmüller TQFT is given in the thesis of Piguet [30, Sec.3] and in references therein, following the work of Andersen and Kashaev.

We end this introduction by mentioning the prior work on face state-integrals [2] and on edge state-integrals [24, 12]. In a certain sense, our paper is a continuation of the previous work, with some interesting new twists, even for the case of the real numbers.

Part I. Local fields

2. Preliminaries

2.1. Quantum dilogarithms over Gaussian groups.

The original motivation for introducing quantum dilogarithms over locally compact abelian groups (in short, LCA groups) equipped with a Gaussian exponential was to define invariants of 3-manifolds using state-integrals which are absolutely convergent, have universal contours of integration (namely copies of the LCA group in question) and lead to topological invariants. The latter can be interpreted as partition functions of complex Chern–Simons theory, or of quantum hyperbolic geometry. The general theory was introduced originally in the joint work of Andersen and the second author [1] and then gradually extended and refined in [21] and [12, App.B].

Abstracting from this, we combine an LCA group and a Gaussian exponential into the notion of a Gaussian group. Let $\mathbb{T}$ be the multiplicative group of complex numbers of absolute value 1.

Definition 2.1. A *Gaussian group* is an LCA group A equipped with a nondegenerate $\mathbb{T}$-valued quadratic form $\langle\cdot\rangle\colon \mathsf{A}\to\mathbb{T}$, i.e., a function that satisfies $\langle x\rangle = \langle -x\rangle$ for any $x\in\mathsf{A}$ and its polarization

$$\langle x;y\rangle := \frac{\langle x+y\rangle}{\langle x\rangle\langle y\rangle}, \quad \forall (x,y)\in\mathsf{A}^2,$$

is a non-degenerate bi-character. The form $\langle\cdot\rangle$ is called a *Gaussian exponential* of A, and the bi-character $\langle\cdot;\cdot\rangle$ is called its *Fourier kernel*.

Note that every Gaussian group is necessarily Pontryagin self-dual, a consequence of the non-degeneracy of its Fourier kernel. Note that the Gaussian exponential and its Fourier kernel satisfy $\langle u\rangle^2 = \langle u; u\rangle$ for all $u \in \mathsf{A}$, which implies that the Fourier kernel determines the square of the Gaussian exponential as is standard in the theory of quadratic forms.

Note also that a Gaussian group A has a canonically normalized Haar measure determined by the condition of the improper integral

$$\int_{\mathsf{A}^2} \langle x; y\rangle \, \mathrm{d}(x, y) = 1 \,. \tag{2}$$

Definition 2.2. A *quantum dilogarithm* over a Gaussian group A is a tempered distribution over A represented by an almost everywhere defined locally integrable function $\varphi \colon \mathsf{A} \to \mathbb{C}$ that satisfies

(1) an *inversion relation*: there exists a non-zero constant $c_\varphi \in \mathbb{C}^\times$ such that

$$\varphi(x)\varphi(-x) = c_\varphi \langle x\rangle \tag{3}$$

for almost all $x \in \mathsf{A}$;

(2) a *pentagon identity*

$$\varphi(x)\varphi(y) = \gamma_{\mathsf{A}} \int_{\mathsf{A}^3} \frac{\langle x - u; y - w\rangle}{\langle u - v + w\rangle} \varphi(u)\varphi(v)\varphi(w) \, \mathrm{d}(u, v, w) \tag{4}$$

for almost all pairs $(x, y) \in \mathsf{A}^2$, where $\gamma_{\mathsf{A}} := \int_{\mathsf{A}} \langle x\rangle \, \mathrm{d}x$ with the integrals defined improperly.

Three examples of quantum dilogarithms for the Gaussian groups $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$ and $\mathbb{Z} \times \mathbb{T}$ have already appeared in the literature, and the corresponding invariants are the Teichmüller TQFT, the Kashaev–Luo–Vartanov invariant and the meromorphic 3D-index, respectively. We briefly describe these examples below.

Example 2.3. The field $\mathbb{R}$ of the real numbers is a Gaussian group with Gaussian exponential $\langle\cdot\rangle : \mathbb{R} \to \mathbb{T}$ given by $\langle x\rangle := e^{\pi i x^2}$ and Fourier kernel $\langle x; y\rangle := e^{2\pi i x y}$ is the usual kernel of the Fourier transform. The Faddeev quantum dilogarithm $\Phi_{\mathsf{b}}(x)$ of [9] for a fixed complex parameter b with non-zero real part is a quantum dilogarithm which was used in [2] to construct a generalized TQFT of face state-integral type. This example is the origin of Definition 2.2, and it shows, in particular, that a given Gaussian group can have many quantum dilogarithms.

Example 2.4. Following [24], $\mathbb{R} \times \mathbb{R}$ is a Gaussian group with the Gaussian exponential

$$\langle x\rangle = e^{2\pi i \dot{x}\ddot{x}}, \qquad x = (\dot{x}, \ddot{x}) \tag{5}$$

and the associated Fourier kernel

$$\langle x; y\rangle = e^{2\pi i(\dot{x}\ddot{y} + \dot{y}\ddot{x})}. \tag{6}$$

A quantum dilogarithm over this Gaussian group is given by

$$\phi(x) = \frac{\Phi_{\mathsf{b}}(\dot{x} + \frac{\ddot{x}}{2})}{\Phi_{\mathsf{b}}(\dot{x} - \frac{\ddot{x}}{2})}, \qquad x = (\dot{x}, \ddot{x}),\ y = (\dot{y}, \ddot{y}) \,. \tag{7}$$

Example 2.5. Following [12, App.A], $\mathbb{T}\times\mathbb{Z}$ is a Gaussian group with the Gaussian exponential given by

$$\langle\cdot\rangle\colon \mathbb{T}\times\mathbb{Z}\to\mathbb{T},\quad \langle z,m\rangle := z^m,\qquad (z,m)\in\mathbb{T}\times\mathbb{Z}, \tag{8}$$

and the Fourier kernel

$$\langle z,m;w,n\rangle := \frac{\langle zw,m+n\rangle}{\langle z,m\rangle\langle w,n\rangle} = z^n w^m\,. \tag{9}$$

A quantum dilogarith φ_q is given by

$$\varphi_q\colon \mathbb{T}\times\mathbb{Z}\to\mathbb{T},\qquad \varphi_q(z,m) = \frac{(-zq^{1-m};q^2)_\infty}{(-z^{-1}q^{1-m};q^2)_\infty} \tag{10}$$

for $|q|<1$, where $(x;q)_\infty = \prod_{k=0}^\infty(1-q^kx)$ is the infinite q-Pochhammer symbol.

Remark 2.6. The pentagon identity (4) should be interpreted as an integral identity of tempered distributions. As such, it can equivalently be written in the form of a distributional integral identity

$$\tilde\varphi(x)\tilde\varphi(y)\langle x;y\rangle = \int_{\mathsf{A}}\tilde\varphi(x-z)\tilde\varphi(z)\tilde\varphi(y-z)\langle z\rangle\,\mathrm{d}z \tag{11}$$

for the (inverse) Fourier transformation

$$\tilde\varphi(x) := \int_{\mathsf{A}}\frac{\varphi(y)}{\langle x;y\rangle}\,\mathrm{d}y\,. \tag{12}$$

Indeed, by using the distributional equality with Dirac's delta-function

$$\int_{\mathsf{A}}\langle x;y\rangle\,\mathrm{d}y = \delta_{\mathsf{A}}(x), \tag{13}$$

we have

$$\begin{aligned}
\tilde\varphi(x)\tilde\varphi(y)\frac{\langle x;y\rangle}{\gamma_{\mathsf{A}}} &= \int_{\mathsf{A}^2}\frac{\varphi(s)\varphi(t)\,\mathrm{d}(s,t)}{\gamma_{\mathsf{A}}\langle x;s-y\rangle\langle y;t\rangle} = \int_{\mathsf{A}^5}\frac{\langle s-u;t-w\rangle\varphi(u)\varphi(v)\varphi(w)}{\langle x;s-y\rangle\langle y;t\rangle\langle u-v+w\rangle}\,\mathrm{d}(u,v,w,s,t)\\
&= \int_{\mathsf{A}^4}\delta_{\mathsf{A}}(s-u-y)\frac{\langle s-u;-w\rangle\varphi(u)\varphi(v)\varphi(w)}{\langle x;s-y\rangle\langle u-v+w\rangle}\,\mathrm{d}(u,v,w,s)\\
&= \int_{\mathsf{A}^3}\frac{\langle y;-w\rangle\varphi(u)\varphi(v)\varphi(w)}{\langle x;u\rangle\langle u-v+w\rangle}\,\mathrm{d}(u,v,w) = \int_{\mathsf{A}^3}\frac{\varphi(u)\varphi(v)\varphi(w)}{\langle x;u\rangle\langle y;w\rangle\langle u-v+w\rangle}\,\mathrm{d}(u,v,w)\\
&= \int_{\mathsf{A}^4}\frac{\langle v;z\rangle\varphi(u)\tilde\varphi(z)\varphi(w)}{\langle x;u\rangle\langle y;w\rangle\langle u-v+w\rangle}\,\mathrm{d}(u,v,w,z)\\
&= \int_{\mathsf{A}^4}\frac{\langle v'+u+w;z\rangle\varphi(u)\tilde\varphi(z)\varphi(w)}{\langle x;u\rangle\langle y;w\rangle\langle v'\rangle}\,\mathrm{d}(u,v',w,z)\\
&= \int_{\mathsf{A}^3}\frac{\varphi(u)\tilde\varphi(z)\varphi(w)\langle z\rangle}{\gamma_{\mathsf{A}}\langle x-z;u\rangle\langle y-z;w\rangle}\,\mathrm{d}(u,w,z) = \frac{1}{\gamma_{\mathsf{A}}}\int_{\mathsf{A}}\tilde\varphi(x-z)\tilde\varphi(z)\tilde\varphi(y-z)\langle z\rangle\,\mathrm{d}(z)\,.
\end{aligned}$$

2.2. **Distributions and delta functions over local fields.** In this section we recall some basic facts about test functions and distributions on local fields F, the latter being examples of locally compact abelian groups. A detailed discussion of these facts may be found for instance in [8, 11, 25].

Let $\mathcal{S}(F)$ denote the Schwartz–Bruhat space $\mathcal{S}(F)$ of Fourier stable complex-valued test functions. In the case of a non-Archimedean local field, the test functions are locally constant compactly supported functions on F, taking finitely many values. Let $\mathcal{S}'(F)$ denote the dual space of tempered distributions. The Fourier transform is an automorphism of the space $\mathcal{S}'(F)$.

We fix a (translationally invariant) Haar measure μ_F on the additive group $(F,+)$ of a local field F with the notation for the differential $\mathrm{d}_F\, x := \mathrm{d}\,\mu_F(x)$. The multiplicative group $\mathsf{B} := (F^\times, \times)$ is also a locally compact abelian group. We fix its Haar measure through the following relation for the differentials

$$\mathrm{d}_{\mathsf{B}}\, x = \frac{\mathrm{d}_F\, x}{\|x\|}, \quad \|x\| := \|x\|_F^d \tag{14}$$

where d is the dimension of the field. For example, we have $d = 1$ for $F = \mathbb{R}$ or $\mathbb{Q}_p$ and $d = 2$ for $F = \mathbb{C}$. One can also define $\|x\|$ through the formula $\mu_F(xA) = \|x\|\mu_F(A)$.

The Dirac delta function $\delta_{\mathsf{B}}(x)$ is a tempered distribution defined by the improper distributional integral

$$\delta_{\mathsf{B}}(x) = \int_{\hat{\mathsf{B}}} \alpha(x)\,\mathrm{d}\alpha \tag{15}$$

where we choose the normalization of the Haar measure on $\hat{\mathsf{B}}$ by the condition (2). In other words, we have

$$\int_{\mathsf{A}^2} \alpha(x)\beta(y)\,\mathrm{d}((\alpha,x),(\beta,y)) = 1 \quad \Leftrightarrow \quad \int_{\mathsf{A}} \alpha(x)\,\mathrm{d}(\alpha,x) = 1\,. \tag{16}$$

We remark that $\delta_{\mathsf{B}}(x) = 0$ unless $x = 1$. The delta functions δ_{B} and δ_F on B and F are related by

$$\delta_{\mathsf{B}}(x) = \delta_F(x-1) \quad \forall x \in \mathsf{B}\,. \tag{17}$$

We begin with a basic lemma on δ-functions.

Lemma 2.7. (a) Let $f\colon \Omega \to F$ be a differentiable function on an open subset $\Omega \subset F$ of F, where the set of zeros $f^{-1}(0)$ is discrete and each zero $a \in f^{-1}(0)$ is simple in the sense that $f'(a) \neq 0$. Then

$$\delta_F(f(x)) = \sum_{a\,:\,f(a)=0} \frac{1}{\|f'(a)\|}\delta_F(x-a)\,, \tag{18}$$

where the derivative is defined in the standard way by using the convergence with respect to the norm.

(b) It follows that

$$\int_{\mathsf{B}} \delta_{\mathsf{B}}\Big(\frac{1+x}{y}\Big)\,\mathrm{d}x = \frac{1}{\|1-y^{-1}\|}\,. \tag{19}$$

Proof. Part (a) is a special case of a more general formula (21) below which can be proven on the basis of the change of variables formula for the integral, see, for example, [18].

For part (b), we use (17) and (14) and part (a) (with the function $f(x) = \frac{1+x}{y} - 1$ with the only zero $a = y - 1$) and compute

$$\int_{\mathsf{B}} \delta_{\mathsf{B}}\Big(\frac{1+x}{y}\Big)\, \mathrm{d}_{\mathsf{B}}\, x = \int_F \delta_F\Big(\frac{1+x}{y} - 1\Big)\frac{\mathrm{d}_F x}{\|x\|} = \int_F \|y\|\delta_F(1+x-y)\frac{\mathrm{d}_F x}{\|x\|} = \frac{\|y\|}{\|y-1\|}\,.$$

□

Below, we will need a multivariate generalization of Equation (18) which is a local field analogue of Grothendieck's residue theorem, and follows from [18, Prop.7.4.1].

Lemma 2.8. Suppose that $f = (f_1, \dots, f_r)$ with $f_i \in F(x)$ with $x = (x_1, \dots, x_r)$ defines a reduced 0-dimensional scheme S_f with $F^\times$ points

$$S_f(F^\times) = \{a \in (F^\times)^r \ | f_i(a) = 0,\ i = 1, \dots r\} \tag{20}$$

which is nondegenerate, i.e., $\mathrm{Jac}(f(a)) := \det(\partial_{z_j} f_i(a)) \neq 0$ for all $a \in S_f$. Then,

$$\prod_{i=1}^r \delta_F(f_i(x)) = \sum_{a \in S_f(F^\times)} \frac{1}{\|\mathrm{Jac}(f(a))\|} \prod_{i=1}^r \delta_F(x_i - a_i)\,. \tag{21}$$

3. A quantum dilogarithm over a local field

3.1. **A Gaussian group associated to a local field.** As in the previous section, let F denote a local field. In characteristic zero, this means that F can be $\mathbb{R}$, $\mathbb{C}$ or a finite extension of the p-adic numbers $\mathbb{Q}_p$. Note that points (singletons) in F have measure zero. The additive group $(F, +)$, the multiplicative group $\mathsf{B} = F^\times$ and its Pontryagin dual $\hat{\mathsf{B}}$ have Haar measures, discussed in detail for instance in [8]. We fix their normalizations as is discussed in Subsection 2.2.

We will denote elements of B by $x, y, \dots$ and elements of $\hat{\mathsf{B}}$ by $\alpha, \beta, \dots$. We will denote the canonical pairing $\hat{\mathsf{B}} \times \mathsf{B} \to \mathbb{T}$ by $(\alpha, x) \mapsto \alpha(x)$. The LCA groups B and $\hat{\mathsf{B}}$ can be combined to define a self-dual LCA group $\mathsf{A} = \hat{\mathsf{B}} \times \mathsf{B}$ which is a Gaussian group with Gaussian exponential

$$\langle \cdot \rangle : \mathsf{A} \to \mathbb{T}, \qquad \langle (\alpha, x) \rangle = \alpha(x) \tag{22}$$

and the associated Fourier kernel

$$\langle \cdot\, ; \cdot \rangle : \mathsf{A}^2 \to \mathbb{T}, \qquad \langle (\alpha, x); (\beta, y) \rangle = \alpha(y)\beta(x)\,. \tag{23}$$

We now define an elementary yet important function on A which will play a key role in this paper, and which by abuse of language we will call a quantum dilogarithm over the local field F.

Definition 3.1. We define

$$\varphi : \hat{\mathsf{B}} \times (\mathsf{B} \setminus \{-1\}) \subset \mathsf{A} \to \mathbb{T}, \qquad \varphi(\alpha, x) = \alpha(1 + x). \tag{24}$$

The value $\varphi(\alpha,-1)$ is undefined, and, in case of need, one can assign any finite value to it [1] because the subset $\hat{\mathsf{B}}\times\{-1\}\subset\mathsf{A}$ has measure zero.

The function φ is clearly a tempered distribution on A.

3.2. **The Fourier transform of φ.** In this section we compute the (inverse) Fourier transform $\tilde{\varphi}$ of φ

Lemma 3.2. The inverse Fourier transform of φ is given by

$$\tilde{\varphi}(\beta,y):=\int_{\mathsf{A}}\frac{\varphi(\alpha,x)}{\langle(\alpha,x);(\beta,y)\rangle}\,\mathrm{d}(\alpha,x)=\frac{1}{\beta(y-1)}\frac{1}{\|1-y^{-1}\|},\qquad(y\neq 1)\,. \tag{25}$$

Proof. We have:

$$\begin{aligned}\tilde{\varphi}(\beta,y)&=\int_{\mathsf{A}}\frac{\varphi(\alpha,x)}{\langle(\alpha,x);(\beta,y)\rangle}\,\mathrm{d}(\alpha,x)=\int_{\mathsf{A}}\frac{\alpha(1+x)}{\alpha(y)\beta(x)}\,\mathrm{d}(\alpha,x)\\&=\int_{\hat{\mathsf{B}}\times\mathsf{B}}\alpha\Big(\frac{1+x}{y}\Big)\beta\Big(\frac{1}{x}\Big)\,\mathrm{d}(\alpha,x)=\int_{\mathsf{B}}\delta_{\mathsf{B}}\Big(\frac{1+x}{y}\Big)\langle(-\beta,x)\rangle\,\mathrm{d}x\\&=\int_{\mathsf{B}}\delta_{\mathsf{B}}\Big(\frac{1+x}{y}\Big)\langle(-\beta,y-1)\rangle\,\mathrm{d}x=\frac{1}{\beta(y-1)}\int_{\mathsf{B}}\delta_{\mathsf{B}}\Big(\frac{1+x}{y}\Big)\,\mathrm{d}x\,.\end{aligned} \tag{26}$$

Part (b) of Lemma 2.7 concludes the proof. ∎

Recall that φ is a tempered distribution on A and so is $\tilde{\varphi}$. The latter is represented by the locally integrable function given in the right hand side of Equation (25) only when evaluated at a test function on A whose support does not contain 1.

3.3. **A pentagon identity for φ.** In this section we give a distributional pentagon identity for φ.

Theorem 3.3. *The function (24) is a quantum dilogarithm i.e., it satisfies the inversion relation (3) and the pentagon identity (11)*

$$\tilde{\varphi}(\alpha,x)\tilde{\varphi}(\beta,y)\alpha(y)\beta(x)=\int_{\mathsf{A}}\tilde{\varphi}(\beta-\gamma,y/z)\tilde{\varphi}(\gamma,z)\tilde{\varphi}(\alpha-\gamma,x/z)\gamma(z)\,\mathrm{d}(\gamma,z) \tag{27}$$

for all $(\alpha,x),(\beta,y)\in\mathsf{A}$.

Proof. The inversion relation is elementary. The proof of the pentagon is inspired by the behavior of the Ptolemy coordinates under a pentagon transformation [15, 14].

Let LHS and RHS denote the left and the right hand sides of (27). Lemma 3.2 implies that

$$\mathrm{LHS}=\frac{f(x)}{\alpha(x-1)}\frac{f(y)}{\beta(y-1)}\alpha(y)\beta(x) \tag{28}$$

$$\mathrm{RHS}=\int_{\hat{\mathsf{B}}\times\mathsf{B}}\frac{f(y/z)}{(\beta-\gamma)(y/z-1)}\frac{f(z)}{\gamma(z-1)}\frac{f(x/z)}{(\alpha-\gamma)(x/z-1)}\gamma(z)\,\mathrm{d}\gamma\,\mathrm{d}z\,, \tag{29}$$

[1]One natural definition could be $\varphi(\alpha,-1)=e^{i\pi(1-\alpha(-1))/4}$ which is consistent with the first property of the quantum dilogarihm $\varphi(\alpha,x)\varphi(-\alpha,1/x)=\alpha(x)$.

where

$$f(x) = \frac{1}{\|1 - x^{-1}\|}\,. \tag{30}$$

Now, we collect the terms in the left hand side with respect to α and β, noting that

$$\frac{1}{\alpha(x-1)}\alpha(y) = \alpha(\frac{1}{x-1})\alpha(y) = \alpha(\frac{y}{x-1})\,. \tag{31}$$

Thus, we have

$$\mathrm{LHS} = f(x)f(y)\alpha(\frac{y}{x-1})\beta(\frac{x}{y-1})\,. \tag{32}$$

Likewise, collecting terms with respect to α, β and γ on the RHS, gives

$$\begin{aligned}\mathrm{RHS} &= \int_{\hat{\mathsf{B}}\times\mathsf{B}} f(y/z)f(z)f(x/z)\frac{\gamma(\frac{z}{z-1}(\frac{y}{z}-1)(\frac{x}{z}-1))}{\beta(y/z-1)\alpha(x/z-1)}\,\mathrm{d}\gamma\,\mathrm{d}z \\ &= \int_{\mathsf{B}} f(y/z)f(z)f(x/z)\frac{\delta_{\mathsf{B}}\left(\frac{z}{z-1}(\frac{y}{z}-1)(\frac{x}{z}-1)\right)}{\beta(y/z-1)\alpha(x/z-1)}\,\mathrm{d}z\,.\end{aligned} \tag{33}$$

Now use the fact that $\frac{z}{z-1}(\frac{y}{z}-1)(\frac{x}{z}-1) = 1$ is equivalent to $z(\frac{y}{z}-1)(\frac{x}{z}-1) = z-1$ which is equivalent to $z = \frac{xy}{x+y-1}$. Substituting this value in the above equation, the terms involving α and β match those of the LHS and we have

$$\frac{\mathrm{RHS}}{\alpha(\frac{y}{x-1})\beta(\frac{x}{y-1})} = f\Big(\frac{x+y-1}{x}\Big)f\Big(\frac{xy}{x+y-1}\Big)f\Big(\frac{x+y-1}{y}\Big)\int_{\mathsf{B}}\delta_{\mathsf{B}}\Big(\frac{(y-z)(\frac{x}{z}-1)}{z-1}\Big)\,\mathrm{d}z\,. \tag{34}$$

Thus, the pentagon identity (27) is equivalent to

$$f(x)f(y) = f\Big(\frac{x+y-1}{x}\Big)f\Big(\frac{xy}{x+y-1}\Big)f\Big(\frac{x+y-1}{y}\Big)\int_{\mathsf{B}}\delta_{\mathsf{B}}\Big(\frac{(y-z)(\frac{x}{z}-1)}{z-1}\Big)\,\mathrm{d}z\,. \tag{35}$$

Note incidentally that the arguments of f are the ones appearing in the 5-term relation for the dilogarithm, hence also in the definition of the Bloch group of F; see [3, 33].

Using the equation (30) for the function f, it follows that Equation (35) is equivalent to

$$\int_{\mathsf{B}}\delta_{\mathsf{B}}\left(\frac{(y-z)(\frac{x}{z}-1)}{z-1}\right)\mathrm{d}z = \left\|\frac{(x-1)(y-1)}{(x+y-1)^2}\right\|\,. \tag{36}$$

We apply part (a) of Lemma 2.7 to the function

$$g(z) = \frac{(y-z)(\frac{x}{z}-1)}{z-1} - 1 \tag{37}$$

with a unique zero:

$$a = \frac{xy}{x+y-1}, \qquad g'(a) = -\frac{(x+y-1)^3}{xy(x-1)(y-1)} \tag{38}$$

and we conclude that

$$\int_{\mathsf{B}}\delta_{\mathsf{B}}\left(\frac{(y-z)(\frac{x}{z}-1)}{z-1}\right)\mathrm{d}z = \frac{1}{\|af'(a)\|} = \left\|\frac{(x-1)(y-1)}{(x+y-1)^2}\right\|. \tag{39}$$

This proves (36) and concludes the formal proof of the theorem. □

4. Angles

In this section we introduce an angled version $\Psi_{a,c}$ of the quantum dilogarithm φ which satisfies an angle dependent pentagon identity (51) as well as the symmetry relations (50a) and (50b). The function $\Psi_{a,c}$ is the building block for the partition function of a tetrahedron and the relation it satisfies will be used to show that the partition function of a triangulation is invariant under 2–3 Pachner moves, and hence a topological invariant.

Recall that angles were used in previous works (see for instance [2, 24, 12]) as complex deformations of real variables. In contrast, in our present paper angles (denoted in general by a, b, c) will be elements of the abelian group

$$\mathsf{C} = \mathbb{R} \times \mathsf{B}\,. \tag{40}$$

An angle $a = (\dot{a}, \ddot{a}) \in \mathsf{C}$ has a real component $\dot{a} \in \mathbb{R}$ and a local field component $\ddot{a} \in \mathsf{B} = F^{\times}$.

We define an involution in C by the formula

$$\bar{a} := (\dot{a}, (\ddot{a})^{-1})\,. \tag{41}$$

element $\bar{a}$ will to be called *conjugate* of a.

When three angles a, b and c are assigned to a triangle, we will always assume that they satisfy

$$a + b + c = \varpi := (1, -1)\,, \tag{42}$$

thus b is expressed in terms of a and c by

$$b = (1, -1) - a - c = (1 - \dot{a} - \dot{c}, -(\ddot{a}\ddot{c})^{-1})\,. \tag{43}$$

Definition 4.1. For $a, c \in \mathsf{C}$, we define a function

$$\Psi_{a,c} \colon \hat{\mathsf{B}} \times (\mathsf{B} \setminus \{\ddot{a}^{-1}\}) \to \mathbb{C}, \qquad \Psi_{a,c}(\alpha, x) = \frac{1}{\alpha\big((1 - \ddot{a}x)\ddot{c}\big)} \frac{\|\ddot{a}x\|^{\dot{c}}}{\|1 - \ddot{a}x\|^{1-\dot{a}}}. \tag{44}$$

Notice, that $\Psi_{a,c}$ is defined almost everywhere on A. We denote by $\bar{\Psi}_{a,c}$ the function of $(\alpha, x) \in \hat{\mathsf{B}} \times (\mathsf{B} \setminus \{\ddot{a}\})$ defined by

$$\bar{\Psi}_{a,c}(\alpha, x) = \overline{\Psi_{\bar{a},\bar{c}}(\alpha, x)} = \alpha\big((1 - x/\ddot{a})/\ddot{c}\big) \frac{\|x/\ddot{a}\|^{\dot{c}}}{\|1 - x/\ddot{a}\|^{1-\dot{a}}}. \tag{45}$$

It follows immediately from the definition that the angle-dependent function $\Psi_{a,c}$ specializes to $\tilde{\varphi}$ through the formula

$$\bar{\Psi}_{0,0}(\alpha, x) = \alpha(x)\tilde{\varphi}(-\alpha, x^{-1})\,. \tag{46}$$

This is a good place to note that the angled version of the quantum dilogarithms of the three Examples 2.3, 2.4 and 2.5 are functions in the Schwartz space $\mathcal{S}(\mathsf{A})$ of A, whereas our function (44) is only a tempered distribution on A under a positivity assumption on angles. This is the content of the next lemma.

Lemma 4.2. For all local fields, including the real and the complex numbers, $\Psi_{a,c}$ is a tempered distribution on A if

$$\dot{a},\ \dot{b},\ \dot{c} \geq 0\,, \tag{47}$$

which is further represented by a locally integrable function when all the above inequalities are strict.

Note that since $\dot{a}+\dot{b}+\dot{c}=1$, the positivity condition (47) is equivalent to

$$1 \geq \dot{a},\, \dot{b},\, \dot{c} \geq 0\,. \tag{48}$$

Proof. When $\dot{a}=\dot{c}=0$, the specialization (46), together with the facts that $\tilde{\varphi}$ is a tempered distribution and $|\alpha(x)|=1$ for all $(\alpha,x)\in \mathsf{A}$ implies that $\Psi_{a,c}$ is a tempered distribution. When $\dot{a},\dot{b},\dot{c}\geq 0$ with at least one positive, satisfying $\dot{a}+\dot{b}+\dot{c}=1$, it follows by the symmetry relation (50a) below, that we can assume that $0<\dot{a},\dot{c}<1$. In this case, the definition of $\Psi_{a,c}$ together with Lemma 4.3 below imply that $\Psi_{a,c}$ is locally integrable. □

The next lemma was observed by Igusa and others [18].

Lemma 4.3. Fix a non-Archimedean local field F and let O_F denote its ring of integers and q denote the size of the residue field. Then,

$$I(s) := \int_{O_F} \|x\|^{s-1} d\mu(x) \tag{49}$$

is absolutely convergent if and only if $\mathrm{Re}(s)>0$, in which case it equals to $\frac{1-q^{-1}}{1-q^{-s}}$.

Proof. The proof is elementary using the fact that $O_F = \{0\} \sqcup_{k=0}^{\infty} \varpi^k O_F^{\times}$ where ϖ is a uniformizer of the local field O_F, and the integral is given by

$$I(s) = (1-q^{-1}) \sum_{k=0}^{\infty} q^{-ks}$$

(where q is the cardinality of the residue field) which is absolutely convergent if and only if $\mathrm{Re}(s)>0$, in which case it equals to $\frac{1-q^{-1}}{1-q^{-s}}$. □

From now on, we will assume that the angles a,b,c with $a+b+c=(1,-1)$ satisfy the *positivity condition* (47). The next two theorems give the main properties of the functions $\Psi_{a,c}$ and $\bar{\Psi}_{a,c}$.

Theorem 4.4. *The functions $\Psi_{a,c}$ and $\bar{\Psi}_{a,c}$ satisfy the symmetry relations*

$$\Psi_{a,c}(-\alpha,1/x)\alpha(x) = \bar{\Psi}_{a,b}(\alpha,x) \tag{50a}$$

$$\int_{\hat{\mathsf{B}}\times\mathsf{B}} \Psi_{a,c}(\beta,y)\alpha(x/y)\beta(y/x)\,\mathrm{d}(\beta,y) = \bar{\Psi}_{b,c}(\alpha,x) \tag{50b}$$

where b satisfies (43).

Proof. We start by the left hand side of (50a)

$$\begin{aligned}
\Psi_{a,c}(-\alpha,1/x)\alpha(x) &= \alpha\big((1-\ddot{a}/x)\ddot{c}\big)\frac{\|\ddot{a}/x\|^{\dot{c}}}{\|1-\ddot{a}/x\|^{1-\dot{a}}}\alpha(x) = \alpha\big((x-\ddot{a})\ddot{c}\big)\frac{\|\ddot{a}/x\|^{\dot{c}}}{\|1-\ddot{a}/x\|^{1-\dot{a}}} \\
&= \alpha\big((x-\ddot{a})\ddot{c}\big)\frac{\|x/\ddot{a}\|^{1-\dot{c}-\dot{a}}}{\|1-x/\ddot{a}\|^{1-\dot{a}}} = \alpha\big(-(1-x/\ddot{a})\ddot{a}\ddot{c}\big)\frac{\|x/\ddot{a}\|^{1-\dot{c}-\dot{a}}}{\|1-x/\ddot{a}\|^{1-\dot{a}}} \\
&= \alpha\big((1-x/\ddot{a})/\ddot{b}\big)\frac{\|x/\ddot{a}\|^{\dot{b}}}{\|1-x/\ddot{a}\|^{1-\dot{a}}} = \bar{\Psi}_{a,b}(\alpha,x).
\end{aligned}$$

Next, we write out the left hand side of (50b), collect the argument of β, integrate over β and rewrite the remaining integral over the Haar measure over F

$$\begin{aligned}\mathrm{LHS}(50\mathrm{b}) &= \int_{\hat{\mathsf{B}}\times\mathsf{B}} (-\beta)\big((1-\ddot{a}y)\ddot{c}\big)\frac{\|\ddot{a}y\|^{\dot{c}}}{\|1-\ddot{a}y\|^{1-\dot{a}}}\alpha(x/y)\beta(y/x)\,\mathrm{d}(\beta,y)\\ &= \int_{\hat{\mathsf{B}}\times\mathsf{B}} \beta\Big(\frac{y/(\ddot{c}x)}{1-\ddot{a}y}\Big)\frac{\|\ddot{a}y\|^{\dot{c}}}{\|1-\ddot{a}y\|^{1-\dot{a}}}\alpha(x/y)\,\mathrm{d}(\beta,y)\\ &= \int_F \delta_F\Big(\frac{y/(\ddot{c}x)}{1-\ddot{a}y}-1\Big)\frac{\|\ddot{a}y\|^{\dot{c}-1}\|\ddot{a}\|}{\|1-\ddot{a}y\|^{1-\dot{a}}}\alpha(x/y)\,\mathrm{d}y\end{aligned}$$

bring the argument of the delta-function to common denominator

$$= \int_F \delta_F\Big(\frac{(1+\ddot{a}\ddot{c}x)y-\ddot{c}x}{(1-\ddot{a}y)\ddot{c}x}\Big)\frac{\|\ddot{a}y\|^{\dot{c}-1}\|\ddot{a}\|}{\|1-\ddot{a}y\|^{1-\dot{a}}}\alpha(x/y)\,\mathrm{d}y$$

and integrate over y (it is fixed by the value $y=y' := x/(\ddot{c}^{-1}+\ddot{a}x)$)

$$\begin{aligned}&= \frac{\|\ddot{a}y'\|^{\dot{c}}}{\|1-\ddot{a}y'\|^{-\dot{a}}}\alpha(x/y') = \frac{\|\ddot{a}x\|^{\dot{c}}}{\|\ddot{c}^{-1}+\ddot{a}x\|^{\dot{c}}\|1-\frac{\ddot{a}x}{\ddot{c}^{-1}+\ddot{a}x}\|^{-\dot{a}}}\alpha(\ddot{c}^{-1}+\ddot{a}x)\\ &= \frac{\|\ddot{a}x\|^{\dot{c}}}{\|\ddot{c}^{-1}+\ddot{a}x\|^{\dot{c}+\dot{a}}\|\ddot{c}\|^{\dot{a}}}\alpha(\ddot{c}^{-1}+\ddot{a}x) = \frac{\|\ddot{a}x\|^{\dot{c}}\|\ddot{c}\|^{\dot{c}}}{\|1+\ddot{c}\ddot{a}x\|^{\dot{c}+\dot{a}}}\alpha\Big(\frac{1+\ddot{c}\ddot{a}x}{\ddot{c}}\Big)\\ &= \frac{\|\ddot{c}\ddot{a}x\|^{\dot{c}}}{\|1+\ddot{c}\ddot{a}x\|^{\dot{c}+\dot{a}}}\alpha\Big(\frac{1+\ddot{c}\ddot{a}x}{\ddot{c}}\Big) = \frac{\|x/\ddot{b}\|^{\dot{c}}}{\|1-x/\ddot{b}\|^{1-\dot{b}}}\alpha\Big(\frac{1-x/\ddot{b}}{\ddot{c}}\Big) = \bar{\Psi}_{b,c}(\alpha,x)\,.\end{aligned}$$

□

Theorem 4.5. *Denoting $\Psi_i := \Psi_{a_i,c_i}$, the following pentagon relation holds*

$$\Psi_1(\alpha,x)\Psi_3(\beta,y) = \int_{\hat{\mathsf{B}}\times\mathsf{B}} \Psi_0(\alpha-\gamma,x/z)\Psi_2(\gamma,z)\Psi_4(\beta-\gamma,y/z)\frac{\alpha(y/z)\beta(x/z)}{\gamma(xy/z^2)}\,\mathrm{d}(\gamma,z) \tag{51}$$

provided that

$$a_3 = a_2+a_4,\quad c_3 = a_0+c_4,\quad c_1 = c_0+a_4,\quad a_1 = a_0+a_2,\quad c_2 = c_1+c_3. \tag{52}$$

Note that Equation (52) implies the balancing condition $b_0+c_2+b_4 = (2,1)$.

Proof. We write out explicitly the right hand side of (51)

$$\begin{aligned}\mathrm{RHS}(51) = \int_{\hat{\mathsf{B}}\times\mathsf{B}} &(\gamma-\alpha)\big((1-\ddot{a}_0x/z)\ddot{c}_0\big)\frac{\|\ddot{a}_0x/z\|^{\dot{c}_0}}{\|1-\ddot{a}_0x/z\|^{1-\dot{a}_0}}(-\gamma)\big((1-\ddot{a}_2z)\ddot{c}_2\big)\frac{\|\ddot{a}_2z\|^{\dot{c}_2}}{\|1-\ddot{a}_2z\|^{1-\dot{a}_2}}\\ &\times(\gamma-\beta)\big((1-\ddot{a}_4y/z)\ddot{c}_4\big)\frac{\|\ddot{a}_4y/z\|^{\dot{c}_4}}{\|1-\ddot{a}_4y/z\|^{1-\dot{a}_4}}\frac{\alpha(y/z)\beta(x/z)}{\gamma(xy/z^2)}\,\mathrm{d}(\gamma,z)\end{aligned}$$

collect the arguments of α,β,γ

$$\begin{aligned}= \int_{\hat{\mathsf{B}}\times\mathsf{B}} &\alpha\Big(\frac{y/\ddot{c}_0}{z-\ddot{a}_0x}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}\|z\|^{1-\dot{a}_0-\dot{c}_0}}{\|z-\ddot{a}_0x\|^{1-\dot{a}_0}}\beta\Big(\frac{x/\ddot{c}_4}{z-\ddot{a}_4y}\Big)\frac{\|\ddot{a}_4y\|^{\dot{c}_4}\|z\|^{1-\dot{a}_4-\dot{c}_4}}{\|z-\ddot{a}_4y\|^{1-\dot{a}_4}}\\ &\times\gamma\left(\frac{\ddot{c}_0\ddot{c}_4(z-\ddot{a}_0x)(z-\ddot{a}_4y)}{\ddot{c}_2(1-\ddot{a}_2z)xy}\right)\frac{\|\ddot{a}_2z\|^{\dot{c}_2}}{\|1-\ddot{a}_2z\|^{1-\dot{a}_2}}\,\mathrm{d}(\gamma,z)\end{aligned}$$

integrate over γ and rewrite the remaining integral over the Haar measure over F

$$= \int_F \alpha\Big(\frac{y/\ddot{c}_0}{z-\ddot{a}_0x}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}\|z\|^{\dot{b}_0}}{\|z-\ddot{a}_0x\|^{1-\dot{a}_0}}\beta\Big(\frac{x/\ddot{c}_4}{z-\ddot{a}_4y}\Big)\frac{\|\ddot{a}_4y\|^{\dot{c}_4}\|z\|^{\dot{b}_4}}{\|z-\ddot{a}_4y\|^{1-\dot{a}_4}} \times \delta_F\left(\frac{\ddot{c}_0\ddot{c}_4(z-\ddot{a}_0x)(z-\ddot{a}_4y)}{\ddot{c}_2(1-\ddot{a}_2z)xy}-1\right)\frac{\|z\|^{\dot{c}_2-1}\|\ddot{a}_2\|^{\dot{c}_2}}{\|1-\ddot{a}_2z\|^{1-\dot{a}_2}}\,\mathrm{d}z$$

bring the common denominator in the delta-function (by using the relation $\ddot{b}_0\ddot{c}_2\ddot{b}_4=1$ which follows from (52)) and collect the powers of $\|z\|$

$$= \int_F \alpha\Big(\frac{y/\ddot{c}_0}{z-\ddot{a}_0x}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}}{\|z-\ddot{a}_0x\|^{1-\dot{a}_0}}\beta\Big(\frac{x/\ddot{c}_4}{z-\ddot{a}_4y}\Big)\frac{\|\ddot{a}_4y\|^{\dot{c}_4}}{\|z-\ddot{a}_4y\|^{1-\dot{a}_4}} \times \delta_F\left(\frac{\ddot{c}_0\ddot{c}_4z(z-\ddot{a}_0x-\ddot{a}_4y-(\ddot{c}_0\ddot{b}_2\ddot{c}_4)^{-1}xy)}{\ddot{c}_2(1-\ddot{a}_2z)xy}\right)\frac{\|z\|^{\dot{b}_0+\dot{c}_2+\dot{b}_4-1}\|\ddot{a}_2\|^{\dot{c}_2}}{\|1-\ddot{a}_2z\|^{1-\dot{a}_2}}\,\mathrm{d}z$$

integrate over z (which is fixed by the value $z=z':=\ddot{a}_0x+\ddot{a}_4y+(\ddot{c}_0\ddot{b}_2\ddot{c}_4)^{-1}xy$)

$$= \alpha\Big(\frac{y/\ddot{c}_0}{z'-\ddot{a}_0x}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}\|x\|}{\|z'-\ddot{a}_0x\|^{1-\dot{a}_0}}\beta\Big(\frac{x/\ddot{c}_4}{z'-\ddot{a}_4y}\Big)\frac{\|\ddot{a}_4y\|^{\dot{c}_4}\|y\|}{\|z'-\ddot{a}_4y\|^{1-\dot{a}_4}}\frac{\|z'\|^{\dot{b}_0+\dot{c}_2+\dot{b}_4-2}\|\ddot{a}_2\|^{\dot{c}_2}\|\ddot{c}_2\|}{\|1-\ddot{a}_2z'\|^{-\dot{a}_2}\|\ddot{c}_0\ddot{c}_4\|}.$$

In the obtained expression, along with (52), we use the equalities

$$\ddot{b}_0\ddot{c}_2\ddot{b}_4=1,\quad \dot{b}_0+\dot{c}_2+\dot{b}_4=2,$$
$$1-\dot{a}_2z'=(1-\ddot{a}_1x)(1-\ddot{a}_3y),$$
$$\frac{y/\ddot{c}_0}{z'-\ddot{a}_0x}=\frac{1/(\ddot{c}_0\ddot{a}_4)}{1-x\ddot{b}_4/(\ddot{c}_0\ddot{b}_2)}=\frac{1/\ddot{c}_1}{1-\ddot{a}_1x},\quad \frac{x/\ddot{c}_4}{z'-\ddot{a}_4y}=\frac{1/(\ddot{a}_0\ddot{c}_4)}{1-y\ddot{b}_0/(\ddot{b}_2\ddot{c}_4)}=\frac{1/\ddot{c}_3}{1-\ddot{a}_3y}$$

and continue the computation as follows:

$$\begin{aligned}\mathrm{RHS}(51) &= \alpha\Big(\frac{1/\ddot{c}_1}{1-\ddot{a}_1x}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}\|x\|\|\ddot{c}_0/(\ddot{c}_1y)\|^{1-\dot{a}_0}}{\|1-\ddot{a}_1x\|^{1-\dot{a}_0-\dot{a}_2}}\\ &\quad\times\beta\Big(\frac{1/\ddot{c}_3}{1-\ddot{a}_3y}\Big)\frac{\|\ddot{a}_4y\|^{\dot{c}_4}\|y\|\|\ddot{c}_4/(\ddot{c}_3x)\|^{1-\dot{a}_4}}{\|1-\ddot{a}_3y\|^{1-\dot{a}_4-\dot{a}_2}}\frac{\|\ddot{a}_2\|^{\dot{c}_2}\|\ddot{c}_2\|}{\|\ddot{c}_0\ddot{c}_4\|}\\ &= \alpha\Big(\frac{1/\ddot{c}_1}{1-\ddot{a}_1x}\Big)\frac{\|x\|^{\dot{c}_0+\dot{a}_4}\|\ddot{a}_0\|^{\dot{c}_0}\|\ddot{c}_0/\ddot{c}_1\|^{1-\dot{a}_0}}{\|1-\ddot{a}_1x\|^{1-\dot{a}_0-\dot{a}_2}}\\ &\quad\times\beta\Big(\frac{1/\ddot{c}_3}{1-\ddot{a}_3y}\Big)\frac{\|y\|^{\dot{c}_4+\dot{a}_0}\|\ddot{a}_4\|^{\dot{c}_4}\|\ddot{c}_4/\ddot{c}_3\|^{1-\dot{a}_4}}{\|1-\ddot{a}_3y\|^{1-\dot{a}_4-\dot{a}_2}}\frac{\|\ddot{a}_2\|^{\dot{c}_2}\|\ddot{c}_2\|}{\|\ddot{c}_0\ddot{c}_4\|}\\ &= \alpha\Big(\frac{1/\ddot{c}_1}{1-\ddot{a}_1x}\Big)\frac{\|\ddot{a}_1x\|^{\dot{c}_1}}{\|1-\ddot{a}_1x\|^{1-\dot{a}_1}}\beta\Big(\frac{1/\ddot{c}_3}{1-\ddot{a}_3y}\Big)\frac{\|\ddot{a}_3y\|^{\dot{c}_3}}{\|1-\ddot{a}_3y\|^{1-\dot{a}_3}}=\Psi_1(\alpha,x)\Psi_3(\beta,y)\,.\end{aligned}$$

This completes the proof of the theorem. □

Remark 4.6. Note that under the specialization of Equation (46), the pentagon identity (27) is a special case of the complex conjugate of (51) where all angles are set to zero.

Let P denote the standard ordered pentagon shown on the left hand side of Figure 1.

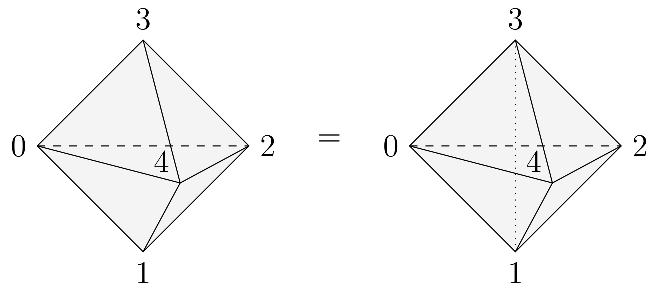

FIGURE 1. The standard ordered 2–3 Pachner move.

Proposition 4.7. The function Ψ satisfies the ordered pentagon identity for all orderings of the vertices of P.

Proof. There are 20 orderings of the vertices of P obtained by applying permutations of its five vertices. For each ordering is associated a pentagon identity. To P itself is associated the identity (51). The five equalities of [23, Eqn.(4.3)] implement the permutations $(0,1)$, $(1,2)$, $(2,3)$, and $(3,4)$ of the symmetric group of the five vertices $\{0,1,2,3,4\}$ in terms of transformations L, M and R. Then, [23, Eqn.(4.8)] expresses the transformations L, M and R in terms of two symmetries S and T of a quantum dilogarithm defined in Eqn.(4.8). i.b.i.d. The symmetries (50a) and (50b) are exactly the S and T transformations of Ψ. It follows that the standard ordered pentagon identity implies all other ordered pentagon identities. □

5. A face-type generalized TQFT

In this section we construct a face-type generalized TQFT using a local field F. In the following constructions, as a combinatorial input, we use the setting of *ordered Δ-complexes*, see [16]. For a Δ-complex X, we let X_i to denote the set of i-dimensional cells of X and $X_{i,j}$ the set of pairs (a,b) where $a \in X_i$ is considered with the ordered Δ-complex structure of the standard i-dimensional simplex Δ^i and $b \in (\Delta^i)_j$. As was explained in [2], the distributional properties of the kinematical kernel requires the assumption that the underlying 3-manifold M obtained from X by removing its vertices satisfies the condition $H_2(M,\mathbb{Z}) = 0$. This ensures that no square of a delta function appears in the kinematical kernel below.

5.1. The face-type partition function.

Let X be an ordered Δ-complex homeomorphic to an oriented pseudo 3-manifold with boundary ∂X with its ordered Δ-complex structure induced from that of X. The kinematical kernel to be defined below depends on only a Gaussian group and on X but not on a quantum dilogarithm.

Given a self-dual LCA group A with a Gaussian exponential $\langle z\rangle$ and the Fourier kernel $\langle z; w\rangle$, we associate to X the following *kinematical kernel*

$$K_X \in \mathcal{S}'(\mathsf{A}^{(\partial X)_2} \times \mathsf{A}^{X_3}), \qquad K_X(y,z) = \int_{x\in\mathsf{A}^{X_2}} \delta_{\mathsf{A}^{(\partial X)_2}}(x|_{(\partial X)_2} - y) \prod_{T\in X_3} K_T(x,z)\,\mathrm{d}x \tag{53}$$

where, for a finite set S and a map $f\colon S \to \mathsf{A}$, we use the notation

$$\delta_{\mathsf{A}^S}(f) := \prod_{s\in S} \delta_{\mathsf{A}}(f(s)), \tag{54}$$

and

$$K_T(x,z) := \langle x_0; z(T)\rangle^{\operatorname{sgn}(T)} \delta_{\mathsf{A}}(x_0 - x_1 + x_2)\delta_{\mathsf{A}}(x_2 - x_3 + z(T)), \quad x_i := x(\partial_i T). \tag{55}$$

Here, $\mathcal{S}'(\mathsf{A}^{(\partial X)_2}\times\mathsf{A}^{X_3})$ denotes the space of tempered distributions on the LCA group $\mathsf{A}^{(\partial X)_2}\times\mathsf{A}^{X_3}$, the dual of the space $\mathcal{S}(\mathsf{A}^{(\partial X)_2}\times\mathsf{A}^{X_3})$ of Schwartz-Bruhat functions on $\mathsf{A}^{(\partial X)_2}\times\mathsf{A}^{X_3}$.

Let C be another LCA group with a distinguished element $\varpi \in \mathsf{C}$. A C-valued angle structure on X is a map $\theta\colon X_{3,1} \to \mathsf{C}$ such that, for any tetrahedron $T \in X_3$, the restriction $\theta(T,\cdot)\colon (\Delta^3)_1 \to \mathsf{C}$ satisfies the same algebraic conditions as the usual (real valued) dihedral angles of an ideal hyperbolic tetrahedron where the value π is replaced by ϖ.

Let us assume now that $\mathsf{A} = \hat{B} \times B$ with $B = F^\times$, and let C be defined as in Section 4. The *dynamical content* of X associated to a C-valued angle structure θ and a (symmetric) angle dependent quantum dilogarithm $\Psi_{a,c}(x)$ over A, with $a, c \in \mathsf{C}$, is defined by

$$D_{X,\theta}(z) \in \mathcal{S}'(\mathsf{A}^{X_3}\times\mathsf{C}^{X_{3,1}}), \quad D_{X,\theta}(z) = \prod_{T\in X_3} D_{T,\theta}(z(T)) \tag{56}$$

where

$$D_{T,\theta} = \begin{cases} \Psi_{a,c} & \text{if } \operatorname{sgn}(T) = +1, \\ \bar{\Psi}_{a,c} & \text{if } \operatorname{sgn}(T) = -1, \end{cases} \tag{57}$$

where a (resp., b, c) is the angle of the edges 01 and 23 (resp., 02 and 13, and 03 and 12) as in Figure 2.

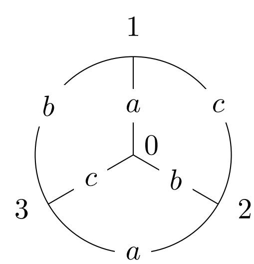

FIGURE 2. The angles of an ideal tetrahedron with ordered vertices.

The associated *partition function* of the pair (X, θ) is the the push-forward along the tetrahedral variables of the product of the kinematical kernel with the dynamical content.

$$\mathcal{F}_{F,X,\theta} = \int_{z\in\mathsf{A}^{X_3}} K_X D_{X,\theta}\, \mathrm{d}z\,. \tag{58}$$

Unlike the case of the three examples 2.3, 2.4 and 2.5 where the kinematical kernel was a distribution and the dynamical content was a function in the Bruhat–Schwartz space and the two were contracted, here both the kinematical kernel and the dynamical content are distributions and we multiply them together, and then we push them forward.

This partition function has three important properties:

(P1) The partition function $\mathcal{F}_{F,X,\theta}$ is invariant under ordered, angled, 2–3 Pachner moves.

(P2) As a function of $\theta = (\dot{\theta}, \ddot{\theta})$, it is distributional on $\ddot{\theta}$, and for every test function $\psi \in \mathcal{S}(\mathcal{A}_X)$ (where $\mathcal{A}_X$ is the affine space where $\ddot{\theta}$ takes values), the function $\dot{\theta} \mapsto IF_{F,X,\theta}$ extends to a meromorphic function of $q^{\dot{\theta}}$.

(P3) The partition functon $\mathcal{F}_{F,X,\theta}$ is angle gauge-invariant (see below Subsection 5.3).

These properties imply the following theorem as was explained exactly in [2] as well as in propositions 3.2, 3.3 and 3.4 of [12].

Theorem 5.1. *The distribution $\mathcal{F}_F$ descends to an invariant $\mathcal{F}_{F,M}(\lambda, \mu)$ of a compact, oriented 3-manifold M satisfying $H_2(M, \mathbb{Z}) = 0$, where $(\lambda, \mu) \in H_1(\partial M, \mathsf{C})$.*

The invariant is distributional on $(\ddot{\lambda}, \ddot{\mu}) \in H_1(\partial M, F^\times)$ and evaluated at a test function, extends to a meromorphic function of $(q^{\dot{\lambda}}, q^{\dot{\mu}})$ where $(\dot{\lambda}, \dot{\mu}) \in H_1(\partial M, \mathbb{R})$.

We call the above invariant a face state-integral, following the fact that the states are assigned to the faces of the triangulation.

In the remaining of the section we discuss the three properties of the partition function. The invariance of the partition function under all ordered angled Pachner moves follows from Proposition 4.7.

5.2. **Integration and point counting.** In this section we discuss property P2. The partition function (58) involves integration on A^{2N} (where $\mathsf{A} = \hat{\mathsf{B}} \times \mathsf{B}$ and $\mathsf{B} = F^\times$) of products of delta functions times a product of φ-functions times a product of positive powers of $\|x_i\|$ and $\|1 - x_i\|$ times a fixed test function ψ. (Recall that test functions on non-Archimedean local fields F are compactly supported and take finitely many values). These integrals reduce to integrals on B^N of products of delta functions of rational functions $f(x) \in F(x)$ of N-variables $x = (x_1, \dots, x_N)$, times a product of positive powers of $\|x_i\|$ and $\|1 - x_i\|$ times a test function. Such a functional integration was developed, among others, by Denef and Loeser, who studied the Igusa local zeta functions from the point of view of counting solutions to polynomial equations modulo $\mathbb{Z}/p^n\mathbb{Z}$. A motivic version of that integration was introduced by Kontsevich in 1995, an arithmetic version of which was given in [7] and a geometric one in [5, 6].

Although the integration is defined analytically, it is expressed in terms of point counting solutions of equations modulo $\mathbb{Z}/p^n\mathbb{Z}$ for fixed p and varying n. This, together with Hironaka's resolution of singularities, inclusion-exclusion, and a local calculation identifies the integrals over A^{2N} in terms of rational functions. In particular, Denef [5, Thm.3.2] proves that if S is a boolean combination of subsets of F^m, with S compact, and $g \in F[x]$, $x = (x_1, \dots, x_m)$, then

$$I(s) = \int_S \|g(x)\|^s \, \mathrm{d}\, x \tag{59}$$

is a rational function of p^{-s}. In [7] a stronger result was proven: the rational function of p^{-s} is independent of p (the characteristic of the residue field of the local field F) for all but finitely many p. Moreover, in [26] Loeser gives a multivariable generalization of the above results for distributional integrals of the form

$$I(s) = \int_S \prod_{j=1}^{r} \|g_j(x)\|^{s_j} \, \mathrm{d}\, x \tag{60}$$

for $s = (s_1, \ldots, s_k)$, defined initially for $\mathrm{Re}(s_j) > 0$, and analytically continued as meromorphic functions with poles on a finite union of linear hyperplanes. This implies property (P2) of the partition function (58).

5.3. **Angle gauge transformations.** In this section we discuss the invariance of the partition function (58) under angled gauge transformation.

Following [2], we let SP_n denote a bipyramid with two vertices labeled 0 and 1 and with basis a polygon P_n with n sides, see Figure 3. Using the edge $e := 01$ of SP_n, we can triangulate SP_n into n positively oriented tetrahedra with common edge e.

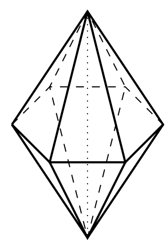

FIGURE 3. The bipyramid SP_n with $n = 6$.

Enumerating the tetrahedra by $i = 1, 2, \ldots, n$ in the cyclic order around the common edge e, the partition function of SP_n is given by

$$\mathcal{F}_{F,SP_n,\theta} = \int_{\mathsf{A}^n} \prod_{i=1}^{n} \langle x_i, u_i | T(a_i, c_i) | y_i, u_{i+1} \rangle \, \mathrm{d}(u_1, \ldots, u_n) \tag{61}$$

where we identify $u_{n+1} = u_1$ and (in this section only) we use the notation

$$\langle x, u | T(a, c) | y, v \rangle := \langle x; v - u \rangle \delta_F(x - y + u) \Psi_{a,c}(v - u). \tag{62}$$

The variables x_i and y_i in (61) are the state variables on the boundary of SP_n respectively associated to the minimal and the next to the minimal faces of the i-th tetrahedron, while a_i and c_i are the angle variables of the i-th tetrahedron associated to the edges connecting the minimal vertex to the next minimal and the maximal vertices respectively.

The *gauge transformation* associated with edge e by amount $\lambda \in \mathsf{C}$ is the transformation of the angle structure $\theta \mapsto \theta'$ given by the simultaneous shift of all angles c_i by the value λ:

$$c_i \mapsto c'_i = c_i + \lambda, \quad \forall i \in \{1, 2, \ldots, n\}. \tag{63}$$

We claim that

$$\mathcal{F}_{F,SP_n,\theta'} = \mathcal{F}_{F,SP_n,\theta} \|\ddot{w}_e\|^{\dot{\lambda}} \tag{64}$$

where

$$w_e = \sum_{i=1}^{n} a_i \tag{65}$$

is the total angle around edge e which is given by contributions from all tetrahedra to the edge e. Indeed, we have

$$\frac{\Psi_{a,c+\lambda}(\alpha,x)}{\Psi_{a,c}(\alpha,x)}=\frac{\|\ddot{a}x\|^{\dot{\lambda}}}{\alpha(\ddot{\lambda})}=\|\ddot{a}\|^{\dot{\lambda}}f_\lambda(\alpha,x) \tag{66}$$

where $f_\lambda\colon \mathsf{A}\to\mathbb{C}^\times$ is a complex (multiplicative) character defined by

$$f_\lambda(\alpha,x)=\frac{\|x\|^{\dot{\lambda}}}{\alpha(\ddot{\lambda})}\,. \tag{67}$$

Thus, we have

$$\begin{aligned}\mathcal{F}_{F,SP_n,\theta'}&=\int_{\mathsf{A}^n}\prod_{i=1}^n\langle x_i,u_i|T(a_i,c_i+\lambda)|y_i,u_{i+1}\rangle\,\mathrm{d}(u_1,\dots,u_n)\\&=\int_{\mathsf{A}^n}\prod_{i=1}^n\langle x_i,u_i|T(a_i,c_i)|y_i,u_{i+1}\rangle\|\ddot{a}_i\|^{\dot{\lambda}}f_\lambda(u_{i+1}-u_i)\,\mathrm{d}(u_1,\dots,u_n)\\&=\Big\|\prod_{i=1}^n\ddot{a}_i\Big\|^{\dot{\lambda}}\int_{\mathsf{A}^n}\prod_{i=1}^n\langle x_i,u_i|T(a_i,c_i)|y_i,u_{i+1}\rangle\frac{f_\lambda(u_{i+1})}{f_\lambda(u_i)}\,\mathrm{d}(u_1,\dots,u_n)=\|\ddot{w}_e\|^{\dot{\lambda}}\mathcal{F}_{F,SP_n,\theta}.\end{aligned} \tag{68}$$

This result implies the following gauge invariance property of the partition function.

Lemma 5.2. The partition function $\mathcal{F}_F$ is invariant under any gauge transformation associated with an internal edge, provided the B-component of the total angle around that edge is of unit norm (in particular, this is the case for a balanced edge corresponding to the total angle $(2,1)$).

6. Computations

6.1. **Preliminaries.** To compute the invariant defined in the previous section, we fix a triangulation X with N ordered tetrahedra and $2N$ faces. We assign variables $z_0,\dots,z_{N-1}\in\mathsf{A}$ to each tetrahedron and $x_0,\dots,x_{2N-1}\in\mathsf{A}$ to each face. Then, we integrate first over the x-variables and then over the z-variables. There are two δ-function linear equations per each tetrahedron, giving rise to a total of $2N$ linear equations for x. Typically, we can solve those uniquely, and hence express x as a $\mathbb{Q}$-linear combination of z. Doing so, we express the invariant as an integral over A^N.

6.2. **The trefoil knot.** Let X be an ideal triangulation of the complement of the trefoil knot in S^3 with two positive tetrahedra T_0 and T_1 with the edge and face identifications given by

tet \ edge	01	02	03	12	13	23
0	0	0	1	0	0	0
1	0	0	1	0	0	0

tet \ face	012	013	023	123
0	0	1	2	3
1	3	2	1	0

(69)

Labeling each face by a variable x_j for $j = 0, \dots, 3$, the delta function equations of the kinematical kernel are

$$-x_0 + x_1 + z_0 = 0, \qquad x_1 - x_2 + x_3 = 0, \qquad x_2 - x_3 + z_1 = 0, \qquad x_0 - x_1 + x_2 = 0$$

with unique solution

$$(x_0, x_1, x_2, x_3) = (z_0 - z_1, -z_1, -z_0, -z_0 + z_1) .$$

The kinematical kernel is

$$K_X(z) = \frac{\langle z_0; z_1\rangle^2}{\langle z_0; z_0\rangle\langle z_1; z_1\rangle} = \prod_{0\le i,j\le 1} \langle z_i; z_j\rangle^{\frac{1}{2}Q_{ij}}, \qquad Q = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} . \tag{70}$$

Indeed, $K_X(z) = \langle x_3; z_0\rangle\langle x_0; z_1\rangle = \langle -z_0 + z_1; z_0\rangle\langle z_0 - z_1; z_1\rangle = \langle z_0; z_1\rangle^2\langle z_0; z_0\rangle^{-1}\langle z_1; z_1\rangle^{-1}$. Writing

$$z_0 = (\alpha, x), \qquad z_1 = (\beta, y)$$

we obtain that

$$K_X(z) = \alpha(x^{-2}y^2)\, \beta(x^2y^{-2}) .$$

The balancing condition on the edges is

$$c_0 + c_1 = (2, 1) \tag{71}$$

from which we can express all angles in terms of a_0, a_1 and c_1. After balancing the angles at all edges, the angle holonomy of half of the longitude is given by

$$\lambda = 2a_0 - 2a_1 - c_0 . \tag{72}$$

The invariant is given by

$$\mathcal{F}_{F,X,\theta} = \int_{\hat{\mathsf{B}}^2\times\mathsf{B}^2} (\alpha(y/x)\beta(x/y))^2\Psi_{a_0,c_0}(\alpha, x)\Psi_{a_1,c_1}(\beta, y)\, \mathrm{d}(\alpha, \beta, x, y) \tag{73}$$

After changing variables to $x \mapsto x/\ddot{a}_0$, $y \mapsto y/\ddot{a}_1$ and integrating out the α and β variables using (15) and (17), the integral becomes

$$\begin{aligned}
\mathcal{F}_{F,X,\theta} &= \int_{\hat{\mathsf{B}}^2\times\mathsf{B}^2} \alpha\Big(\frac{y^2/\ddot{c}_0}{(1-\ddot{a}_0x)x^2}\Big)\frac{\|\ddot{a}_0x\|^{\dot{c}_0}}{\|1-\ddot{a}_0x\|^{1-\dot{a}_0}}\beta\Big(\frac{x^2/\ddot{c}_1}{(1-\ddot{a}_1y)y^2}\Big)\frac{\|\ddot{a}_1y\|^{\dot{c}_1}}{\|1-\ddot{a}_1y\|^{1-\dot{a}_1}}\, \mathrm{d}(\alpha, \beta, x, y)\\
&= \int_{\hat{\mathsf{B}}^2\times\mathsf{B}^2} \alpha\Big(\frac{y^2(\ddot{a}_0/\ddot{a}_1)^2/\ddot{c}_0}{(1-x)x^2}\Big)\frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\beta\Big(\frac{x^2(\ddot{a}_1/\ddot{a}_0)^2/\ddot{c}_1}{(1-y)y^2}\Big)\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\, \mathrm{d}(\alpha, \beta, x, y)\\
&\overset{(15),(17)}{=} \int_{\mathsf{B}^2} \delta_F\Big(\frac{y^2(\ddot{a}_0/\ddot{a}_1)^2/\ddot{c}_0}{(1-x)x^2} - 1\Big)\frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\delta_F\Big(\frac{x^2(\ddot{a}_1/\ddot{a}_0)^2/\ddot{c}_1}{(1-y)y^2} - 1\Big)\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\, \mathrm{d}(x, y)\\
&\overset{(14)}{=} \int_{F^2} \delta_F\Big(\frac{y^2(\ddot{a}_0/\ddot{a}_1)^2/\ddot{c}_0}{(1-x)x^2} - 1\Big)\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\delta_F\Big(\frac{x^2(\ddot{a}_1/\ddot{a}_0)^2/\ddot{c}_1}{(1-y)y^2} - 1\Big)\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\, \mathrm{d}(x, y) .
\end{aligned}$$

Using the edge-balancing equations we find that the delta function equations take the form

$$1 - x = \varepsilon x^{-2}y^2, \qquad 1 - y = \varepsilon^{-1}\, x^2y^{-2} \tag{74}$$

for $\varepsilon := \ddot{\lambda}$. This, together with the edge-balancing conditions implies that

$$\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}} = \frac{1}{\|\varepsilon\|^{\dot{\theta}_\mu}}\|x\|^{e_0}\|y\|^{e_1} \tag{75}$$

where

$$\begin{aligned}(e_0, e_1) &:= (\dot{c}_0 - 1, \dot{c}_1 - 1) - (1 - \dot{a}_0, 1 - \dot{a}_1)Q \\ &= (\dot{c}_0, \dot{c}_1) + (\dot{a}_0, \dot{a}_1)Q - (1,1) - Q.(1,1) \\ &= \dot{\lambda}(-1,1) + (-1,-1)\end{aligned} \tag{76}$$

and

$$\dot{\mu} = (1,-1).(1-\dot{a}_0, 1-\dot{a}_1) = -\dot{a}_0 + \dot{a}_1\,. \tag{77}$$

The above discussion, combined with Lemma (2.8), implies that $I_F(3_1, \theta)$ depends only on $\lambda = (\dot{\lambda}, \ddot{\lambda})$ and $\dot{\theta}_\mu$, and is given by

$$\mathcal{F}_{F,3_1}(\varepsilon, s, t) = \frac{1}{\|\varepsilon\|^t} \sum_{(x,y)\in X_\varepsilon(F^\times)} \frac{\|x\|^{-s-1}\|y\|^{s-1}}{\|\mathrm{Jac}(f(x,y))\|} \tag{78}$$

where $\varepsilon = \ddot{\lambda} \in F^\times$, $s = \dot{\lambda} \in \mathbb{R}$ and $t = \dot{\mu} \in \mathbb{R}$.

Remark 6.1. Note that the invariant of 3_1 is chiral. Indeed, the mirror image of 3_1, obtained by changing the orientation, corresponds to transformation

$$\ddot{\lambda} \mapsto 1/\ddot{\lambda}, \quad \dot{\lambda} \mapsto \dot{\lambda}, \quad \dot{\mu} \mapsto -\dot{\mu}\,.$$

Thus, our invariant is sensitive to the orientation of X.

6.3. **The figure-eight knot.** Let X be an ideal triangulation of the complement of the figure-eight knot in S^3 with one positive tetrahedron T_0 and one negative tetrahedron T_1 with the edge and face identifications given by

tet \ edge	01	02	03	12	13	23
0	0	1	0	1	1	0
1	1	0	1	0	0	1

tet \ face	012	013	023	123
0	0	1	2	3
1	2	3	0	1

(79)

Labeling each face by a variable x_j for $j = 0, \dots, 3$, the delta function equations of the kinematical kernel are

$$-x_0 + x_1 + z_0 = 0, \qquad -x_2 + x_3 + z_1 = 0,$$
$$x_1 - x_2 + x_3 = 0, \qquad -x_0 + x_1 + x_3 = 0,$$

with unique solution

$$(x_0, x_1, x_2, x_4) = (z_0 + z_1, z_1, z_0 + z_1, z_0)\,.$$

The kinematical kernel takes the form

$$K_X(z) = \frac{\langle z_0; z_0\rangle}{\langle z_1; z_1\rangle} = \prod_{0\le i,j\le 1} \langle z_i; z_j\rangle^{\frac{1}{2}Q_{ij}}, \qquad Q = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\,. \tag{80}$$

Writing

$$z_0 = (\alpha, x), \qquad z_1 = (\beta, y)$$

we obtain that

$$K_X(z) = \alpha(x^2)\,\beta(y^{-2})\,.$$

The balancing condition on the edges is

$$2a_0 + c_0 + 2b_1 + c_1 = (2,1) \tag{81}$$

from which we can express all angles in terms of a_0, a_1 and c_1. After balancing the angles at all edges, the angle holonomy of half of the longitude is given by

$$\lambda = 2a_0 + c_0 - \varpi = 2a_0 + c_0 - \varpi\,. \tag{82}$$

The integral is given by

$$\begin{aligned}
\mathcal{F}_{F,X,\theta} &= \int_{\hat{\mathsf{B}}^2\times\mathsf{B}^2} \alpha\Big(\frac{x^2}{(1-\ddot{a}_0 x)\ddot{c}_0}\Big)\beta\Big(\frac{y^{-2}(1-y\ddot{a}_1^{-1})}{\ddot{c}_1}\Big)\frac{\|\ddot{a}_0 x\|^{\dot{c}_0}}{\|1-\ddot{a}_0 x\|^{1-\dot{a}_0}}\frac{\|\ddot{a}_1 y\|^{\dot{c}_1}}{\|1-\ddot{a}_1 y\|^{1-\dot{a}_1}}\,\mathrm{d}(\alpha,\beta,x,y)\\
&= \int_{\hat{\mathsf{B}}^2\times\mathsf{B}^2} \alpha\Big(\frac{x^2}{(1-x)\ddot{c}_0\ddot{a}_0^2}\Big)\beta\Big(\frac{1-y}{y^2\ddot{c}_1\ddot{a}_1^2}\Big)\frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\,\mathrm{d}(\alpha,\beta,x,y)\\
&\overset{(15),(17)}{=} \int_{\mathsf{B}^2} \delta_F\Big(\frac{x^2}{(1-x)\ddot{c}_0\ddot{a}_0^2}-1\Big)\delta_F\Big(\frac{1-y}{y^2\ddot{c}_1\ddot{a}_1^2}-1\Big)\frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\,\mathrm{d}(x,y)\\
&\overset{(14)}{=} \int_{F^2} \delta_F\Big(\frac{x^2}{(1-x)\ddot{c}_0\ddot{a}_0^2}-1\Big)\delta_F\Big(\frac{1-y}{y^2\ddot{c}_1\ddot{a}_1^2}-1\Big)\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\,\mathrm{d}(x,y)\\
&= \int_{F^2} \delta_F\Big(-\frac{x^2}{(1-x)\ddot{\lambda}}-1\Big)\delta_F\Big(-\frac{1-y}{y^2\ddot{\lambda}}-1\Big)\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\,\mathrm{d}(x,y)\,,
\end{aligned} \tag{83}$$

where the second equality comes from a change of variables $x \mapsto \ddot{x}/a_0$ and $y \mapsto y\ddot{a}_1$, the third equality comes from integrating out the α, β and γ variables using (15) and (17), and the last equality follows from the edge-balancing equations (81).

The vanishing of the arguments of the three delta functions gives the gluing equations

$$X_\varepsilon \;:\; 1-x = -\varepsilon^{-1}x^2, \qquad 1-y = -\varepsilon\, y^2 \tag{84}$$

defining a 1-dimensional affine scheme X equipped with a map $X(F) \to F^\times$ with fiber X_ε for $\varepsilon = \ddot{\lambda} \in F^\times$. Using Equations (84), the edge-balancing equations and the meridian $\mu = a_0 - a_1$, we compute that

$$\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}} = \frac{1}{\|\ddot{\lambda}\|^{\dot{\mu}}}\|x\|^{\dot{\lambda}-2}\|y\|^{\dot{\lambda}+2}\,. \tag{85}$$

The above discussion, combined with Lemma (2.8), implies that $I_F(4_1,\theta)$ depends only on $\lambda = (\dot{\lambda},\ddot{\lambda})$ and $\dot{\mu}$, and is given by

$$\mathcal{F}_{F,4_1}(\varepsilon,s,t) = \frac{1}{\|\varepsilon\|^t}\sum_{(x,y)\in X_\varepsilon(F^\times)}\frac{\|x\|^{\dot{\lambda}-2}\|y\|^{\dot{\lambda}+2}}{\|\mathrm{Jac}(f(x,y))\|} \tag{86}$$

where $\varepsilon = \ddot{\lambda} \in F^\times$, $s = \dot{\lambda} \in \mathbb{R}$ and $t = \dot{\mu} \in \mathbb{R}$ and $f = (f_1, f_2)$ where each f_i is the argument of the delta function equations.

6.4. **The 5_2 knot.** Consider the triangulation of the complement of the 5_2 knot with three positively oriented tetrahedra T_j for $j = 0, 1, 2$ with the edge and face-pairings given by

tet \ edge	01	02	03	12	13	23
0	0	1	1	0	2	2
1	2	0	1	1	1	2
2	0	2	1	2	0	1

tet \ face	012	013	023	123
0	0	1	2	3
1	4	5	1	2
2	3	0	5	4

(87)

Labeling each face by a variable x_j for $j = 0, \dots, 5$, the delta function equations of the kinematical kernel are

$$-x_0 + x_1 + z_0 = 0, \qquad -x_4 + x_5 + z_1 = 0, \qquad x_0 - x_3 + z_2 = 0,$$
$$x_1 - x_2 + x_3 = 0, \qquad -x_1 + x_2 + x_5 = 0, \qquad x_0 + x_4 - x_5 = 0,$$

with unique solution

$$(x_0, x_1, x_2, x_4, x_4, x_5) = (-z_1, -z_0 - z_1, -z_0 - 2z_1 + z_2, -z_1 + z_2, 2z_1 - z_2, z_1 - z_2)\,.$$

The kinematical kernel is

$$K_X(z) = \prod_{0 \leq i,j \leq 2} \langle z_i; z_j \rangle^{\frac{1}{2} Q_{ij}}, \qquad Q = \begin{pmatrix} 0 & -2 & 1 \\ -2 & -4 & 3 \\ 1 & 3 & -2 \end{pmatrix}. \tag{88}$$

Writing

$$z_0 = (\alpha, x), \qquad z_1 = (\beta, y), \qquad z_2 = (\gamma, z)$$

we obtain that

$$K_X(z) = \alpha(y^{-2} z)\, \beta(x^{-2} y^{-4} z^3)\, \gamma(x y^3 z^{-2})\,.$$

The balancing condition on two edges edges is

$$b_0 + c_0 + b_1 + 2c_1 + a_2 + c_2 = (2, 1), \qquad a_0 + b_0 + 2a_1 + b_2 + c_2 = (2, 1) \tag{89}$$

from which we can express all angles in terms of a_0, a_1, a_2 and c_1. After balancing the angles at all edges, the angle holonomy of half of the longitude is given by

$$\lambda = 2a_0 + 4a_1 - 3a_2 - c_1\,. \tag{90}$$

The integral is given by

$$
\begin{aligned}
\mathcal{F}_{F,X,\theta} &= \int_{\hat{\mathsf{B}}^3\times\mathsf{B}^3} \alpha\Big(\frac{z}{(1-\ddot{a}_0x)\ddot{c}_0y^2}\Big)\beta\Big(\frac{z^3}{(1-\ddot{a}_1y)\ddot{c}_1x^2y^4}\Big)\gamma\Big(\frac{xy^3}{(1-\ddot{a}_2y)\ddot{c}_2z^2}\Big) \\
&\quad\times \frac{\|\ddot{a}_0x\|^{\dot{c}_0}}{\|1-\ddot{a}_0x\|^{1-\dot{a}_0}}\frac{\|\ddot{a}_1y\|^{\dot{c}_1}}{\|1-\ddot{a}_1y\|^{1-\dot{a}_1}}\frac{\|\ddot{a}_2z\|^{\dot{c}_2}}{\|1-\ddot{a}_2z\|^{1-\dot{a}_2}}\,\mathrm{d}(\alpha,\beta,\gamma,x,y,z) \\
&= \int_{\hat{\mathsf{B}}^3\times\mathsf{B}^3} \alpha\Big(\frac{\ddot{c}_0^{-1}\ddot{a}_1^2\ddot{a}_2^{-1}z}{(1-x)y^2}\Big)\beta\Big(\frac{\ddot{c}_1^{-1}\ddot{a}_0^2\ddot{a}_1^4\ddot{a}_2^{-3}z^3}{(1-y)x^2y^4}\Big)\gamma\Big(\frac{\ddot{c}_2^{-1}\ddot{a}_0^{-1}\ddot{a}_1^{-3}\ddot{a}_2^2xy^3}{(1-y)z^2}\Big) \\
&\quad\times \frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\frac{\|z\|^{\dot{c}_2}}{\|1-z\|^{1-\dot{a}_2}}\,\mathrm{d}(\alpha,\beta,\gamma,x,y,z) \\
&\overset{(15),(17)}{=} \int_{\mathsf{B}^3} \delta_F\Big(\frac{\ddot{c}_0^{-1}\ddot{a}_1^2\ddot{a}_2^{-1}z}{(1-x)y^2}-1\Big)\delta_F\Big(\frac{\ddot{c}_1^{-1}\ddot{a}_0^2\ddot{a}_1^4\ddot{a}_2^{-3}z^3}{(1-y)x^2y^4}-1\Big)\delta_F\Big(\frac{\ddot{c}_2^{-1}\ddot{a}_0^{-1}\ddot{a}_1^{-3}\ddot{a}_2^2xy^3}{(1-y)z^2}-1\Big) \\
&\quad\times \frac{\|x\|^{\dot{c}_0}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1}}{\|1-y\|^{1-\dot{a}_1}}\frac{\|z\|^{\dot{c}_2}}{\|1-z\|^{1-\dot{a}_2}}\,\mathrm{d}(x,y,z) \\
&\overset{(14)}{=} \int_{F^3} \delta_F\Big(\frac{\ddot{c}_0^{-1}\ddot{a}_1^2\ddot{a}_2^{-1}z}{(1-x)y^2}-1\Big)\delta_F\Big(\frac{\ddot{c}_1^{-1}\ddot{a}_0^2\ddot{a}_1^4\ddot{a}_2^{-3}z^3}{(1-y)x^2y^4}-1\Big)\delta_F\Big(\frac{\ddot{c}_2^{-1}\ddot{a}_0^{-1}\ddot{a}_1^{-3}\ddot{a}_2^2xy^3}{(1-y)z^2}-1\Big) \\
&\quad\times \frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\frac{\|z\|^{\dot{c}_2-1}}{\|1-z\|^{1-\dot{a}_2}}\,\mathrm{d}(x,y,z)\,.
\end{aligned}
\tag{91}
$$

where the second equality comes from a change of variables $x \mapsto \ddot{x}/a_0$, $y \mapsto y/\ddot{a}_1$ and $z \mapsto z/\ddot{a}_2$ and the third equality comes from integrating out the α, β and γ variables using (15) and (17).

The $F^\times$-component of the edge-balancing equations (89) implies that

$$(\ddot{c}_0^{-1}\ddot{a}_1^2\ddot{a}_2^{-1},\ddot{c}_1^{-1}\ddot{a}_0^2\ddot{a}_1^4\ddot{a}_2^{-3},\ddot{c}_2^{-1}\ddot{a}_0^{-1}\ddot{a}_1^{-3}\ddot{a}_2^2) = (1,\ddot{\lambda},\ddot{\lambda}^{-1})\,. \tag{92}$$

This means that the vanishing of the arguments of the three delta functions gives the gluing equations

$$X_\varepsilon \;:\; 1-x=y^{-2}z, \qquad 1-y=\varepsilon\, x^{-2}y^{-4}z^3, \qquad 1-z=\varepsilon^{-1}\, xy^3z^{-2} \tag{93}$$

definining a 1-dimensional affine scheme X equipped with a map $X(F)\to F^\times$ with fiber X_ε for $\varepsilon=\ddot{\lambda}\in F^\times$.

On the other hand, the $\mathbb{R}$-component of the edge-balancing equations (89) implies that

$$-(\dot{c}_0,\dot{c}_1,\dot{c}_2)-(\dot{a}_0,\dot{a}_1,\dot{a}_2)Q = \dot{\lambda}(0,1,-1)\,. \tag{94}$$

This, combined with Equations (93), implies that

$$\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\frac{\|z\|^{\dot{c}_2-1}}{\|1-z\|^{1-\dot{a}_2}} = \frac{1}{\|\varepsilon\|^{\dot{\mu}}}\|x\|^{e_0}\|y\|^{e_1}\|z\|^{e_2} \tag{95}$$

where

$$
\begin{aligned}
(e_0,e_1,e_2) &:= (\dot{c}_0-1,\dot{c}_1-1,\dot{c}_2-1)-(1-\dot{a}_0,1-\dot{a}_1,1-\dot{a}_2)Q \\
&= (\dot{c}_0,\dot{c}_1,\dot{c}_2)+(\dot{a}_0,\dot{a}_1,\dot{a}_2)Q-(1,1,1)-Q.(1,1,1) \\
&= \dot{\lambda}(0,1,-1)+(0,2,-3)
\end{aligned}
\tag{96}
$$

and

$$\dot{\theta}_\mu = (0,1,-1).(1-\dot{a}_0, 1-\dot{a}_1, 1-\dot{a}_2) = -\dot{a}_1 + \dot{a}_2 \,. \tag{97}$$

The above discussion, combined with Lemma (2.8), implies that $I_F(5_2, \theta)$ depends only on $\lambda = (\dot{\lambda}, \ddot{\lambda})$ and $\dot{\mu}$, and is given by

$$\mathcal{F}_{F,5_2}(\varepsilon, s, t) = \frac{1}{\|\varepsilon\|^t} \sum_{(x,y,z)\in X_\varepsilon(F^\times)} \frac{\|x\|^{s+2}\|y\|^{-s-3}}{\|\mathrm{Jac}(f(x,y,z))\|} \tag{98}$$

where $\varepsilon = \ddot{\lambda} \in F^\times$, $s = \dot{\lambda} \in \mathbb{R}$ and $t = \dot{\mu} \in \mathbb{R}$.

Let us comment on the geometry of the 1-dimensional scheme X defined by Equations (93). The projection map $X(F) \to F^\times$ which sends (x, y, z, ε) to ε has degree 7, equal to the degree of the A-polynomial of 5_2 with respect to M. In fact, the map $X(F) \to F^\times$ factors through a map $X(F) \to X_A(F) \xrightarrow{L} F^\times$ where $X(F) \to X_A(F)$ is a birational map, $A(M, L)$ is the A-polynomial of 5_2 and X_A is the corresponding curve.

6.5. **The $(-2, 3, 7)$ pretzel knot.** We next discuss the case of the $(-2, 3, 7)$ pretzel knot. Although this knot has is scissors congruent and the same cubic trace field as the 5_2 knot, its character variety is more interesting. Consider the triangulation of the complement of the $(-2, 3, 7)$ pretzel knot with isometry signature `eLAkaccddjgnqw`. It has four positively oriented tetrahedra T_j for $j = 0, 1, 2, 4$ with the edge and face-pairings given by

tet \ edge	01	02	03	12	13	23
0	0	1	2	0	1	0
1	3	0	2	1	3	1
2	1	3	2	1	0	3
3	3	1	0	3	1	3

tet \ face	012	013	023	123
0	0	1	2	3
1	3	4	1	5
2	5	2	4	6
3	7	3	6	7

(99)

Using this data, and following the steps of the computation as was done in Section 6.4 for the 5_2 knot, we find that the kinematical kernel is

$$K_X(z) = \prod_{0\le i,j\le 3} \langle z_i; z_j \rangle^{\frac{1}{2}Q_{ij}}, \qquad Q = \begin{pmatrix} 2 & -1 & -1 & -2 \\ -1 & -8 & 9 & 1 \\ -1 & 9 & -8 & 1 \\ -2 & 1 & 1 & 4 \end{pmatrix} . \tag{100}$$

The edge-balancing equations (with one edge removed) are

$$\begin{aligned} 2a_0 + b_1 + b_2 + c_0 + c_3 &= (2,1), \\ a_1 + a_2 + 2b_0 + 2b_3 + c_1 + c_2 &= (2,1), \\ c_0 + c_1 + c_2 &= (2,1), \end{aligned} \tag{101}$$

which may be solved to express all angles in terms of a_0, a_1, a_2, a_3 and c_1. After balancing the angles at all edges, the angle holonomy of half of the longitude is given by

$$\lambda = a_0 + 8a_1 - 9a_2 - a_3 - c_1 \,. \tag{102}$$

The $F^\times$-component of the edge-balancing equations (101) implies that

$$(\ddot{c}_0^{-1}\ddot{a}_0^{-2}\ddot{a}_1\ddot{a}_2\ddot{a}_3^2, \ddot{c}_1^{-1}\ddot{a}_0\ddot{a}_1^8\ddot{a}_2^{-9}\ddot{a}_3^{-1}, \ddot{c}_2^{-1}\ddot{a}_0\ddot{a}_1^{-9}\ddot{a}_2^8\ddot{a}_3^{-1}, \ddot{c}_3^{-1}\ddot{a}_0^2\ddot{a}_1^{-1}\ddot{a}_2^{-1}\ddot{a}_3^{-4}) = (1, \ddot{\lambda}, \ddot{\lambda}^{-1}, 1) \,, \tag{103}$$

giving the gluing equations

$$X_\varepsilon \ : \quad \begin{aligned} & 1-x = x^2y^{-1}z^{-1}w^{-2}, \quad 1-y = \varepsilon\, x^{-1}y^{-8}z^9w, \\ & 1-z = \varepsilon^{-1}\, x^{-1}y^{-9}z^8, \quad 1-w = x^{-2}yzw^4\,. \end{aligned} \tag{104}$$

On the other hand, the $\mathbb{R}$-component of the edge-balancing equations (101) implies that

$$-(\dot{c}_0,\dot{c}_1,\dot{c}_2,\dot{c}_3) - (\dot{a}_0,\dot{a}_1,\dot{a}_2,\dot{a}_3)Q = \dot{\lambda}(0,1,-1,0)\,. \tag{105}$$

This, combined with Equations (104), implies that

$$\frac{\|x\|^{\dot{c}_0-1}}{\|1-x\|^{1-\dot{a}_0}}\frac{\|y\|^{\dot{c}_1-1}}{\|1-y\|^{1-\dot{a}_1}}\frac{\|z\|^{\dot{c}_2-1}}{\|1-z\|^{1-\dot{a}_2}}\frac{\|w\|^{\dot{c}_3-1}}{\|1-z\|^{1-\dot{a}_3}} = \frac{1}{\|\varepsilon\|^{\dot{\mu}}}\|x\|\,\|y\|^{-\dot{\lambda}-2}\|z\|^{\dot{\lambda}}\|w\|^{-3}\,. \tag{106}$$

The above, combined with (2.8), implies that $I_F((-2,3,7),\theta)$ depends only on $\lambda = (\dot{\lambda},\ddot{\lambda})$ and $\dot{\mu}$, and is given by

$$\mathcal{F}_{F,(-2,3,7)}(\varepsilon,s,t) = \frac{1}{\|\varepsilon\|^t}\sum_{(x,y,z,w)\in X_\varepsilon(F^\times)}\frac{\|x\|\,\|y\|^{-s-2}\|z\|^s\|w\|^{-3}}{\|\mathrm{Jac}(f(x,y,z,w))\|} \tag{107}$$

where $\varepsilon = \ddot{\lambda} \in F^\times$, $s = \dot{\lambda} \in \mathbb{R}$ and $t = \dot{\mu} \in \mathbb{R}$.

The geometry of the 1-dimensional scheme X defined by Equations (104) is the following. The projection map $X(F) \to F^\times$ which sends (x,y,z,w,ε) to ε has degree 55, equal to that of the A-polynomial of the $(-2,3,7)$ knot with respect to the meridian. and in fact, the map $X \to F^\times$ factors through a map $X(F) \to X_A(F) \xrightarrow{L} F^\times$ where $A(M,L)$ is the A-polynomial of $(-2,3,7)$ pretzel knot and X_A is the corresponding curve.

6.6. **Point counts of zero-dimensional schemes.** The distributional invariant $\mathcal{F}_F$ computed above leads, after evaluation at a test function, to point-counts of zero-dimensional schemes. In this section we list three elementary facts about point counts of reduced zero-dimensional schemes X which were pointed out to us by Frank Calegari.

Suppose X is a reduced 0-dimensional scheme over $\mathbb{Q}$. Then, there exists a finite set S of rational primes such that for all local fields F with discrete valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$ where q is a power of a prime not in S, we have:

$$|X(F)| = |X(\mathcal{O}_F)| = |X(\mathbb{F}_q)|\,. \tag{108}$$

The first equality follows from the fact that X can be spread as a 0-dimensional scheme on $\mathbb{Z}[1/S]$, and the second follows from Hensel's lemma.

The second fact is that the point counts $|X(\mathbb{F}_p)|$ for $p \not\in S$ determine the point count $X(\overline{\mathbb{Q}})$. This follows from Chebotarev density theorem.

A third fact is that when a component $Y(\overline{\mathbb{Q}})$ of $X(\overline{\mathbb{Q}})$ is defined over a Galois field K which is disjoint from that of the other components, then we can find a positive density set of primes such that

$$|X(\mathbb{Q}_p)| = |Y(\mathbb{Q}_p)|\,, \tag{109}$$

i.e., the $\mathbb{Q}_p$-count on X equals to that of the Y and it is isolated from that of $X \setminus Y$.

7. A pair of edge-type generalized TQFTs

In the previous Section 5 we constructed a face-type generalized TQFT using a quantum dilogarithm φ (24) on the Gaussian group $\mathsf{A} = \hat{\mathsf{B}} \times \mathsf{B}$ (with $\mathsf{B} = F^\times$) associated to a local field F. In this section we construct a pair of edge-type generalized TQFTs using the Weil transform of φ, one with respect to $\hat{\mathsf{B}}$ and another with respect to B. In favorable circumstances each of these Weil transforms leads to a edge-type generalized TQFT whose states are placed in the edges of the tetrahedra (and not in the faces, as was the case of Section 5). When additional symmetries are found, the triangulations are unordered (yet oriented) and the weights of the tetrahedra manifestly depend only on the combinatorial information of the triangulation encoded by the Neumann–Zagier matrices [27]. These edge-type generalized TQFTs are sometimes called of Turaev–Viro type, because their partition functions are edge state-integrals whose state-variables are on the edges of the ideal triangulation, just like the original Turaev–Viro invariants [32]. Two examples of such generalized TQFTs using the Gaussian groups $\mathbb{R}\times\mathbb{R}$ and $\mathbb{Z}\times S^1$ from Examples 2.4 and 2.5 are described in [24] and [12], respectively.

Our goal in this section is to define, in addition to the face-type generalized TQFT of Section 5, two more edge-type generalized TQFTs using the two Weil transformations.

7.1. The $\hat{\mathsf{B}}$-Weil transform.

In this section we compute the $\hat{\mathsf{B}}$-Weil transform of the quantum dilogarithm (44) following the Appendix B of [12]. It is the function $g_{a,c} : \mathsf{B}^2 \to \mathbb{C}$ defined by

$$g_{a,c}(x,z) = g_{a,c}((\alpha,x),(\gamma,z)) := \alpha(z)\int_{\hat{\mathsf{B}}} \bar{\Psi}_{a,c}(-\alpha-\beta,1/x)\langle(\beta,1);(\gamma,z)\rangle\,\mathrm{d}\beta \tag{110}$$

where the independence of the above definition from α and γ is assured from the invariance properties of the Weil transformation (see Appendix B of [12] for details).

Lemma 7.1. We have:

$$g_{a,c}(x,z) = f_{\dot{a},\dot{c}}(1/(x\ddot{a}), z\ddot{c}), \quad f_{\dot{a},\dot{c}}(x,z) := \|x\|^{\dot{c}}\|z\|^{\dot{a}}\delta_F(x+z-1)\,. \tag{111}$$

The function $g_{a,c}$ satisfies the symmetries

$$g_{a,c}(x,z) = g_{b,a}(y,x) = g_{c,b}(z,y), \quad a+b+c=\varpi, \quad xyz=1. \tag{112}$$

Proof. By using the formula

$$\bar{\Psi}_{a,c}(\alpha,x) = \alpha\big((1-x/\ddot{a})/\ddot{c}\big)\frac{\|x/\ddot{a}\|^{\dot{c}}}{\|1-x/\ddot{a}\|^{1-\dot{a}}},$$

we compute:

$$\begin{aligned}
g_{a,c}(x,z) &= \alpha(z)\int_{\hat{\mathsf{B}}}(-\alpha-\beta)\big((1-(x\ddot{a})^{-1})/\ddot{c}\big)\frac{\|1/(x\ddot{a})\|^{\dot{c}}}{\|1-(x\ddot{a})^{-1}\|^{1-\dot{a}}}\beta(z)\,\mathrm{d}\beta \\
&= \int_{\hat{\mathsf{B}}}(\alpha+\beta)\Big(\frac{z\ddot{c}}{1-(x\ddot{a})^{-1}}\Big)\frac{\|x\ddot{a}\|^{-\dot{c}}}{\|1-(x\ddot{a})^{-1}\|^{1-\dot{a}}}\,\mathrm{d}\beta = \delta_{\mathsf{B}}\Big(\frac{z\ddot{c}}{1-(x\ddot{a})^{-1}}\Big)\frac{\|x\ddot{a}\|^{-\dot{c}}}{\|1-(x\ddot{a})^{-1}\|^{1-\dot{a}}} \\
&= \delta_F\Big(\frac{z\ddot{c}}{1-(x\ddot{a})^{-1}}-1\Big)\frac{\|x\ddot{a}\|^{-\dot{c}}}{\|z\ddot{c}\|^{1-\dot{a}}} = \frac{\|z\ddot{c}\|^{\dot{a}}}{\|x\ddot{a}\|^{\dot{c}}}\delta_F\Big(z\ddot{c}+\frac{1}{x\ddot{a}}-1\Big) = f_{\dot{a},\dot{c}}(1/(x\ddot{a}),z\ddot{c})\,.
\end{aligned}$$

This concludes the proof of Equation (111). The rest follows easily from part this equation using part (a) of Lemma 2.7. ☐

7.2. **The B-edge TQFT.** Based on the function $g_{a,c}(x,z)$, one can formulate a generalized TQFT model of the Turaev–Viro type (like the ones in [24] and [12]) using *unordered* triangulations, and placing integration variables (lying in $F^\times$) at the edges of triangulations (as opposed to the faces and tetrahedra done in the previous section), with the angle-dependent symmetric tetrahedral Boltzmann weights of a tetrahedron T obtained under the substitutions $x \to \frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}$ and $z \to \frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}$ into $g_{a,c}(x,z)$. This results into the tetrahedral weight

$$W_{a,c}(T,x) = \|x_{0,1}x_{2,3}\ddot{c}\|^{\dot{a}}\|x_{0,2}x_{1,3}\|^{\dot{b}}\|x_{0,3}x_{1,2}/\ddot{a}\|^{\dot{c}}\delta_F\Big(x_{0,1}x_{2,3}\ddot{c} + \frac{x_{0,3}x_{1,2}}{\ddot{a}} - x_{0,2}x_{1,3}\Big) \tag{113}$$

where $x_{i,j}$ is the B-valued state variable on the geometric edge opposite to the edge $\partial_i\partial_j T$ of the tetrahedron T. Indeed, when $z = \frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}$, we have:

$$\begin{aligned}
\frac{\|z\ddot{c}\|^{\dot{a}}}{\|x\ddot{a}\|^{\dot{c}}}\delta_F\Big(z\ddot{c} + \frac{1}{x\ddot{a}} - 1\Big) &= \frac{\|\frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}\ddot{c}\|^{\dot{a}}}{\|\frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}\ddot{a}\|^{\dot{c}}}\delta_F\Big(\frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}\ddot{c} + \frac{1}{\frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}\ddot{a}} - 1\Big) \\
&= \frac{\|x_{0,1}x_{2,3}\ddot{c}\|^{\dot{a}}\|x_{0,2}x_{1,3}\|^{1-\dot{a}}}{\|x_{0,2}x_{1,3}\ddot{a}\|^{\dot{c}}\|x_{0,3}x_{1,2}\|^{-\dot{c}}}\delta_F\Big(x_{0,1}x_{2,3}\ddot{c} + \frac{x_{0,3}x_{1,2}}{\ddot{a}} - x_{0,2}x_{1,3}\Big) \\
&= \|x_{0,1}x_{2,3}\ddot{c}\|^{\dot{a}}\|x_{0,2}x_{1,3}\|^{\dot{b}}\|x_{0,3}x_{1,2}/\ddot{a}\|^{\dot{c}}\delta_F\Big(x_{0,1}x_{2,3}\ddot{c} + \frac{x_{0,3}x_{1,2}}{\ddot{a}} - x_{0,2}x_{1,3}\Big) \\
&= \|x_{0,2}x_{1,3}\ddot{a}\|^{\dot{b}}\|x_{0,3}x_{1,2}\|^{\dot{c}}\|x_{0,1}x_{2,3}/\ddot{b}\|^{\dot{a}}\delta_F\Big(x_{0,2}x_{1,3}\ddot{a} + \frac{x_{0,1}x_{2,3}}{\ddot{b}} - x_{0,3}x_{1,2}\Big)
\end{aligned}$$

where the last equality reflects the invariance of the weight under the cyclic permutation $(1,a) \mapsto (2,b) \mapsto (3,c) \mapsto (1,a)$.

This results into a face-type generalized TQFT whose partition function we denote by $\mathcal{E}_{\mathsf{B}}$.

Remark 7.2. The arguments in the delta functions in (113) are very similar to the defining equations of Zickert's enhanced Ptolemy variety of an ideal triangulation $\mathcal{T}$, but modified by the angle data; see [34]. This fact can be the key point for explaining the possible relation between the generalized TQFT invariant of a 3-manifold constructed using the tetrahedral weight (113) and the A-polynomial of a knot [4]. We will illustrate this with the example of the 4_1 knot.

Example 7.3. The partition function of the B-edge generalized TQFT for the ideal triangulation of the complement of the figure eight knot with two ideal tetrahedra is given by the integral

$$\mathcal{E}_{\mathsf{B},4_1}(\lambda,\mu) = \int_{\mathsf{B}} g_{a_0,c_0}(x,1/x^2)\bar{g}_{a_1,c_1}(1/x,x^2)\,\mathrm{d}x \tag{114}$$

which we can calculate by using (111), the balancing condition $2a_0+c_0=2a_1+c_1$ and the definitions of the longitude $\lambda=a_0-b_0$ and the meridian $\mu=a_0-a_1$

$$\begin{aligned}\mathcal{E}_{\mathsf{B},4_1}(\lambda,\mu) &= \int_{\mathsf{B}} f_{\dot{a}_0,\dot{c}_0}\big(\tfrac{1}{x\ddot{a}_0},\tfrac{\ddot{c}_0}{x^2}\big) f_{\dot{a}_1,\dot{c}_1}\big(x\ddot{a}_1,\tfrac{x^2}{\ddot{c}_1}\big)\,\mathrm{d}x = \int_{\mathsf{B}} f_{\dot{a}_0,\dot{c}_0}\big(\tfrac{1}{x},\tfrac{\ddot{c}_0\ddot{a}_0^2}{x^2}\big) f_{\dot{a}_1,\dot{c}_1}\big(\tfrac{x\ddot{a}_1}{\ddot{a}_0},\tfrac{x^2}{\ddot{c}_1\ddot{a}_0^2}\big)\,\mathrm{d}x\\
&= \int_{\mathsf{B}} f_{\dot{a}_0,\dot{c}_0}\big(\tfrac{1}{x},-\tfrac{\ddot{\lambda}}{x^2}\big) f_{\dot{a}_1,\dot{c}_1}\big(\tfrac{x}{\ddot{\mu}},-\tfrac{x^2}{\ddot{\lambda}\ddot{\mu}^2}\big)\,\mathrm{d}x\\
&= \int_{\mathsf{B}} \frac{\|\ddot{\lambda}\|^{\dot{a}_0-\dot{a}_1}\|x\|^{2\dot{a}_1+\dot{c}_1}}{\|\ddot{\mu}\|^{2\dot{a}_1+\dot{c}_1}\|x\|^{2\dot{a}_0+\dot{c}_0}}\delta_{\mathsf{B}}\big(\tfrac{1}{x}-\tfrac{\ddot{\lambda}}{x^2}\big)\delta_{\mathsf{B}}\big(\tfrac{x}{\ddot{\mu}}-\tfrac{x^2}{\ddot{\lambda}\ddot{\mu}^2}\big)\,\mathrm{d}x\\
&= \int_F \frac{\|\ddot{\lambda}\|^{\dot{\mu}}\|x\|}{\|\ddot{\mu}\|^{\dot{\lambda}}}\delta_F\big(x-\ddot{\lambda}-x^2\big)\delta_F\big(x-\tfrac{x^2}{\ddot{\lambda}\ddot{\mu}}-\ddot{\mu}\big)\,\mathrm{d}x\\
&= \int_F \frac{\|\ddot{\lambda}\|^{\dot{\mu}}\|x\|}{\|\ddot{\mu}\|^{\dot{\lambda}}}\delta_F\big(x-\ddot{\lambda}-x^2\big)\delta_F\big(x-\tfrac{x-\ddot{\lambda}}{\ddot{\lambda}\ddot{\mu}}-\ddot{\mu}\big)\,\mathrm{d}x\\
&= \frac{\|\ddot{\lambda}\|^{\dot{\mu}}\|\ddot{\mu}-1/\ddot{\mu}\|}{\|\ddot{\mu}\|^{\dot{\lambda}}}\delta_F\Big((\ddot{\mu}-\ddot{\lambda})(1-\tfrac{1}{\ddot{\lambda}\ddot{\mu}})-(\ddot{\mu}-1/\ddot{\mu})^2\Big) = \frac{\|\ddot{\lambda}\|^{\dot{\mu}}}{\|\ddot{\mu}\|^{\dot{\lambda}}}\|\ddot{\mu}-1/\ddot{\mu}\|\delta_F\big(A_{4_1}(\ddot{\lambda},\ddot{\mu})\big)\end{aligned}$$

where

$$A_{4_1}(L,M) := L+L^{-1}+(M-M^{-1})^2-M-M^{-1} \tag{115}$$

is the A-polynomial of the figure-eight knot [4].

7.3. **The B-Weil transform.** We now consider the B-Weil transform of the quantum dilogarithm (44). It is the function $h_{a,c}:\hat{\mathsf{B}}^2\to\mathbb{C}$ defined by

$$h_{a,c}(\alpha,\gamma)=h_{a,c}((\alpha,x),(\gamma,z)) := \gamma(x)\int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,1/(xy))\langle(0,y);(\gamma,z)\rangle\,\mathrm{d}y\,. \tag{116}$$

Lemma 7.4. We have:

$$h_{a,c}(\alpha,\gamma)=\int_{\mathsf{B}}\alpha\Big(\frac{\ddot{c}}{1-y}\Big)\frac{\|y\|^{\dot{c}}}{\|1-y\|^{1-\dot{a}}\gamma(y\ddot{a})}\,\mathrm{d}y. \tag{117}$$

Proof. We compute:

$$\begin{aligned}h_{a,c}(\alpha,\gamma) &= \gamma(x)\int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,1/(xy))\gamma(y)\,\mathrm{d}y = \gamma(x)\int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,1/y)\gamma(y/x)\,\mathrm{d}y\\
&= \int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,1/y)\gamma(y)\,\mathrm{d}y = \int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,y)\gamma(1/y)\,\mathrm{d}y\\
&= \int_{\mathsf{B}}\alpha\Big(\frac{\ddot{c}}{1-y/\ddot{a}}\Big)\frac{\|y/\ddot{a}\|^{\dot{c}}}{\|1-y/\ddot{a}\|^{1-\dot{a}}}\gamma(1/y)\,\mathrm{d}y = \int_{\mathsf{B}}\alpha\Big(\frac{\ddot{c}}{1-y}\Big)\frac{\|y\|^{\dot{c}}}{\|1-y\|^{1-\dot{a}}\gamma(y\ddot{a})}\,\mathrm{d}y\end{aligned}$$

□

Lemma 7.5. (a) The functions $g_{a,c}(x,z)$ are $h_{a,c}(\alpha,\gamma)$ are related to each other by a Fourier transformation

$$h_{a,c}(\alpha,\gamma)=\int_{\mathsf{B}^2}\frac{\gamma(x)}{\alpha(z)}g_{a,c}(x,z)\,\mathrm{d}(x,z)\,, \tag{118}$$

which corresponds to a duality symmetry of the beta pentagon relations [22].
(b) The function $h_{a,c}$ satisfies the symmetries

$$h_{a,c}(\alpha,\gamma)=h_{b,a}(\beta,\alpha)=h_{c,b}(\gamma,\beta),\quad a+b+c=\varpi,\quad \alpha+\beta+\gamma=0. \tag{119}$$

Proof. From (110) we have

$$g_{a,c}(x,z)=\int_{\hat{\mathsf{B}}}\bar{\Psi}_{a,c}(-\beta,1/x)\beta(z)\,\mathrm{d}\beta \quad\Leftrightarrow\quad \bar{\Psi}_{a,c}(-\alpha,1/x)=\int_{\mathsf{B}}\frac{g_{a,c}(x,z)}{\alpha(z)}\,\mathrm{d}z \tag{120}$$

and from (116)

$$h_{a,c}(\alpha,\gamma)=\int_{\mathsf{B}}\bar{\Psi}_{a,c}(-\alpha,1/y)\gamma(y)\,\mathrm{d}y \quad\Leftrightarrow\quad \bar{\Psi}_{a,c}(-\alpha,1/x)=\int_{\hat{\mathsf{B}}}\frac{h_{a,c}(\alpha,\gamma)}{\gamma(x)}\,\mathrm{d}\gamma \tag{121}$$

so that

$$\int_{\mathsf{B}}\frac{g_{a,c}(x,z)}{\alpha(z)}\,\mathrm{d}z=\int_{\hat{\mathsf{B}}}\frac{h_{a,c}(\alpha,\gamma)}{\gamma(x)}\,\mathrm{d}\gamma \quad\Leftrightarrow\quad h_{a,c}(\alpha,\gamma)=\int_{\mathsf{B}^2}\frac{\gamma(x)}{\alpha(z)}g_{a,c}(x,z)\,\mathrm{d}(x,z). \tag{122}$$

To show the cyclic symmetries (119), we compute

$$\begin{aligned}
h_{a,c}(\alpha,\gamma)&=\int_{\mathsf{B}^2}\frac{\gamma(x)}{\alpha(z)}g_{a,c}(x,z)\,\mathrm{d}(x,z)=\int_{\mathsf{B}^2}\frac{\gamma(x)}{\alpha(z)}g_{b,a}((xz)^{-1},x)\,\mathrm{d}(x,z)\\
&=\int_{\mathsf{B}^2}\frac{\gamma(x)}{\alpha(u/x)}g_{b,a}(1/u,x)\,\mathrm{d}(x,u)=\int_{\mathsf{B}^2}\frac{(\alpha+\gamma)(x)}{\alpha(u)}g_{b,a}(1/u,x)\,\mathrm{d}(x,u)\\
&=\int_{\mathsf{B}^2}\frac{\alpha(y)}{\beta(x)}g_{b,a}(y,x)\,\mathrm{d}(x,y)=h_{b,a}(\beta,\alpha)\,.
\end{aligned}$$

□

7.4. **The $\hat{\mathsf{B}}$-edge generalized TQFT.** Based on the function $h_{a,c}(\alpha,\gamma)$, one can formulate a second face-type generalized TQFT using *unordered* triangulations, and placing $\hat{\mathsf{B}}$-integration variables at the edges of the triangulations with the angle-dependent symmetric tetrahedral Boltzmann weights of a tetrahedron T obtained under the substitutions $\alpha\to\alpha_{0,2}+\alpha_{1,3}-\alpha_{0,3}-\alpha_{1,2}$ and $\gamma\to\alpha_{0,1}+\alpha_{2,3}-\alpha_{0,2}-\alpha_{1,3}$ into $h_{a,c}(\alpha,\gamma)$. This model whose partition function we denote by $\mathcal{E}_{\hat{\mathsf{B}}}$ will be discussed in more detail for the case of the field $F=\mathbb{R}$ in the next section.

Part II. The field of the real numbers

8. The field of the real numbers

8.1. **The quantum dilogarithm.** In this section we discuss the quantum dilogarithm (24) in more detail for the case of the field $F=\mathbb{R}$. Then, $\mathsf{B}=\mathbb{R}^\times$ is isomorphic to $\mathbb{R}\times\mathbb{Z}/2\mathbb{Z}$ by the map that sends x to $(\sqrt{2}\log|x|,\mathrm{sgn}(x))$ where $\mathrm{sgn}(x)=1$ (resp., -1) when $x>0$ (resp., $x<0$) and the choice of the constant $\sqrt{2}$ is fixed by the condition of matching the Haar measures on $\mathbb{R}^\times$ and on $\mathbb{R}\times\mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{R}$ (and $\mathbb{Z}/2\mathbb{Z}$) is a self-dual LCA group, so is B.

Thus, we can identify $\hat{\mathsf{B}}$ with B and denote its elements simply by $x, y, \dots \in \mathbb{R}^\times$ instead of $\alpha, \beta, \dots$.

Summarizing our discussion, when $F = \mathbb{R}$, we have isomorphisms

$$\hat{\mathsf{B}} \simeq \mathsf{B} = \mathbb{R}^\times \simeq \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}\,. \tag{123}$$

For $x \in \mathbb{R}^\times$, we will define $\varepsilon_x := 0$ (resp., 1) when $x > 0$ (resp., $x < 0$), and we will define $\ell_x := 4\pi \log|x|$. (The factor of 4π is included here to simplify equation (130) below.) Then, we have:

$$x = (-1)^{\varepsilon_x} e^{\frac{\ell_x}{4\pi}}, \qquad (x \in \mathbb{R}^\times)\,. \tag{124}$$

Using the fact that $\varepsilon_x = (1 - \mathrm{sgn}(x))/2$ where $\mathrm{sgn}(x) = 1$ for $x > 0$ and -1 for $x < 0$, it follows that

$$\varepsilon_{xy} = \varepsilon_x + \varepsilon_y - 2\varepsilon_x\varepsilon_y, \qquad \ell_{xy} = \ell_x + \ell_y \tag{125}$$

for all $x, y \in \mathbb{R}^\times$. The Gaussian exponential (22) of $\mathsf{A} = \mathbb{R}^\times \times \mathbb{R}^\times$ is given by

$$\langle x, y\rangle := (-1)^{\varepsilon_x\varepsilon_y} e^{4\pi i \log|x|\log|y|} = (-1)^{\varepsilon_x\varepsilon_y} e^{\frac{i}{4\pi}\ell_x\ell_y} \tag{126}$$

and the evaluation map $\hat{\mathsf{B}} \times \mathsf{B} \to \mathbb{T}$ is given by $(y, x) \mapsto \langle y, x\rangle$. Equation (125) implies that

$$\langle x, y\rangle = \langle y, x\rangle, \qquad \langle x, yy'\rangle = \langle x, y\rangle\langle x, y'\rangle \tag{127}$$

for all $x, y, y' \in \mathbb{R}^\times$ and satisfies, in particular, $\langle x, 1\rangle = 1$ for all $x \in \mathbb{R}^\times$. With the above identifications, the quantum dilogarithm (24) and its angled version (44) are given by

$$\varphi : \mathbb{R}^\times \setminus \{-1\} \times \mathbb{R}^\times \to \mathbb{T}, \qquad \varphi(y, x) = (-1)^{\varepsilon_y \varepsilon_{1+x}} |1 + x|^{i\ell_y}\,, \tag{128}$$

$$\Psi_{a,c} : \mathbb{R}^\times \setminus \{\ddot{a}^{-1}\} \times \mathbb{R}^\times \to \mathbb{C}^\times, \qquad \Psi_{a,c}(y, x) = (-1)^{\varepsilon_y \varepsilon_{(1-\ddot{a}x)\ddot{c}}} e^{-\frac{i}{4\pi}\ell_y \ell_{(1-\ddot{a}x)\ddot{c}}} \frac{|\ddot{a}x|^{\dot{c}}}{|1 - \ddot{a}x|^{1-\dot{a}}} \tag{129}$$

where $a = (\dot{a}, \ddot{a}) \in \mathbb{R} \times \mathbb{R}^\times$ and likewise c. When $\dot{a}, \dot{c} > 0$ with $\dot{a} + \dot{c} < 1$, then we have the bound $\Psi_{a,c}(y, x) = O(e^{\frac{\ell_x}{4\pi}(\dot{a}+\dot{c}-1)})$ (resp., $O(e^{\frac{\ell_x}{4\pi}\dot{c}})$) when $\ell_x \gg 0$ (resp., $\ell_x \ll 0$) which implies exponential decay at infinity in the x-direction and boundedness in the y-direction. Moreover, the function $\Psi_{a,c}(y, x)$ which has a singularity at $x = \ddot{a}^{-1}$ is locally integrable since it behaves like $\varepsilon^{\dot{a}-1}$ for $x = \ddot{a}^{-1} + \varepsilon$. Hence, $\Psi_{a,c}(y, x)$ is locally integrable and polynomially bounded at infinity, hence a tempered distribution on A [31, Thm.V.10].

Likewise, the functions $g_{a,c}$ and $h_{a,c}$ defined explicitly below are tempered distributions on A since they are partial and full Fourier transforms (see Equations (110) and (116)).

8.2. **The function $h_{a,c}$.** In this section we compute explicitly the function $h_{a,c}$ for $F = \mathbb{R}$.

Theorem 8.1. (a) When $F = \mathbb{R}$, the function $h_{a,c}$ of Equation (117) is given by

$$\begin{aligned} h_{a,c}(x, y) &= \frac{\langle x, \ddot{c}\rangle}{\langle y, \ddot{a}\rangle}\big(\mathrm{B}(\dot{a} - i\ell_x, \dot{c} - i\ell_y) + (-1)^{\varepsilon_x}\mathrm{B}(\dot{a} - i\ell_x, \dot{b} + i\ell_{xy}) + (-1)^{\varepsilon_y}\mathrm{B}(\dot{c} - i\ell_y, \dot{b} + i\ell_{xy})\big) \\ &= \sqrt{2\pi}\frac{\langle x, \ddot{c}\rangle}{\langle y, \ddot{a}\rangle}(-1)^{\varepsilon_x\varepsilon_y}\Gamma_{\varepsilon_x}(\dot{a} - i\ell_x)\Gamma_{\varepsilon_y}(\dot{c} - i\ell_y)\Gamma_{\varepsilon_{xy}}(\dot{b} + i\ell_{xy}) \end{aligned} \tag{130}$$

for $x, y \in \mathbb{R}^\times$, where

$$\Gamma_n(z) := \sqrt{\frac{2}{\pi}}\Gamma(z)\cos(\pi(n-z)/2), \quad n \in \{0,1\}\,. \tag{131}$$

(b) The function $h_{a,c}$ satisfies the symmetries

$$h_{a,c}(x,z) = h_{b,a}(y,x) = h_{c,b}(z,y), \qquad a+b+c=(1,-1), \quad xyz=1\,. \tag{132}$$

(c) Moreover, we have

$$h_{a,c}(x,y) = 2\frac{\langle x, \ddot{c}\rangle}{\langle y, \ddot{a}\rangle}\mathrm{BC}(\tfrac{\pi}{2}(\varepsilon_x + i\ell_x - \dot{a}), \tfrac{\pi}{2}(\varepsilon_y + i\ell_y - \dot{b}))\,\mathrm{B}(\dot{a} - i\ell_x, \dot{c} - i\ell_y) \tag{133}$$

for all $x, y \in \mathbb{R}^\times$ where

$$\mathrm{BC}(x,y) := \frac{\cos(x)\cos(y)}{\cos(x+y)} \tag{134}$$

is a trigonometric version of the beta function.

Equation (130) is similar the equation expressing the Venetziano amplitude as a sum of three beta functions and appears in p-adic string theory, see eg [10, Eqn.(2)].

Proof. For the first part, when $F = \mathbb{R}$, the general formula

$$h_{a,c}(\alpha,\gamma) = \int_{\mathsf{B}} \alpha\Big(\frac{\ddot{c}}{1-y}\Big)\frac{\|y\|^{\dot{c}}}{\|1-y\|^{1-\dot{a}}\gamma(y\ddot{a})}\,\mathrm{d}_{\mathsf{B}}y,$$

with $\alpha = x = (-1)^{\varepsilon_x}e^{\frac{\ell_x}{2\pi}}$ and $\gamma = y = (-1)^{\varepsilon_y}e^{\frac{\ell_y}{2\pi}}$ takes the form

$$\begin{aligned} h_{a,c}(x,y) &= \sum_{\epsilon\in\{\pm1\}}\int_0^\infty \frac{\big\langle x, \frac{\ddot{c}}{1-\epsilon u}\big\rangle u^{\dot{c}-1}}{\big|1-\epsilon u\big|^{1-\dot{a}}\big\langle y, \epsilon u\ddot{a}\big\rangle}\,\mathrm{d}u \\ &= \frac{\langle x, \ddot{c}\rangle}{\langle y, \ddot{a}\rangle}\sum_{k=0}^{1}\int_0^\infty \big|1-(-1)^k u\big|^{\dot{a}-1-i\ell_x}(-1)^{\varepsilon_x\varepsilon_{1-(-1)^k u}+k\varepsilon_y}u^{\dot{c}-1-i\ell_y}\,\mathrm{d}u \end{aligned}$$

so that

$$\begin{aligned} &h_{a,c}(x,y)\langle y, \ddot{a}\rangle/\langle x, \ddot{c}\rangle \\ &= \int_0^\infty |1-u|^{\dot{a}-1-i\ell_x}(-1)^{\varepsilon_x\varepsilon_{1-u}}u^{\dot{c}-1-i\ell_y}\,\mathrm{d}u + (-1)^{\varepsilon_y}\int_1^\infty v^{\dot{a}-1-i\ell_x}(v-1)^{\dot{c}-1-i\ell_y}\,\mathrm{d}v \\ &= \int_0^1 |1-u|^{\dot{a}-1-i\ell_x}u^{\dot{c}-1-i\ell_y}\,\mathrm{d}u + (-1)^{\varepsilon_x}\int_1^\infty |1-u|^{\dot{a}-1-i\ell_x}u^{\dot{c}-1-i\ell_y}\,\mathrm{d}u \\ &\quad + (-1)^{\varepsilon_y}\int_0^1 t^{\dot{b}-1+i(\ell_x+\ell_y)}(1-t)^{\dot{c}-1-i\ell_y}\,\mathrm{d}t \\ &= \mathrm{B}(\dot{a}-i\ell_x, \dot{c}-i\ell_y) + (-1)^{\varepsilon_x}\int_0^1 (1-t)^{\dot{a}-1-i\ell_x}t^{\dot{b}-1+i(\ell_x+\ell_y)}\,\mathrm{d}t + (-1)^{\varepsilon_y}\,\mathrm{B}(\dot{c}-i\ell_y, \dot{b}+i\ell_{xy}) \\ &= \mathrm{B}(\dot{a}-i\ell_x, \dot{c}-i\ell_y) + (-1)^{\varepsilon_x}\,\mathrm{B}(\dot{a}-i\ell_x, \dot{b}+i\ell_{xy}) + (-1)^{\varepsilon_y}\,\mathrm{B}(\dot{c}-i\ell_y, \dot{b}+i\ell_{xy}) \end{aligned}$$

where

$$\mathrm{B}(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t. \tag{135}$$

is the Euler beta function and

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{136}$$

is the Γ-function. Finally, by using the formula

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \Gamma(x)\Gamma(y)\Gamma(1-x-y)\frac{\sin(\pi(x+y))}{\pi},$$

we write the singed sum of the three beta functions as a product of three Γ_n functions as follows:

$$\begin{aligned}&\mathrm{B}(\dot a - i\ell_x, \dot c - i\ell_y) + (-1)^{\varepsilon_x}\,\mathrm{B}(\dot a - i\ell_x, \dot b + i\ell_{xy}) + (-1)^{\varepsilon_y}\,\mathrm{B}(\dot c - i\ell_y, \dot b + i\ell_{xy})\\ &\quad= \frac{1}{\pi}\Gamma(\dot a - i\ell_x)\Gamma(\dot c - i\ell_y)\Gamma(\dot b + i\ell_{xy})\Big(\sin(\pi(\dot b + i\ell_{xy})) + (-1)^{\varepsilon_x}\sin(\pi(\dot c - i\ell_y))\\ &\qquad + (-1)^{\varepsilon_y}\sin(\pi(\dot a - i\ell_x))\Big) = \sqrt{2\pi}(-1)^{\varepsilon_x\varepsilon_y}\Gamma_{\varepsilon_x}(\dot a - i\ell_x)\Gamma_{\varepsilon_y}(\dot c - i\ell_y)\Gamma_{\varepsilon_{xy}}(\dot b + i\ell_{xy}).\end{aligned}$$

This concludes the proof of the first part. The second part follows from Equation (112). The third part follows from the second equality of Equation (130) together with the inversion relation

$$\Gamma_n(z)\Gamma_n(1-z) = 1, \qquad n \in \{0,1\}. \tag{137}$$

of the normalized Γ-functions. □

8.3. **A tetrahedral weight based on the function $h_{a,c}$.** The Boltzmann weights are obtained by the substitutions $x = \frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}$ and $y = \frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}$ in $h_{a,c}(x,y)$. The triangulations are now oriented but unordered, and using Theorem 8.1, it follows that the tetrahedral weight of a tetrahedron is

$$\begin{aligned}W_{a,c}(T,x) &= \sqrt{2\pi}\frac{\mathrm{G}_{\dot a}\left(\frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}\right)\mathrm{G}_{\dot b}\left(\frac{x_{0,3}x_{1,2}}{x_{0,1}x_{2,3}}\right)\mathrm{G}_{\dot c}\left(\frac{x_{0,1}x_{2,3}}{x_{0,2}x_{1,3}}\right)}{\langle -\ddot a, x_{0,1}x_{2,3}\rangle\langle -\ddot b, x_{0,2}x_{1,3}\rangle\langle -\ddot c, x_{0,3}x_{1,2}\rangle}\\ &= 2\frac{\mathrm{BC}(\frac{\pi}{2}(\varepsilon_A + i\ell_A - \dot a), \frac{\pi}{2}(\varepsilon_B + i\ell_B - \dot b))\,\mathrm{B}(\dot a - i\ell_A, \dot b - i\ell_B)}{\langle -\ddot a, x_{0,1}x_{2,3}\rangle\langle -\ddot b, x_{0,2}x_{1,3}\rangle\langle -\ddot c, x_{0,3}x_{1,2}\rangle}\end{aligned} \tag{138}$$

where

$$\mathrm{G}_t(u) := i^{\varepsilon_u}\Gamma_{\varepsilon_u}(t - i\ell_u), \quad \forall (t,u) \in \mathbb{R}_{>0}\times\mathbb{R}^\times \tag{139}$$

and

$$A := \frac{x_{0,2}x_{1,3}}{x_{0,3}x_{1,2}}, \quad B := \frac{x_{0,3}x_{1,2}}{x_{0,1}x_{2,3}}\,. \tag{140}$$

8.4. **Definition of the edge state-integral.** In this section we define a Turaev–Viro type generalized TQFT based on the tetrahedral weight (138) for the self-dual LCA group $\mathbb{R}^\times$. This follows closely the definition of the of the KLV invariant and of the meromorphic 3D-index (using an analogous function for the self-dual LCA groups $\mathbb{R}\times\mathbb{R}$ and $\mathbb{T}$, respectively [24, 12].

Fix an ideal triangulation $\mathcal{T}$ of an oriented 3-manifold with N tetrahedra T_i for $i = 1,\dots,N$. The invariant is defined as follows:

(a) Assign variables $x_i \in \mathbb{R}^\times$ for $i = 1,\dots,N$ to N edges of $\mathcal{T}$.
(b) Choose a strictly positive pre-angle structure $\theta = (a,b,c)$ at each tetrahedron, where $a+b+c = (1,-1)$ and $\dot{a},\dot{b},\dot{c} > 0$. Here, a is the angle of the 01 and 23 edges, b is the angle of the 02 and 13 edges, and c is the angle of the 03 and 12 edges.
(c) The weight $B(T,x,\theta) := W_{a,c}(T,x)$ of a tetrahedron T is given by Equation (138).
(d) Define

$$\mathcal{E}_{\hat{\mathbb{R}},\mathcal{T},\theta} := \int_{(\mathbb{R}^\times)^N} \prod_{i=1}^N B(T_i,x,\theta)\delta(x_N)d\mu(x) \tag{141}$$

where $d\mu(x)$ is the normalized Haar measure on $(\mathbb{R}^\times)^N$.

An oriented unordered ideal tetrahedron has shape z, $z' = 1/(1-z)$ and $z'' = 1-1/z$ placed at the pair of opposite edges 01 and 23, 02 and 13, or 03 and 12, respectively. Recall the Neumann-Zagier matrices $\overline{A}$, $\overline{B}$ and $\overline{C}$ of an oriented, unordered ideal triangulation [27], whose rows and columns are indexed by the edges and the tetrahedra of the triangulation, respectively. The (i,j) entry of each of these matrices is the number of times the shape z_j (or, z'_j, or z''_j) appears around the edge e_i of the triangulation.

The next proposition shows that that the integral (141) (and even the integrand) depends only on the Neumann–Zagier matrices of the gluing equations of the triangulation $\mathcal{T}$. We use the shorthand notation M_j for the j-th column of a matrix M and $x^v = \prod_{i=1}^N x_i^{v_i}$.

Proposition 8.2. With the above notation and for $i = j,\dots,N$, we have:

$$B(T_j,x,\theta) = \sqrt{2\pi}\frac{\mathrm{G}_{\dot{a}_j}\left(x^{(\overline{B}-\overline{C})_j}\right)\mathrm{G}_{\dot{b}_j}\left(x^{(\overline{C}-\overline{A})_j}\right)\mathrm{G}_{\dot{c}_j}\left(x^{(\overline{A}-\overline{B})_j}\right)}{\langle -\ddot{a}_j, x^{\overline{A}_j}\rangle\langle -\ddot{b}_j, x^{\overline{B}_j}\rangle\langle -\ddot{c}_j, x^{\overline{C}_j}\rangle} \tag{142}$$

It follows that $\mathcal{E}_{\hat{\mathbb{R}},\mathcal{T},\theta}$ depends on only the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ and θ. Moreover,

$$\prod_{j-1}^N \langle -\ddot{a}_j, x^{\overline{A}_j}\rangle\langle -\ddot{b}_j, x^{\overline{B}_j}\rangle\langle -\ddot{c}_j, x^{\overline{C}_j}\rangle = 1 \tag{143}$$

for all (x,θ). Hence, this factor can be removed from the integrand of (141).

Its proof follows mutandis-mutandis from [12, Prop.3.1]. To show identity (143), we use (127) which implies that $\langle -\ddot{a}_j x^{\overline{A}_j}\rangle = \prod_i \langle \ddot{a}_j^{\overline{A}_{ij}}, x_i\rangle^{-1}$, as well as the fact that θ satisfies the edge-balancing conditions $\prod_i \ddot{a}_j^{\overline{A}_{ij}} \ddot{b}_j^{\overline{B}_{ij}} \ddot{c}_j^{\overline{C}_{ij}} = 1$ for all $i = 1,\dots,N$.

Following the arguments of [12, Sec.3] *mutatis mutandis*, we obtain an invariant $\mathcal{E}_{\hat{\mathbb{R}},M}(\lambda,\mu)$ for a 3-manifold M with torus boundary.

8.5. **Examples.** We now illustrate the above invariant for the case of the complement of the 4_1 knot. The next lemma expresses the invariant of the 4_1 knot in terms of an integral of the beta function.

Lemma 8.3. The invariant of the 4_1 is given by

$$\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) = \frac{1}{2\pi i}\int_{\epsilon-i\mathbb{R}} \frac{\mathrm{B}(z,z-\dot\lambda)\,\mathrm{B}(z+\dot\mu,z+\dot\mu-\dot\lambda)}{\cos(\pi(z-\frac{\dot\lambda}{2}))\cos(\pi(z+\dot\mu-\frac{\dot\lambda}{2}))}\big(\cos(\pi\tfrac{\dot\lambda}{2})\big)^2\,\mathrm{d}z \tag{144}$$

where $\epsilon>0$.

Proof. Using the ideal triangulation of the 4_1 knot with two tetrahedra, with the balancing condition $2a_0+c_0=2a_1+c_1$ and the definitions $\lambda := 2a_0+c_0-\varpi$ and $\mu := a_0-a_1$, we have

$$\begin{aligned}
\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) &= \int_{\mathbb{R}^\times} h_{a_0,c_0}(x,1/x^2)\bar{h}_{a_1,c_1}(1/x,x^2)\frac{\mathrm{d}x}{|x|}\\
&= \frac{1}{2}\int_{\mathbb{R}^\times}\frac{\langle x,\ddot{c}_0\ddot{a}_0^2\rangle}{\langle x,\ddot{c}_1\ddot{a}_1^2\rangle}\frac{\Gamma_{\varepsilon_x}(\dot{a}_0-i\ell_x)\Gamma_{\varepsilon_x}(\dot{b}_0-i\ell_x)}{\Gamma_0(1-\dot{c}_0-2i\ell_x)}\frac{\Gamma_{\varepsilon_x}(\dot{a}_1-i\ell_x)\Gamma_{\varepsilon_x}(\dot{b}_1-i\ell_x)}{\Gamma_0(1-\dot{c}_1-2i\ell_x)}\,\mathrm{d}\ell_x\\
&= \frac{1}{\pi}\sum_{\epsilon\in\{0,1\}}\int_{\mathbb{R}}\prod_{k\in\{0,1\}}\mathrm{B}(\dot{a}_k-it,\dot{b}_k-it)\frac{\cos(\frac{\pi}{2}(\epsilon+it-\dot{a}_k))\cos(\frac{\pi}{2}(\epsilon+it-\dot{b}_k))}{\sin(\frac{\pi}{2}(\dot{c}_k+2it))}\,\mathrm{d}t\\
&= \int_{\dot{a}_1-i\mathbb{R}}\frac{\mathrm{B}(z,z-\dot\lambda)\,\mathrm{B}(z+\dot\mu,z+\dot\mu-\dot\lambda)}{\pi\cos(\pi(z-\frac{\dot\lambda}{2}))\cos(\pi(z+\dot\mu-\frac{\dot\lambda}{2}))}\\
&\times\sum_{\epsilon\in\{0,1\}}\cos(\tfrac{\pi}{2}(\epsilon-z))\cos(\tfrac{\pi}{2}(\epsilon+\dot\lambda-z))\cos(\tfrac{\pi}{2}(\epsilon-\dot\mu-z))\cos(\tfrac{\pi}{2}(\epsilon+\dot\lambda-\dot\mu-z))\,\mathrm{d}iz\\
&= \frac{i}{2\pi}\int_{\dot{a}_1-i\mathbb{R}}\mathrm{B}(z,z-\dot\lambda)\,\mathrm{B}(z+\dot\mu,z+\dot\mu-\dot\lambda)\left(1+\frac{\big(\cos(\pi\frac{\dot\lambda}{2})\big)^2}{\cos(\pi(z-\frac{\dot\lambda}{2}))\cos(\pi(z+\dot\mu-\frac{\dot\lambda}{2}))}\right)\mathrm{d}z\,.
\end{aligned}$$

It remains to remove the 1 above. The integral $\int_{\dot{a}_1-i\mathbb{R}}\mathrm{B}(z,z-\dot\lambda)\,\mathrm{B}(z+\dot\mu,z+\dot\mu-\dot\lambda)\,\mathrm{d}z$ is *not* absolutely convergent since the integrand is $O(1/|z|)$ for large z. Yet, the integral is distributionally well-defined and in fact vanishes; see Lemma 9.2 below. This concludes the proof of the lemma. □

The next lemma uses the Fourier transform relation between the g and the h-functions (118) and expresses the invariant of the 4_1 knot in terms of a period of its A-polynomial.

Lemma 8.4. We have:

$$\mathcal{E}_{\mathbb{R},4_1}(\lambda,\mu) = \int_{-\infty}^{1}\frac{|1-x|^{\dot\mu}(|y+x-1|^{\dot\lambda}+|-y+x-1|^{\dot\lambda})}{|x|^{2\dot\mu+\dot\lambda}2^{\dot\lambda}}\frac{dx}{y} \tag{145}$$

where $y=\sqrt{(1-x)(1-x+4x^2)}$.

Proof. Using the triangulation of 4_1 with two tetrahedra, and its balancing edge conditions, for all local fields F we have:

$$\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) = \int_{\hat{\mathsf{B}}} h_{a_0,c_0}(\alpha,-2\alpha)\bar{h}_{a_1,c_1}(-\alpha,2\alpha)\,\mathrm{d}\alpha. \tag{146}$$

We now use the Fourier transform relation (118) and obtain that

$$\begin{aligned}\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) &= \int_{\mathsf{B}^4\times\hat{\mathsf{B}}} g_{a_0,c_0}(x,y)(-\alpha)(x^2yu^2v)\bar{g}_{a_1,c_1}(u,v)\,\mathrm{d}(x,y,u,v)\,\mathrm{d}\alpha \\ &= \int_{\mathsf{B}^4} \delta_F(x^2yu^2v-1)g_{a_0,c_0}(x,y)\bar{g}_{a_1,c_1}(u,v)\,\mathrm{d}(x,y,u,v).\end{aligned}$$

Using the balancing condition $2a_0+c_0=2a_1+c_1$ and the explicit form of $g_{a,c}(x,z)$ given in (111), we can reduce the number of integrations from four to two:

$$\begin{aligned}\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) &= \int_{\mathsf{B}^4} \delta_F(x^2yu^2v-1)f_{\dot{a}_0,\dot{c}_0}(1/x,y)\bar{f}_{\dot{a}_1,\dot{c}_1}(1/u,v)\,\mathrm{d}(x,y,u,v) \\ &= \int_{\mathsf{B}^2} \delta_F\big(1-(1-x)(1-u)/(xu)^2\big)\|x\|^{\dot{c}_0}\|1-x\|^{\dot{a}_0-1}\|u\|^{\dot{c}_1}\|1-u\|^{\dot{a}_1-1}\,\mathrm{d}(x,u) \\ &= \int_{F^2} \delta_F\big((xu)^2-(1-x)(1-u)\big)\|x\|^{\dot{c}_0+1}\|1-x\|^{\dot{a}_0-1}\|u\|^{\dot{c}_1+1}\|1-u\|^{\dot{a}_1-1}\,\mathrm{d}(x,u)\,.\end{aligned}$$

Using the delta-function, we can replace the norm of $1-u$ in terms of the norms of x, $1-x$ and u, and then use the definitions of the longitude $\lambda=a_0-b_0$ and the meridian $\mu=a_0-a_1$, and the change of variables $v=xu$ and $y=2xv-x+1$ to obtain that

$$\begin{aligned}\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) &= \int_{F^2} \delta_F\big((xu)^2-(1-x)(1-u)\big)\big\|\tfrac{1-x}{x^2}\big\|^{\dot{\mu}}\|xu\|^{\dot{\lambda}}\,\mathrm{d}(x,u) \\ &= \int_{F^2} \delta_F\big(v^2-(1-x)(1-\tfrac{v}{x})\big)\frac{\|1-x\|^{\dot{\mu}}\|v\|^{\dot{\lambda}}}{\|x\|^{2\dot{\mu}+1}}\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{F^2} \delta_F\Big(y^2-(1-x)(1-x+4x^2)\Big)\frac{\|1-x\|^{\dot{\mu}}\|y+x-1\|^{\dot{\lambda}}}{\|x\|^{2\dot{\mu}+\dot{\lambda}}\|2\|^{\dot{\lambda}-1}}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned} \tag{147}$$

When $F=\mathbb{R}$, $1-x+4x^2>0$ for all real x, thus the delta function imposes the condition that $x<1$. Part (a) of Lemma 2.7 for

$$p_{4_1}(x,y)=y^2-(1-x)(1-x+4x^2) \tag{148}$$

thought of as a function of y with roots $y=\pm\sqrt{(1-x)(1-x+4x^2)}$ and derivative $\frac{\partial}{\partial y}p_{4_1}(x,y)=2y$, implies that

$$\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu)=\int_{-\infty}^{1}\frac{|1-x|^{\dot{\mu}}(|y+x-1|^{\dot{\lambda}}+|-y+x-1|^{\dot{\lambda}})}{|x|^{2\dot{\mu}+\dot{\lambda}}2^{\dot{\lambda}}}\frac{dx}{y}\,,$$

where $y=\sqrt{(1-x)(1-x+4x^2)}$. Over $\mathbb{C}$, Equation (148) defines an elliptic curve with Weiestrass form

$$Y^2=-X^3+\frac{1}{3}X+\frac{322}{27} \tag{149}$$

and j-invariant $-\frac{1}{15}$. This elliptic curve is isomorphic to the A-polynomial curve (115) of the 4_1 knot. ∎

Remark 8.5. We can give an alternative period formula

$$\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) = \int_{\mathbb{R}} \frac{\left|(1-\frac{7}{4}t^2+t^4)^{1/2}-t\right|^{\dot\lambda}}{|t-t^{-1}|^{2\dot\mu+\dot\lambda}(1-\frac{7}{4}t^2+t^4)^{1/2}}\,\mathrm{d}t\,. \tag{150}$$

for $\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu)$. To obtain this, use Equation (147) and do the change of variables $y = t\sqrt{1-x+4x^2}$ to obtain

$$\begin{aligned}
\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu) &= \int_{\mathbb{R}^2} \delta_F\Big(y^2-(1-x)(1-x+4x^2)\Big)\frac{|1-x|^{\dot\mu}|y+x-1|^{\dot\lambda}}{|x|^{2\dot\mu+\dot\lambda}2^{\dot\lambda-1}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^2} \delta_F\Big(t^2-1+x\Big)\frac{|1-x|^{\dot\mu}\left|t\sqrt{1-x+4x^2}+x-1\right|^{\dot\lambda}}{2^{\dot\lambda-1}|x|^{2\dot\mu+\dot\lambda}\sqrt{1-x+4x^2}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \frac{\left|(1-\frac{7}{4}t^2+t^4)^{1/2}-t\right|^{\dot\lambda}}{|t-t^{-1}|^{2\dot\mu+\dot\lambda}(1-\frac{7}{4}t^2+t^4)^{1/2}}\,\mathrm{d}t
\end{aligned}$$

This expresses the function $\mathcal{E}_{\hat{\mathbb{R}},4_1}(\lambda,\mu)$ in terms of a period of the elliptic curve $y^2=4-7t^2+4t^4$. The latter has j-invariant $13997521/225$, hence it is not isomorphic to the A-polynomial curve.

Combining the above lemmas 8.3 and 8.4 when $\lambda=\mu=0$, we obtain Equation (1) stated in the introduction.

9. Fourier transforms of the Euler Γ and B-functions

In this section, which is independent of quantum dilogarithms and of local fields, we give some complementary properties of the Fourier transform of the Γ and B-functions which explain the relation of multidimensional Mellin-Barnes integrals to periods of algebraic varieties. Our results are similar to the work of Passare–Tsikh–Cheshel [29] who relate periods of families of Calabi-Yau manifolds to Mellin–Barnes integrals and explains why the explicit Mellin–Barnes integrals that appear in [17] are periods of the A-polynomial curves.

We begin with an elementary lemma (which ought to be better-known) that computes the inverse Fourier transform of the Γ and the B-functions.

Lemma 9.1. For $a>0$ and x real we have:

$$\Gamma(a+2\pi i x) = \int_{\mathbb{R}} e^{2\pi i x s}e^{as-e^s}\,\mathrm{d}s \tag{151}$$

For $a,b>0$ and x,y real we have:

$$\mathrm{B}(a+2\pi i x, a+2\pi i y) = \int_{\mathbb{R}^3} \frac{e^{-2\pi i(xu+yv)}}{(1+e^s)^a(1+e^{-s})^b}\delta(u-\log(1+e^s))\delta(v-\log(1+e^{-s}))\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v \tag{152}$$

as well as

$$\mathrm{B}(a+2\pi i x, a+2\pi i x) = \int_{\mathbb{R}^2} \frac{e^{-2\pi i x t}}{(1+e^s)^a(1+e^{-s})^a}\delta(t-\log(1+e^s)\log(1+e^{-s}))\,\mathrm{d}s\,\mathrm{d}t \tag{153}$$

Proof. The definition of the Γ-function and the change of variables $t = e^s$ gives:

$$\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{\mathrm{d}t}{t} = \int_{\mathbb{R}} e^{sz} e^{-e^s} ds\,.$$

Then, substitute $z = a + 2\pi i x$ to obtain (151), and observe that the integral is absolutely convergent (in fact, the integrand is exponentially decaying for real s with $|s|$ large.

For the next identity, use

$$\mathrm{B}(z,w) = \int_0^1 (1-t)^z t^w \frac{\mathrm{d}t}{t(t-1)} \tag{154}$$

and observe that

$$\frac{\mathrm{d}t}{t(t-1)} = \Big(\frac{1}{t} + \frac{1}{1-t}\Big)\,\mathrm{d}t = \mathrm{d}(\log(t) - \log(1-t)) = \mathrm{dlog}\Big(\frac{t}{1-t}\Big) = \mathrm{d}s$$

where $s = \log\big(\frac{t}{1-t}\big)$ satisfies $t = 1/(1+e^{-s})$ and $1-t = 1/(1+e^s)$. The change of variables from t to s gives

$$\begin{aligned}\mathrm{B}(z,w) &= \int_{\mathbb{R}} (1+e^s)^{-a}(1+e^{-s})^{-b} e^{-2\pi i(x\log(1+e^s)+y\log(1+e^{-s}))}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^3} \frac{e^{-2\pi i(xu+yv)}}{(1+e^s)^a(1+e^{-s})^b}\delta(u-\log(1+e^s))\delta(v-\log(1+e^{-s}))\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

which concludes the proof of Equation (152). The proof of (153) is similar and left to the reader. □

An application of the above lemma is the vanishing of the following distributional integral.

Lemma 9.2. For all $a > 0$, we have:

$$\int_{a-i\mathbb{R}} \mathrm{B}(z,z)^2\,\mathrm{d}z = 0\,. \tag{155}$$

Note that $|\,\mathrm{B}(z,z)^2| = O(|z|^{-1})$ for $z = a - ix$ with t real and $|x|$ large, hence the integral (155) is not absolutely convergent.

Proof. Equation (153) expresses $\mathrm{B}(z,z)$ as a double integral distributionally

$$\mathrm{B}(a+2\pi i x, a+2\pi i x) = \int_{\mathbb{R}} \Big(\frac{e^s}{(1+e^s)^2}\Big)^a e^{-2\pi i x\log(1+e^s)\log(1+e^{-s})}\,\mathrm{d}s\,.$$

Using this identity twice for $z = a - ix$, we obtain that the integral I of Equation (155) is given by

$$I = \int_{\mathbb{R}^3} \Big(\frac{e^s}{(1+e^s)^2}\Big)^a e^{-2\pi i x\log(1+e^s)\log(1+e^{-s})} \Big(\frac{e^t}{(1+e^t)^2}\Big)^a e^{-2\pi i x\log(1+e^t)\log(1+e^{-t})}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x$$

Now perform the x-integral, which is a single Fourier transform, to obtain that

$$I = \int_{\mathbb{R}^3} \Big(\frac{e^s}{(1+e^s)^2}\Big)^a \Big(\frac{e^t}{(1+e^t)^2}\Big)^a \delta(\log(1+e^s)\log(1+e^{-s}) + \log(1+e^t)\log(1+e^{-t}))\,\mathrm{d}s\,\mathrm{d}t\,.$$

Since $\log(1+e^s)\log(1+e^{-s}) + \log(1+e^t)\log(1+e^{-t}) > 0$ for all real s and t, the distribution vanishes, and the result follows. □

Consider the integral

$$\kappa_{5_2}^{\mathrm{HKS}} = \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}^2} \mathrm{B}^2\big(\tfrac{1}{2}-x,\tfrac{1}{3}+x-y\big)\,\mathrm{B}\big(\tfrac{1}{2}-x,\tfrac{1}{3}+2y\big)\,\mathrm{d}x\,\mathrm{d}y \tag{156}$$

from [17, Fig.10.2], whose numerical value is $\kappa_{5_2}^{\mathrm{HKS}} = .534186\dots$.

Proposition 9.3. We have:

$$\begin{aligned}\kappa_{5_2}^{\mathrm{HKS}} = \int_{\mathbb{R}^6} & e^{-\frac{1}{2}u_1-\frac{1}{3}v_1-\frac{1}{2}u_2-\frac{1}{3}v_2-\frac{1}{2}u_3-\frac{1}{3}v_3}\delta(u_1-v_1+u_2-v_2+u_3)\delta(v_1+v_2-2v_3)\\ &\times\prod_{j=1}^{3}\delta(e^{-u_j}+e^{-v_j}-1)\,\mathrm{d}u_j\,\mathrm{d}v_j\,.\end{aligned} \tag{157}$$

Letting $(z_j,w_j)=(e^{-u_j},e^{-v_j})$ for $j=1,2,3$ consider the curve $X(\mathbb{C}^\times)$ in $(\mathbb{C}^\times)^6$ with defining equations

$$\begin{aligned} z_j+w_j&=1 \quad\ j=1,2,3,\\ z_1w_1^{-1}z_2w_2^{-1}z_3 = 1,& \quad w_1w_2w_3^{-2}=1\,. \end{aligned} \tag{158}$$

Then, $\kappa_{5_2}^{\mathrm{HKS}}$ is an integral over the cycle $X(\mathbb{R}_+)$.

Proof. Using (153), we have:

$$\begin{aligned}\mathrm{B}\big(\tfrac{1}{2}-ix,\tfrac{1}{3}+ix-iy\big) &= \int_{\mathbb{R}^3}\frac{e^{ixu_1-i(x-y)v_1}}{(1+e^{s_1})^{\frac{1}{2}}(1+e^{-s_1})^{\frac{1}{3}}}\\ &\qquad \delta(u_1-\log(1+e^{s_1}))\delta(v_1-\log(1+e^{-s_1}))\,\mathrm{d}s_1\,\mathrm{d}u_1\,\mathrm{d}v_1\\ \mathrm{B}\big(\tfrac{1}{2}-ix,\tfrac{1}{3}+2iy\big) &= \int_{\mathbb{R}^3}\frac{e^{ixu_3-2iyv_3}}{(1+e^{s_1})^{\frac{1}{2}}(1+e^{-s_1})^{\frac{1}{3}}}\\ &\qquad \delta(u_3-\log(1+e^{s_3}))\delta(v_3-\log(1+e^{-s_3}))\,\mathrm{d}s_3\,\mathrm{d}u_3\,\mathrm{d}v_3\end{aligned}$$

Insert twice the first equation and once the second equation to (156) to obtain a 11-dimensional integral representation of $\kappa_{5_2}^{\mathrm{HKS}}$. Now, do the x and y integration (which is a Fourier transform), which reduces the integral to a 9-dimensional one and inserts the product of two delta functions $\delta(u_1-v_1+u_2-v_2+u_3)\delta(v_1+v_2-2v_3)$ in the integrand.

Using part (a) of Lemma 2.7, we see that for every function of Schwartz-Bruhat class g, we have

$$\int_{\mathbb{R}} g(s)\delta(u-\log(1+e^s))\delta(v-\log(1+e^{-s}))\,\mathrm{d}s = g(\log(e^u-1))\delta(e^{-u}+e^{-v}-1)\,.$$

Applying the above identity concludes the proof of (157).

We now study the solution to the delta function equations. In the complex torus $(\mathbb{C}^\times)^6$, with coordinates $(z_1,z_2,z_3,w_1,w_2,w_3)$, where $(z_j,w_j)=(e^{-u_j},e^{-v_j})$ for $j=1,2,3$, the equations (158) define a curve $X(\mathbb{C}^\times)$ given by the equation

$$-1+2z_1-z_1^2+2z_2-2z_1z_2-z_2^2+z_1^2z_2^2-z_1^3z_2^2-z_1^2z_2^3+z_1^3z_2^3=0\,. \tag{159}$$

The above equation has discriminant with respect to z_2 a polynomial in z_1 with real roots at $-6.44292\dots$, 0 and 1. Moreover, the points $X(\mathbb{R}_+)$ in the curve with coordinates in $\mathbb{R}_+^6$ are parametrized by $z_1\in(0,1)$, $z_2=z_2(z_1)$ being the unique real branch of (159) for all $z_1\in\mathbb{R}$, and $z_3 = \frac{1-z_1-z_2+z_1z_2}{z_1z_2}$. Combined with (157), this expresses $\kappa_{5_2}^{\mathrm{HKS}} = \int_{X((0,1))} \omega$ where ω is a holomorphic differential form on the curve (159). A final computation identifies this curve with the A-polynomial of the 5_2 knot. In particular, this gives a proof of [13, Eqn.(18)]. □

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International Center for Mathematics, Department of Mathematics, Southern University of Science and Technology, Shenzhen, China
http://people.mpim-bonn.mpg.de/stavros
Email address: stavros@mpim-bonn.mpg.de

Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland
http://www.unige.ch/math/folks/kashaev
Email address: Rinat.Kashaev@unige.ch