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G-functions and multisum versus holonomic sequences

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Abstract

The purpose of the paper is three-fold: (a) we prove that every sequence which is a multidimensional sum of a balanced hypergeometric term has an asymptotic expansion of Gevrey type-1 with rational exponents, (b) we construct a class of G-functions that come from enumerative combinatorics, and (c) we give a counterexample to a question of Zeilberger that asks whether holonomic sequences can be written as multisums of balanced hypergeometric terms. The proofs utilize the notion of a G-function, introduced by Siegel, and its analytic/arithmetic properties shown recently by André.

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1. Introduction

1.1. Balanced multisum sequences are of Nilsson type

The purpose of the paper is three-fold:

(a) we prove that every sequence which is a multidimensional sum of a balanced hypergeometric term has an asymptotic expansion of Gevrey type-1 with rational exponents,
(b) we construct a class of G-functions that come from enumerative combinatorics, and
(c) we give a counterexample to a question of Zeilberger that asks whether holonomic sequences can be written as multisums of balanced hypergeometric terms.

The proofs utilize the notion of a G-function, introduced by Siegel, and its analytic/arithmetic properties shown recently by André. Let us begin by introducing the notion of a (balanced) multisum sequence.

Definition 1.1. A (*balanced*) *multisum sequence* (a_n) is a sequence of complex numbers of the form

$$a_n = \sum_{k \in \operatorname{supp}(\mathfrak{t}_{n,\bullet})} \mathfrak{t}_{n,k} \tag{1}$$

where $\mathfrak{t}$ is a (balanced) term and the sum is over a finite set that depends on $\mathfrak{t}$.

Definition 1.2. A *term* $\mathfrak{t}_{n,k}$ in variables (n,k) where $k = (k_1, \dots, k_r)$ is an expression of the form:

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{k_i} \prod_{j=1}^{J} A_j(n,k)!^{\epsilon_j} \tag{2}$$

where $C_i \in \overline{\mathbb{Q}}$ for $i = 0, \dots, r$, $\epsilon_j = \pm 1$ for $j = 1, \dots, J$, and A_j are integral linear forms in the variables (n,k) such that for every $n \in \mathbb{N}$, the set

$$\operatorname{supp}(\mathfrak{t}_{n,\bullet}) := \{ k \in \mathbb{Z}^r \mid A_j(n,k) \geqslant 0,\ j = 1, \dots, J \} \tag{3}$$

is finite. We will call a term *balanced* if in addition it satisfies the *balance condition*:

$$\sum_{j=1}^{J} \epsilon_j A_j = 0. \tag{4}$$

For example, the *Apéry sequence* (see [22]) is a balanced multisum given by:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \left(\frac{(n+k)!}{k!^2 (n-k)!} \right)^2. \tag{5}$$

Multisum sequences appear frequently in enumeration questions; for numerous examples, see [12,26]. A key problem is to study the asymptotics of a (balanced) multisum sequence. This is a classical problem that has been discussed by several authors for decades, see [3,12,26,30]. Parsing through the literature, in numerous examples of balanced multisum examples, a certain rationality of the leading exponents of n was found by accident, with no explanation. Understanding this rationality lead to the results of our paper.

To explain this rationality, let us introduce one more definition.

Definition 1.3. We say that a sequence (a_n) is of *Nilsson type* if it has an asymptotic expansion of the form

$$a_n \sim \sum_{\lambda,\alpha,\beta} \lambda^{-n} n^{\alpha} \big(\log(n)\big)^{\beta} f_{\lambda,\alpha,\beta}\left(\frac{1}{n}\right) \tag{6}$$

where the summation is over a finite set of triples (λ, α, β), $\lambda \in \overline{\mathbb{Q}}$ is an algebraic number, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, and $f_{\lambda,\alpha,\beta}(z)$ is Gevrey-1 power series, i.e., the coefficient of z^k in $f_{\lambda,\alpha,\beta}(z)$ is bounded by C^n for some $C > 0$.

Now we can state our first result which does not seem to be covered by the existing literature on asymptotic expansions of sequences.

Theorem 1. *Every balanced multisum sequence is of Nilsson type.*

1.2. G-functions

The proof of Theorem 1 utilizes the notion of a G-function, introduced by Siegel in [24] with motivation being arithmetic problems in elliptic integrals, and transcendence problems in number theory. For further information about G-functions and their properties, see [1,4,10,24,27].

Definition 1.4. We say that series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a *G-function* if

(a) the coefficients a_n are algebraic numbers,
(b) there exists a constant $C > 0$ so that for every $n \in \mathbb{N}$ the absolute value of every conjugate of a_n is less than or equal to C^n,
(c) the common denominator of $a_0, \ldots, a_n$ is less than or equal to C^n,

(d) $G(z)$ is holonomic, i.e., it satisfies a linear differential equation with coefficients polynomials in z.

Remark 1.5. In [1], André calls a series that satisfies (a)–(c), an *arithmetic Gevrey*-0 series.

Our next theorem is a construction of G-functions from balanced terms. Notice that it is easy to generate examples of balanced terms; see for example the Apèry sequence above.

Theorem 2. *If (a_n) is a balanced multisum sequence, the generating series $G(z)=\sum_{n=0}^{\infty} a_n z^n$ is a G-function.*

G-functions can be constructed by arithmetic, or by geometry. Theorem 2 offers a new construction. For further discussion, see Section 3.

The proof of Theorem 2 also gives the following result, which may be of interest to enumerative combinatorial problems that often lead to sequences of integers.

Theorem 3. *The generating series of an integer-valued, exponentially bounded holonomic sequence is a G-function.*

1.3. Holonomic sequences are not balanced multisums

We now come to the third part of the paper, which compares multisum and holonomic sequences.

Definition 1.6. A sequence (a_n) is *holonomic* (i.e., *D-finite* in the sense of [25]) if it satisfies a linear recursion relation with polynomial coefficients. In other words, there exist $d \in \mathbb{N}$ and $P_j(n) \in \overline{\mathbb{Q}}[n]$ (where $\overline{\mathbb{Q}}$ denotes the set of algebraic numbers) for $j = 1, \dots, d$, so that for every $n \in \mathbb{N}$ we have:

$$P_d(n)a_{n+d} + \cdots + P_0(n)a_n = 0. \tag{7}$$

The following is a fundamental theorem of Wilf–Zeilberger.

Theorem 4. *(See [29,31].) Every multisum sequence is holonomic.*

The above theorem has a constructive proof with several computer implementations, see [20,21] and [29]. The converse was widely accepted as a reasonable conjecture, communicated to the author by Zeilberger. Our goal is to give a counterexample, and give an obstruction for the converse to hold.

Theorem 5. *Consider the holonomic sequence (a_n) defined by*

$$(2n+1)a_{n+2} - (7n+11)a_{n+1} + (2n+1)a_n = 0. \tag{8}$$

with initial conditions $a_0 = 0$, $a_1 = 1$. Then, (a_n) is not a balanced multisum.

To understand why the converse to Theorem 4 fails, and why the example given by (8) is not pathological (but rather typical), let us look at the asymptotic expansion of an exponentially bounded holonomic sequence. It follows from Birkhoff and Trjitzinsky and Turrittin that a holonomic sequence is almost of Nilsson type, i.e., it satisfies (6) where the exponents α are algebraic, but not necessarily rational numbers. This also typically happens in the analysis of linear ODE, where the above exponents are known as *Frobenius exponents*. For more details, see Section 2.4. On the other hand, balanced multisum sequences have rational exponents according to Theorem 1. This proves and explains Theorem 5.

2. Proofs

2.1. Proof of Theorem 2

Let us begin with the following alternative presentation of a balanced term.

Lemma 2.1. *Every balanced term* $\mathfrak{t}$ *can be written in the form:*

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^r C_i^{k_i} \prod_{j=1}^J \binom{B_j(n,k)}{D_j(n,k)}^{\epsilon_j} \tag{9}$$

where $C_i \in \overline{\mathbb{Q}}$ *for* $i = 0, \dots, r$, $\epsilon_j = \pm 1$ *for* $j = 1, \dots, J$, *and* B_j, D_j *are integral linear forms in the variables* (n,k).

Proof. Consider a balanced term $\mathfrak{t}$ given by (2), where the linear forms A_j satisfy the balance condition (4). Let $J^{\pm} = \{j \in J \mid \epsilon_j = \pm 1\}$ and consider the linear form $A(n,k)$ defined by:

$$A(n,k) = \sum_{j \in J^+} A_j(n,k) = \sum_{j \in J^-} A_j(n,k),$$

where the second equality follows from the balance condition. Then, multiply and divide the balanced term by $A(n,k)!$, and rearrange the factors into a ratio of multibinomial coefficients as follows:

$$\begin{aligned}
\prod_{j=1}^J \bigl(A_j(n,k)!\bigr)^{\epsilon_j} &= \frac{\prod_{j \in J^+} A_j(n,k)!}{\prod_{j \in J^-} A_j(n,k)!} \\
&= \frac{\prod_{j \in J^+} A_j(n,k)!}{\prod_{j \in J^-} A_j(n,k)!} \frac{A(n,k)!}{A(n,k)!} \\
&= \frac{\binom{A(n,k)}{A_j \mid j \in J^-}}{\binom{A(n,k)}{A_j \mid j \in J^+}}.
\end{aligned}$$

Now, write the multibinomial coefficient as a product of binomial coefficients. The result follows. □

The next lemma from number theory is well known (see [22, p. 198] and also [24]) and follows easily from Chebytchev's theorem. Below, lcm denotes the *least common multiple*.

Lemma 2.2. *(See [22,24].) There exists $C > 0$ so that*

$$\operatorname{lcm}\left(\binom{n}{0}, \dots, \binom{n}{n}\right) < C^n \tag{10}$$

for all $n \in \mathbb{N}$.

Proof. Let $\operatorname{ord}_p m$ denote the maximal power of a prime number p that divides a natural number m. Then, for every natural number n and integer a and b with $0 \leqslant b \leqslant a \leqslant n$ and every prime number p we have:

$$\operatorname{ord}_p \binom{a}{b} \leqslant \left[\frac{\log a}{\log p}\right] - \operatorname{ord}_p b \leqslant \operatorname{ord}_p \operatorname{lcm}(1, \dots, a) - \operatorname{ord}_p b \leqslant \operatorname{ord}_p \operatorname{lcm}(1, \dots, n).$$

Thus,

$$\operatorname{lcm}\left(\binom{n}{0}, \dots, \binom{n}{n}\right) \leqslant \operatorname{lcm}(1, \dots, n).$$

On the other hand, it is known that

$$\log \operatorname{lcm}(1, \dots, n) = O(n).$$

For a detailed discussion, see [22, p. 198]. □

We are now ready to give the proof of Theorem 2.

Proof of Theorem 2. Fix a balanced term $\mathfrak{t}_{n,k}$ as in (9), and the corresponding sequence (a_n) of (1). We will show that conditions (a)–(c) of Definition 1.4 are satisfied. Condition (a) is obvious.

Using $\binom{a}{b} \leqslant 2^a$, and the fact that the set (3) is a subset of $[-Kn, Kn]^r \cap \mathbb{Z}^r$ for some $K > 0$, Eq. (9) implies that there exists a constant $C > 0$ so that

$$|\mathfrak{t}_{n,k}| < C^n$$

for all (n, k) and for all complex conjugates of $\mathfrak{t}_{n,k}$. Summing up with respect to k in Eq. (1), and using the fact that the summation set has polynomial size in n, it follows (after possibly enlarging C) that

$$|a_{\mathfrak{t},n}| < C^n$$

for all $n > 0$ and for all complex conjugates of a_n. This proves condition (b) of Definition 1.4.

Condition (c) follows from Eq. (9), Lemma 2.2 and the fact that the summation set (3) is bounded polynomially by n.

Condition (d) follows from Wilf–Zeilberger's Theorem 4. □

2.2. The local monodromy of a G-function

In this section we will make little distinction between a convergent power series, its analytic continuation, and the corresponding function. Recall that a power series is *holonomic* if it satisfies a linear differential equation $PG(z) = 0$ where $P \in \overline{\mathbb{Q}}\langle z, d/dz\rangle$ is a linear differential operator with coefficients in $\overline{\mathbb{Q}}[z]$. By the theory of differential equations (see for example [15,19]), a holonomic function has analytic continuation as a multivalued analytic function in $\mathbb{C} \setminus \Lambda$, where Λ is a finite set of algebraic numbers. The following theorem follows from a combination of results of Katz, André and Chudnovsky; see [1,7,17] and also [6] for a detailed exposition.

Theorem 6. *(See [1,7,17].) The local monodromy T of a G-function around a singularity is quasi-unipotent. In other words,*

$$\left(T^r - 1\right)^s = 0 \tag{11}$$

for some nonzero natural numbers r and s.

It follows that the local expansion of $G(z)$ around a singularity $\lambda \in \Lambda$ is a finite sum of series of the form:

$$\sum_{\alpha,\beta} c_{\alpha,\beta}(z-\lambda)^{\alpha}\left(\log(z-\lambda)\right)^{\beta} h_{\alpha,\beta}(z-\lambda) \tag{12}$$

where $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, $c_{\alpha,\beta} \in \mathbb{C}$ and $h_{\alpha,\beta}(w)$ are convergent germs at $w = 0$. In fact, André shows that $h_{\alpha,\beta}(w)$ are G-functions themselves. Said differently, the G-functions that come from arithmetic

(a) are *regular holonomic* (i.e., the power series $h_{\alpha,\beta}(w)$ above are convergent at $w = 0$), and
(b) have *rational exponents* (denoted by $\{\alpha\}$ above).

On the other hand, the generating series of a generic exponentially bounded holonomic sequence (a_n) will not be in general regular holonomic, nor will they have rational exponents. This explains Theorem 5.

Remark 2.3. Power series of the form (12) are known in the literature as *Nilsson series*; see [18].

2.3. The Taylor series of a G-function and Theorem 1

The following lemma is a well-known application of Cauchy's theorem; see for example [16, Thm. A].

Lemma 2.4. *(See [16, Thm. A].) If $\alpha \in \mathbb{C} \setminus \mathbb{N}$, $\beta \in \mathbb{N}$, and*

$$(1-z)^{\alpha}\left(\log(1-z)\right)^{\beta} = \sum_{n=0} a_n z^n$$

then

$$a_n = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(\left(\log(n)\right)^{\beta}\phi_0(n) + \cdots + \left(\log(n)\right)^{0}\phi_\beta(n)\right)$$

where $\phi_j(z)$ for $j = 0, \ldots, \beta$ are Gevrey-1 series with rational coefficients.

Recall that a series $\sum_{n=0}^{\infty} a_n z^n$ is *Gevrey*-1 (resp. *arithmetic Gevrey*-1) if $\sum_{n=0}^{\infty}(a_n/n!)z^n$ is convergent at $z = 0$ (resp. a G-function). Lemma 2.4 and a deformation of the contour argument implies the following. See also [8, Sec. 7].

Proposition 2.5. *If $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a G-function, then (a_n) is of Nilsson type, where λ in Eq. (6) are the singularities of $G(z)$, and α, β, $f_{\lambda,\alpha,\beta}$ are determined by the local monodromy of $G(z)$ at $z = \lambda$.*

Remark 2.6. In fact, Jungen's proof combined with André's theorem that the series $h_{\alpha,\beta}(w)$ are G-functions, implies that the series $f_{\lambda,\alpha,\beta}(z)$ of Eq. (6) are arithmetic Gevrey-1.

2.4. *Proof of Theorem 5*

The next lemma is well known.

Lemma 2.7. *If (a_n) is holonomic, the generating series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is holonomic.*

Birkhoff and Trjitzinsky, followed by Turrittin (see [3,23,28] and [2, Eqn. 1.3]) prove the following result concerning the asymptotic expansion of a holonomic sequence.

Proposition 2.8. *If $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holonomic function, then*

$$a_n \sim \sum_{\lambda,\alpha,\beta,s} n!^{s}\lambda^{-n} n^{-\alpha-1}\left(\log(n)\right)^{\beta} f_{\lambda,\alpha,\beta,s}\left(\frac{1}{n}\right) \tag{13}$$

where λ lies in a subset of the finite set of singularities of $G(z)$, s lies in a finite set of nonpositive rational numbers, and α, β are the exponents in the local expansion of $G(z)$ around λ, and $f_{\lambda,\alpha,\beta,s}(z)$ are Gevrey-1.

2.5. *The exponents of the sequence of Theorem 1*

It remains to compute the exponents of the holonomic function $G(z)$ associated to the sequence of Eq. (8). One way to solve this problem is to convert the holonomic equation (8) into a differential equation for the generating series and compute the exponents of the differential equation using *Frobenius's method*; see [15,19]. In addition, one needs to show that the corresponding constants $c_{\alpha,\beta}$ in (12) are nonvanishing. An alternative way is to relate the exponents of the generating series $G(z)$ of a sequence (a_n) to the asymptotic expansion of the sequence itself.

Consider the sequence (a_n) given by (8) and its generating series $G(z)$. Converting the recursion relation for (a_n) into a differential equation for $G(z)$ we obtain that $G(z)$ satisfies the inhomogeneous differential equation:

$$z\big(z^2 - 7z + 2\big)G'(z) + \big(z^2 - 4z - 3\big)G(z) + z = 0, \quad G(0) = 0. \tag{14}$$

If we wish, we can divide by z and differentiate once to get a linear second order differential equation for $G(z)$. The singularities Λ of $G(z)$ is a subset of the roots of $z(z^2 - 7z + 2)$. I.e., we have:

$$\Lambda \subset \left\{0, \frac{1}{4}(7 \pm \sqrt{33})\right\}. \tag{15}$$

Frobenius's method gives that the exponent at $\lambda_\pm = \frac{1}{4}(7 \pm \sqrt{33})$ is given by

$$\alpha_\pm = -1 \pm \frac{5}{2}\sqrt{\frac{3}{11}} \tag{16}$$

which is non-rational. It is easy to compute that $\beta = 0$. It remains to argue that the so-called *Stokes constant* $c_{\alpha_\pm,\beta} \neq 0$. One can do an explicit numerical computation in the spirit of [11, Sec. 4], using *Padé approximants* and working in the so-called *Borel plane*.

Alternatively, we may argue as follows. If $G(z)$ is analytic at $\frac{1}{4}(7 + \sqrt{33})$, then by Galois invariance and Eq. (15), it follows that $G(z)$ is entire. Lemma 2.9 below implies that $G(z)$ is a polynomial. It follows that $a_n = 0$ for sufficiently large n. The recursion relation (8) implies that $a_n = 0$ for all $n \in \mathbb{N}$, an obvious contradiction.

The next lemma was communicated to us by Y. André, and is a useful way of deducing the existence of singularities of G-functions. For a detailed discussion, see also [6].

Lemma 2.9. *Every entire G-function is a polynomial.*

Proof. According to Chudnovsky and Katz, a G-function is a solution of a *Fuchsian* differential equation, i.e., regular singular in $\mathbb{C} \cup \{\infty\}$. An entire G-function does not have any monodromy at finite distance, hence it does not have any monodromy at infinity as well. According to a classical result of Schlesinger, any solution of a Fuchsian differential equation which is invariant under the global monodromy group is a rational function. If, moreover, it is entire, then it is a polynomial function. □

3. Further discussion

Theorem 2 may be viewed as a way of constructing holonomic G-functions from enumerative combinatorics. There are two well-known sources of G-functions: from *arithmetic* (see Theorem 6 and also [1,4,10,17]), and from *geometry*, related to the regularity of the *Gauss–Manin connection*. For the latter, see for example, [5,9,17]. In all cases (combinatorics, geometry and arithmetic), the constructed G-functions are regular holonomic with rational exponents.

The G-functions obtained geometry and arithmetic are closely related. The main conjecture is that all G-functions come from geometry. For a discussion of this topic, and for a precise formulation of the Bombieri–Dwork Conjecture, see the survey papers of [4,17] and also [27, p. 8]. Our question is motivated by Theorem 2 and Bombieri–Dwork Conjecture of [27, p. 8].

Question 1. If (a_n) is an integer-valued, exponentially bounded holonomic sequence, does it follow that it is a multisum sequence?

Our next question compares the G-functions of Theorem 2 with those that come from geometry.

Question 2. Does every G-function of Theorem 2 come from geometry?

In [13] this was shown to be true when the balanced term $\mathfrak{t}_{n,k}$ is *special*, i.e., it is a product of binomials of linear forms of (n, k) (in other words, $\epsilon_j = +1$ for all $j = 1, \dots, J$ in Eq. (9).

Finally, let us point out that the proof of Theorems 1 and 2 is not constructive. In particular, it would be nice to be able to compute the singularities of the generating series of a balanced multisum sequence directly from the balanced term $\mathfrak{t}$. With this in mind, the author developed an efficient ansatz for the asymptotics of balanced multisum sequences; see [13]. When $r = 1$ in Eq. (2) (i.e., for single-sums), the ansatz can be proven using the Euler–MacLaurin formula and various ideas of resurgence; see [14].

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