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Alternating knots, planar graphs, and q-series

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Abstract Recent advances in Quantum Topology assign q-series to knots in at least three different ways. The q-series are given by generalized Nahm sums (i.e., special q-hypergeometric sums) and have unknown modular and asymptotic properties. We give an efficient method to compute those q-series that come from planar graphs (i.e., reduced Tait graphs of alternating links) and compute several terms of those series for all graphs with at most 8 edges drawing several conclusions. In addition, we give a graph-theory proof of a theorem of Dasbach-Lin which identifies the coefficient of q^k in those series for $k = 0, 1, 2$ in terms of polynomials on the number of vertices, edges, and triangles of the graph.

Keywords Knots · Colored Jones polynomial · Stability · Index · q-series · q-hypergeometric series · Nahm sums · Planar graphs · Tait graphs

Mathematics Subject Classification Primary 57N10 · Secondary 57M25

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1 Introduction

1.1 q-series in quantum knot theory

Recent developments in Quantum Topology associate q-series to a knot K in at least three different ways are as follows:

- via stability of the coefficients of the colored Jones polynomial of K,
- via the 3D index of K, and
- via the conversion of state-integrals of the quantum dilogarithm to q-series.

The first method is developed of alternating knots in detail, see [1,3,4] and also [13]. The second method uses the 3D index of an ideal triangulation introduced in [6,7], with necessary and sufficient conditions for its convergence established in [9] and its topological invariance (i.e., independence of the ideal triangulation) for hyperbolic 3-manifolds with torus boundary proven in [11]. The third method was developed in [12].

In all three methods, the q-series are multi-dimensional q-hypergeometric series of generalized Nahm type; see [13, Sect. 1.1]. Their modular and the asymptotic properties remain unknown. Some empirical results and relations among these q-series are given in [15,16].

The paper focuses on the q-series obtained by the first method. For some alternating knots, the q-series obtained by the first method can be identified with a finite product of unary theta or false theta series; see [1,2]. This was observed independently by the first author and Zagier in 2011 for all alternating knots in the Rolfsen table [18] up to the knot 8_4. Ideally, one might expect this to be the case for all alternating knots. For the knot 8_5, however, the first 100 terms of its q-series failed to identify it with a reasonable finite product of unary theta or false theta series. This computation was performed by the first author at the request of Zagier, and the result was announced in [10, Sect. 6.4].

The purpose of the paper was to give the details of the above computation and to extend it systematically to all alternating knots and links with at most 8 crossings. Our computational approach is similar to the computation of the index of a knot given in [11, Sect. 7].

1.2 Rooted plane graphs and their q-series

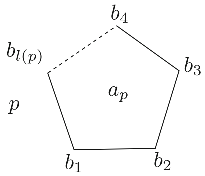
By *planar graph*, we mean an abstract graph, possibly with loops and multiple edges, which can be embedded on the plane. A *plane graph* (also known as a planar map) is an embedding of a planar graph to the plane. A *rooted plane map* is a plane map together with the choice of a vertex of the unbounded region.

In [13], Le and the first author introduced a function

$$\Phi : \{\text{Rooted plane graphs}\} \longrightarrow \mathbb{Z}[[q]], \qquad G \mapsto \Phi_G(q).$$

For the precise relation between $\Phi_G(q)$ and the colored Jones function of the corresponding alternating link L_G, see Sect. 2. To define $\Phi_G(q)$, we need to introduce some notation. An *admissible state* (a, b) of G is an integer assignment a_p for each face p of G and b_v for each vertex v of G such that $a_p + b_v \geq 0$ for all pairs (v, p), where v is a vertex of p. For the unbounded face p_∞, we set $a_\infty = 0$, and thus $b_v = a_\infty + b_v \geq 0$ for all $v \in p_\infty$. We also set $b_v = 0$ for a fixed vertex v of p_∞. In the formulas below, v and w will denote vertices of G, and p is the face of G and p_∞ is the unbounded face. We also write $v \in p$, $vw \in p$ if v is a vertex and vw is an edge of p.

For a polygon p with $l(p)$ edges and vertices $b_1, \ldots, b_{l(p)}$ in counterclockwise order,



we define

$$\gamma(p) = l(p)a_p^2 + 2a_p(b_1 + b_2 + \cdots + b_{l(p)}).$$

Let

$$A(a, b) = \sum_p \gamma(p) + 2 \sum_{e=(v_i v_j)} b_{v_i} b_{v_j}, \tag{1}$$

where the p-summation (here and throughout the paper) is over the set of *bounded* faces of G and the e-summation is over the set of edges $e = (v_i v_j)$ of p, and

$$B(a, b) = 2 \sum_v b_v + \sum_p (l(p) - 2)a_p, \tag{2}$$

where the v-summation is over the set of vertices of G and the p-summation is over the set of bounded faces of G.

Definition 1.1 [13] With the above notation, we define

$$\Phi_G(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod_{(p,v):v\in p} (q)_{a_p+b_v}}, \tag{3}$$

where the sum is over the set of all admissible states (a, b) of G, and in the product $(p, v) : v \in p$ means a pair of face p and vertex v such that p contains v. Here, c_2 is the number of edges of G and

$$(q)_\infty = \prod_{n=1}^{\infty} (1 - q)^n = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \ldots$$

Convergence of the q-series of Eq. (3) in the formal power series ring $\mathbb{Z}[[q]]$ is not obvious, but was shown in [13]. Below, we give effective (and actually optimal) bounds for convergence of $\Phi_G(q)$. To phrase them, let $b_p = \min\{b_v : v \in p\}$, where p denotes a face of G.

Theorem 1.2 (a) *We have*

$$A(a,b) = \sum_p \left(l(p)(a_p + b_p)^2 + 2(a_p + b_p) \sum_{v \in p} (b_v - b_p) + \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) \right) + \sum_{vv' \in p_\infty} b_v b_{v'} \tag{4}$$

Each term in the above sum is manifestly nonnegative.

(b) *$B(a,b)$ can also be written as a finite sum of manifestly nonnegative linear forms on (a,b).*

(c) *If $\frac{1}{2}(A(a,b) + B(a,b)) \leq N$ for some natural number N, then for every i and every j, there exist c_i, c_i' and c_j, c_j' (computed effectively from G) such that*

$$c_i N \leq b_i \leq c_i' N, \qquad c_j' \sqrt{N} \leq a_j \leq c_j N + c_j' \sqrt{N}\,.$$

For a detailed illustration of the above Theorem, see Sect. 5.1.

1.3 Properties of the q-series of a planar graph

The next lemma summarizes some properties of the series $\Phi_G(q)$. Part (a) of the next lemma is taken from [13, Theorem 1.7] [13, Lemma 13.2]. Parts (b) and (c) were observed in [1] and [13] and follow easily from the behavior of the colored Jones polynomial under disjoint union and under a connected sum. Note that we use the normalization that the colored Jones polynomial of the unknot is 1. Part (d) was proven in [1] and [13, Lemma 13.3].

Lemma 1.3 *[1, 13]*

(a) *The series $\Phi_G(q)$ depends only on the abstract planar graph G and not on the rooted plane map.*

(b) *If $G = G_1 \sqcup G_2$ is disconnected, then*

$$(1-q)\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)\,.$$

(c) *If G has a separating edge (also known as a bridge) e and $G \setminus \{e\} = G_1 \sqcup G_2$, then*

$$\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)\,.$$

(d) *If G is a planar graph (possibly with multiple edges and loops) and G' denotes the corresponding simple graph obtained by removing all loops and replacing all edges of multiplicity more than with edges of multiplicity one, then*

$$\Phi_G(q) = \Phi_{G'}(q) \,.$$

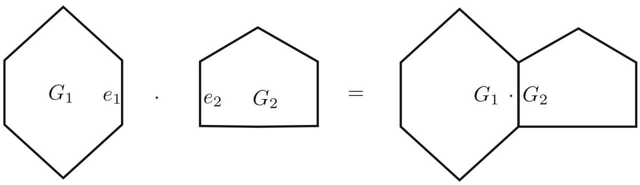
So, we can focus our attention to simple, connected planar graphs. In the remaining of the paper, unless otherwise stated, G will denote a *simple* planar graph. Let $\langle f(g)\rangle_k$ denote the coefficient of q^k of $f(q) \in \mathbb{Z}[[q]]$. The next theorem was proven in [8] using properties of the Kauffman bracket skein module. We give an independent proof using combinatorics of planar graphs in Sect. 4. Our proof allows us to compute the coefficient of q^3 in $\Phi_G(q)$, observing a new phenomenon related to induced embeddings, and guesses the coefficients of q^4 and q^5 in $\Phi_G(q)$. This is discussed in a subsequent publication [14].

Theorem 1.4 *[8] If G is a planar graph, we have*

$$\langle \Phi_G(q)\rangle_0 = 1 \tag{5a}$$
$$\langle \Phi_G(q)\rangle_1 = c_1 - c_2 - 1 \tag{5b}$$
$$\langle \Phi_G(q)\rangle_2 = \frac{1}{2}\Big((c_1 - c_2)^2 - 2c_3 - c_1 + c_2\Big)\,, \tag{5c}$$

where c_1, c_2, and c_3 denote the number of vertices, edges, and 3-cycle of G.

If G_1 and G_2 are the two planar graphs with distinguished boundary edges e_1 and e_2, let $G_1 \cdot G_2$ denote their edge connected sum along $e_1 = e_2$ depicted as follows:



Let P_r denote a planar polygon with r edges when $r \geq 3$, and let P_2 denote the connected graph with two vertices and one edge, a reduced form of a bigon. For a positive natural number b, consider the unary theta (when b is odd) and false theta series (when b is even) $h_b(q)$ is given by

$$h_b(q) = \sum_{n\in\mathbb{Z}} \varepsilon_b(n)\, q^{\frac{b}{2}n(n+1)-n},$$

where

$$\varepsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd} \\ 1 & \text{if } b \text{ is even and } n \geq 0 \\ -1 & \text{if } b \text{ is even and } n < 0 \end{cases}.$$

Observe that

$$h_1(q) = 0, \qquad h_2(q) = 1, \qquad h_3(q) = (q)_\infty \,.$$

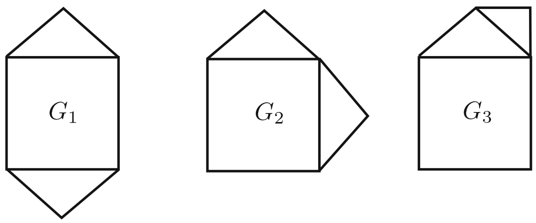


Fig. 1 Three graphs G_1, G_2, and G_3, and the corresponding alternating links $L8a8$, $L8a8$, and 8_{13}

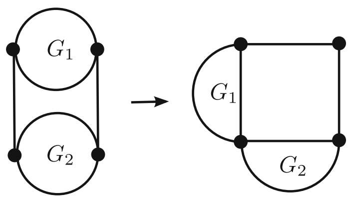


Fig. 2 A flyping move on a planar graph

The following lemma (observed independently by Armond-Dasbach) follows from the Nahm sum for $\Phi_G(q)$ combined with a q-series identity (see Eq. (16) below). This identity was proven by Armond-Dasbach [1, Theorem 3.7] and Andrews [2].

Lemma 1.5 *For all planar graphs G and natural numbers $r \geq 3$, we have*

$$\Phi_{G \cdot P_r}(q) = \Phi_G(q)\Phi_{P_r}(q) = \Phi_G(q)h_r(q) .$$

Question 1.6 Is it true that for all planar graphs G_1 and G_2, we have

$$\Phi_{G_1 \cdot G_2}(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)?$$

As an illustration of Lemma 1.5, for the three graphs of Fig. 1, we have

$$\Phi_{L8a8}(q) = \Phi_{8_{13}}(q) = h_4(q)h_3(q)^2 .$$

Remark 1.7 Observe that the alternating planar projections of the graphs G_1 and G_2 of Fig. 1 are related by a *flype move* [17, Fig. 1].

Flyping a planar alternating link projection corresponds to the operation on graphs shown in Fig. 2.

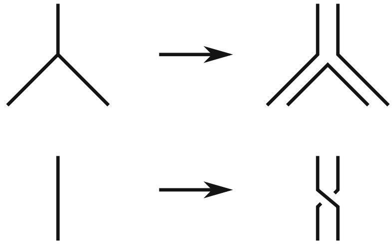
If the planar graphs G and G' are related by flyping, then $\Phi_G(q) = \Phi_{G'}(q)$, since the corresponding alternating links are isotopic.

2 The connection between $\Phi_G(q)$ and alternating links

In this section, we explain connection between $\Phi_G(q)$ and the colored Jones function of the alternating link L_G following [13].

2.1 From planar graphs to alternating links

Given a planar graph G (possibly with loops or multiple edges), there is an alternating planar projection of a link L_G given by



2.2 From alternating links to planar (Tait) graphs

Given a diagram D of a *reduced alternating non-split* link L, its Tait graph can be constructed as follows: the diagram D gives rise to a polygonal complex of $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Since D is alternating, it is possible to label each polygon by a color b (black) or w (white) such that at every crossing, the coloring looks as follows in Fig. 3.

There are exactly two ways to color the regions of D with black and white colors. In this note, we will work with the one whose unbounded region has color w. In each b-colored polygon (in short, b-polygon), we put a vertex and connect two of them with an edge if there is a crossing between the corresponding polygons. The resulting graph is a planar graph called the Tait graph associated with the link diagram D. Note that the Tait graph is always planar but not necessarily reduced. Although the reduction of the Tait graph may change the alternating link and its colored Jones polynomial, it does not change the limit of the shifted colored Jones function in Theorem 2.1 because of Lemma 1.3.

2.3 The limit of the shifted colored Jones function

When L is an alternating link, the colored Jones polynomial $J_{L,n}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ (normalized to be 1 at the unknot, and colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$ [13]) has the lowest q-monomial with coefficient ± 1, and after dividing by this monomial, we obtain the *shifted* colored Jones polynomial

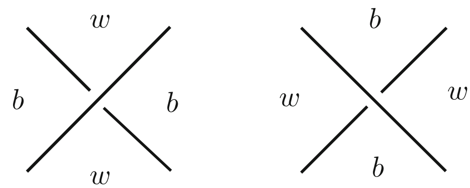


Fig. 3 The checkerboard coloring of a link diagram

$\hat{J}_{L_G,n}(q) \in 1 + q\mathbb{Z}[q]$. Let $\langle f(q)\rangle_N$ denotes the coefficient of q^N in $f(q)$. The limit $f(q) = \lim_n f_n(q) \in \mathbb{Z}[[q]]$ of a sequence of polynomials $f_n(q) \in \mathbb{Z}[q]$ is defined as follows [13]. For every natural number N, there exists a natural number $n_0(N)$ such that $\langle f_n(q)\rangle_N = \langle f(q)\rangle_N$ for all $n \geq n_0(N)$.

Theorem 2.1 *[13, Theorem 1.10] Let L be an alternating link projection and G be its Tait graph. Then, the following limit exists:*

$$\lim_{n\to\infty} \hat{J}_{L,n}(q) = \Phi_G(q) \in \mathbb{Z}[[q]]. \tag{6}$$

Remark 2.2 (a) The convergence statement in the above theorem holds in the following strong form [13]: for every natural number N, and for $n > N$, we have

$$\langle \hat{J}_{L,n}(q)\rangle_N = \langle \Phi_G(q)\rangle_N \,. \tag{7}$$

(b) $\Phi_G(q)$ is the *reduced* version of the one in [13, Theorem 1.10] and differs from the unreduced version $\Phi_G^{\mathrm{TQFT}}(q)$ by

$$\Phi_G(q) = (1-q)\Phi_G^{\mathrm{TQFT}}(q)\,,$$

where

$$\Phi_G^{\mathrm{TQFT}}(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod\limits_{(p,v):v\in p} (q)_{a_p+b_v}}, \tag{8}$$

and the summation (a,b) is over all admissible states where we do not assume that $b_v = 0$ for a fixed vertex v in the unbounded face of G.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Part (a) follows from completing the square in Eq. (1):

$$\begin{aligned}
A(a,b) &= \sum_p \left(l(p)a_p^2 + 2a_p\left(\sum_{v\in p} b_v\right)\right) + 2\sum_{e=(v_iv_j)} b_{v_i}b_{v_j} \\
&= \sum_p \left(l(p)(a_p+b_p)^2 + 2a_p\left(\sum_{v\in p} b_v - l(p)b_p\right) - l(p)b_p^2 + 2\sum_{e=(v_iv_j)} b_{v_i}b_{v_j}\right) \\
&= \sum_p \left(l(p)(a_p+b_p)^2 + 2(a_p+b_p)\left(\sum_{v\in p} b_v - l(p)b_p\right)\right. \\
&\quad \left. + \sum_{e=(v_iv_j)\in p} (b_{v_i}-b_p)(b_{v_j}-b_p)\right) + \sum_{e=(v_iv_j)\in p_\infty} b_{v_i}b_{v_j}.
\end{aligned}$$

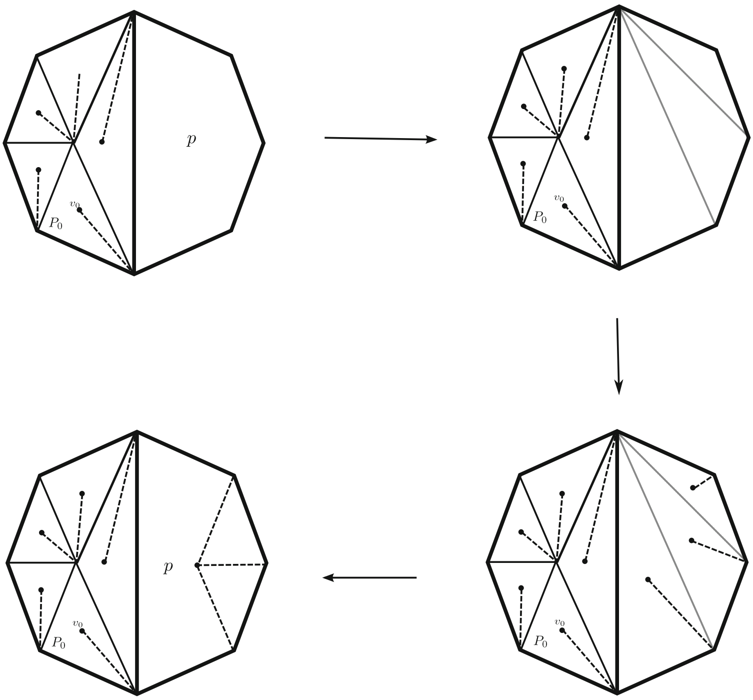
For the remaining parts of Theorem 1.2, fix a 2-connected planar graph G, a vertex v_0 of G and a bounded face p_0 of G that contains v_0.

Lemma 3.1 *There exists a graph* Γ *which depends on* G, v_0, *and* p_0 *such that*

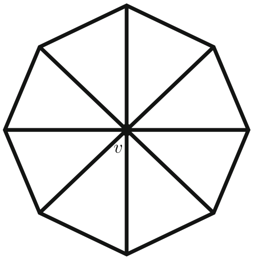
- *The vertices of* Γ *are vertices of* G *as well as one vertex* v_p *for each bounded face* p *of* G.
- *The edges of* Γ *are of the form* vv_p, *where* v *is a vertex of* G *and* p *is a bounded face that contains* v.
- $v_0 v_{p_0}$ *is an edge of* Γ.
- *Every vertex* v *in* G *has degree* n_v *in* Γ *where*

$$n_v = \begin{cases} 2 & \text{if } v \text{ is not a boundary vertex} \\ \leq 2 & \text{if } v \text{ is a boundary vertex} \end{cases}.$$

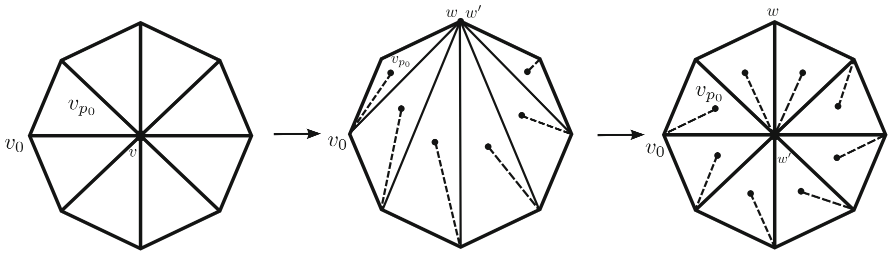
Proof First, we can assume that each face p of G is a triangle. Indeed, if a face p is not a triangle, we can divide it into a union of triangles by creating new edges inside p. Once we have succeeded in constructing a Γ for the resulted graph, we can remove the added edges in p and collapse all the interior vertices of the newly created triangles in p into one single vertex v_p. The figures below illustrate the above process.



Now, assuming that all faces of G are triangles, let us proceed by induction on the number of vertices of G. If there is no interior vertex in G then since the unbounded face p_∞ is also a triangle, then G itself is a triangle and we are done. Therefore, let us assume that there is an interior vertex v of G. Locally, the graph at v looks like the following:



Next, we remove v and all of the edges incident to it from G and denote the resulted face by p. Let w be a vertex of p and connect w to each of the vertices of p by an edge. Denote the resulted graph by G_w. By induction hypothesis, there exists a graph Γ_w for G_w. At w, make another copy of the vertex called w'. Now, drag w' into the interior of p while keeping it connected to vertices of p and at the same time, delete the edges that are incident to w and that lie in the interior of p. This has to be done in such a way that all the vertices of Γ_w still lie in the interior of the new triangles that have w' as a vertex. Create two new vertices in the interior of the two triangles in p that contain w as a vertex and connect them to w'. The resulted graph satisfies the requirements of the lemma. The figures below explain the process. □



Proof (of part (b) of Theorem 1.2) We can decompose $B(a, b)$ into a finite sum of nonnegative terms as follows:

$$B(a,b) = \sum_{\hat{e}=(vv_p)} (a_p + b_v) + \sum_v (2 - n_v) b_v, \tag{9}$$

where the summation is over all edges of Γ. □

Corollary 3.2 *For a pair* (p, v)*, where* p *is a face of* G *and* v *is a vertex of* p*, the* $B(a, b) \geq a_p + b_v$*.*

Proof This is a direct consequence of Eq. (9) since by Lemma 3.1, there exists a graph Γ that contains vv_p as an edge. □

Proof (of part (c) of Theorem 1.2) Let us prove the linear bound on the b_v first. Let us set $b_{v_0} = 0$, where v_0 is a boundary vertex of G. Let p_0 be a bounded face that contains v_0, so we have $a_{p_0} + b_{v_0} \geq 0$. Since $0 \leq B(a, b) \leq 2N$ by part (b) of Theorem 1.2 and Corollary 3.2, we have that $0 \leq a_{p_0} + b_{v_0} \leq 2N$. Since $b_{v_0} = 0$ this means that $0 \leq a_{p_0} \leq 2N$. Similarly, if v is another vertex of p_0, then by Corollary 3.2, we have $0 \leq a_{p_0} + b_v \leq 2N$ which implies that $-2N \leq b_v \leq 2N$. Let G' be the graph obtained from G by removing the boundary edges of p_0. Choose a face p' of G' and a vertex $v' \in p'$ that also belongs to the removed face p_0. Repeating the above process with (p', v'), we have that $-4N \leq b_{v''} \leq 4N$ for any $v'' \in p'$. Continuing this process until all faces of g are covered, we have that $|b_v| \leq dN$ for all vertices v of G.

To prove the bound for the a_p's, note that from part (a) of Theorem 1.2, we have that $\frac{e(p)}{2}(a_p + b_v)^2 \leq N$ for all bounded faces p and all vertices v of G. This implies that $|a_p + b_v| \leq \sqrt{\frac{2}{e_p}}\sqrt{N}$. Since $|b_v| \leq dN$ this implies that $|a_p| \leq \sqrt{\frac{2}{e_p}}\sqrt{N} + dN$. For the lower bound of a_p, note that since $a_p + b_v \geq 0$, we have $a_p \geq -b_v \geq -dN$. □

4 The coefficients of 1, q, and q^2 in $\Phi_G(q)$

4.1 Some lemmas

In this section, we prove Theorem 1.4, using the unreduced series $\Phi_G^{\mathrm{TQFT}}(q)$ of Eq. (8). Our admissible states (a, b) in this section do not satisfy the property that $b_v = 0$ for some vertex v of the unbounded face of G.

Since $A(a, b) + B(a, b) \geq 0$ for an admissible state (a, b) with equality if and only if $(a, b) = (0, 0)$ (as shown in Theorem 1.2), it follows that the coefficient of q^0 in $\Phi_G(q)$ is 1. For the remaining of the proof of Theorem 1.4, we will use several lemmas.

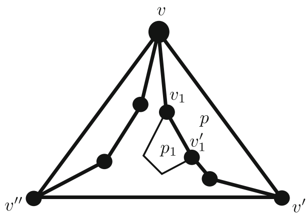
Lemma 4.1 *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

(1) $b_v = b_{v'} = 1$ *where vv' is an edge of p_∞,*
(2) $a_p + b_p = 0$ *for any face p of G, and*
(3) $(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ *for any face p of G and edge $v_1 v_2$ of p,*

then

- $b_v \geq 1$ *for all vertices v,*
- $a_p = -1$ *for all faces $p \neq p_\infty$, and*
- $B(a, b) \geq 2 + V_\infty$.

Proof Let p be the bounded face that contains v, v'. We have $(b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 1$ since $b_v = b_{v'} = 1$. (2) then implies that $a_p = -b_p = -1$, and thus $b_w \geq b_p = 1$ for all $w \in p$. Let $v_1 v_1'$ be another edge of p and let $p_1 \neq p$ be a



face that contains v_1v_1'. Since $(b_{v_1} - b_p)(b_{v_1'} - b_p) = 0$, we have $\min\{b_{v_1}, b_{v_1'}\} = b_p = 1$. So from $(b_{v_1} - b_{p_1})(b_{v_1'} - b_{p_1}) = 0$, we have that $b_{p_1} = 1$. Therefore, $a_{p_1} = -1$ and $b_w \geq b_{p_1} = 1$ for any vertex $w \in p_1$. By a similar argument, we can show that $b_v \geq 1$ for every vertex v and $a_p = -1$ for every face p of G. Let $p_1, p_2, \ldots, p_f$ be the bounded faces of G, where $f = F_G - 1$. Then, from Eq. (2), we have

$$\begin{aligned}
B(a,b) &= -\sum_{j=1}^{f}(l(p_j) - 2) + 2\sum_{v} b_v \\
&\geq -\sum_{j=1}^{f} l(p_j) + 2f + 2c_1 \\
&= -(2c_2 - V_\infty) + 2F_G - 2 + 2c_1 \\
&= 2(c_1 - c_2 + F_G) - 2 + V_\infty \\
&= 2 + V_\infty.
\end{aligned}$$

□

The proof of the next lemma is similar to the one of Lemma 4.1 and is, therefore, omitted.

Lemma 4.2 *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

(1) $b_v = b_{v'} = 0$ *and* $(b_v - b_p)(b_{v'} - b_p) = 1$ *where p is a boundary face and vv' is a boundary edge that belongs to p,*
(2) $a_p + b_p = 0$ *for any face p of G, and*
(3) $(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ *for any face p of G and edge v_1v_2 not on the boundary of p,*

then $b_w \geq -1$ for all vertices w, $a_p = 1$ for all faces $p \neq p_\infty$, and $B(a, b) \geq V_\infty - 2$. Furthermore, $B(a, b) = V_\infty - 2$ if and only if

- $b_v = 0$ *for all boundary vertices v and $b_w = -1$ for all other vertices w.*
- $a_p = 1$ *for all faces p.*

Lemma 4.3 *Let G be a 2-connected planar graph, p_0 be a boundary face, and (a, b) be an admissible state such that*

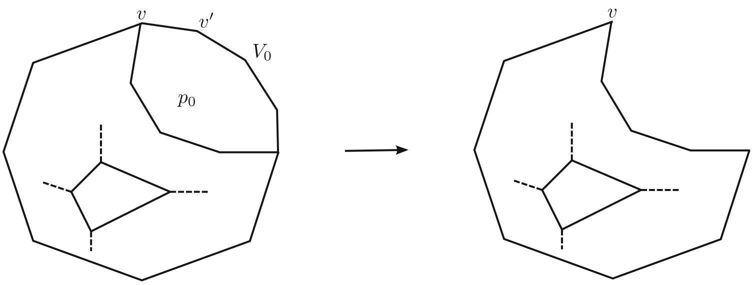
(1) $a_{p_0} + b_{p_0} = 0$,

(2) *There exists a boundary edge* vv' *of* p_0 *such that* $b_v b_{v'} = 0$ *and* $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$*, and*
(3) *Let* G_0 *be the graph obtained from* G *by deleting the boundary edges of* p_0*, and let* (a_0, b_0) *be the restriction of the admissible state* (a, b) *on* G_0*.*

Then,

(a) (a_0, b_0) *is an admissible state for* G_0*,*
(b) $A(a_0, b_0) = A(a, b) - \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'}$*,*
(c) $B(a_0, b_0) = B(a, b) - 2 \sum_{v \in V_0} b_v$*, where* V_0 *is the set of boundary vertices of* p_0 *that do not belong to any other bounded face,*
(d) $B(a, b) \geq 2 \sum_{v \in V_0} b_v$*,*
(e) *If furthermore* $B(a, b) \leq 1$*, then* $A(a, b) = A(a_0, b_0)$*,* $B(a, b) = B(a_0, b_0)$*.*

Proof From (2), we have either $b_v = 0$ or $b_{v'} = 0$, and it follows from $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$ that $b_{p_0} = 0$. This means that we have $b_v \geq 0$ for all $v \in p_0$. This implies (a). Furthermore, (1) implies that $a_{p_0} = 0$, and thus $A(a, b) - A(a_0, b_0) = l(p_0)a_{p_0}^2 + 2a_{p_0}(\sum_{v \in p_0} b_v) + \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'} = \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'}$ and $B(a, b) - B(a_0, b_0) = a_{p_0} + 2 \sum_{v \in V_0} b_v = 2 \sum_{v \in V_0} b_v$. This proves (b) and (c). (d) follows from (c) since we have $0 \leq B(a_0, b_0) = B(a, b) - 2 \sum_{v \in V_0} b_v$, and (e) is a consequence of (b), (c), and (d) since $1 \geq B(a, b) \geq 2 \sum_{v \in V_0} b_v$ implies that $\sum_{v \in V_0} b_v = 0$.



□

4.2 The coefficient of q in $\Phi_G(q)$

We need to find the admissible states (a, b) such that $\frac{1}{2}(A(a, b) + B(a, b)) = 1$. Parts (a) and (b) of Theorem 1.2 imply that $A(a, b), B(a, b) \in \mathbb{N}$. Thus, if $\frac{1}{2}(A(a, b) + B(a, b)) = 1$, then we have the following cases:

$A(a, b)$	2	1	0
$B(a, b)$	0	1	2

Case 1: $(A(a,b), B(a,b)) = (2,0)$. Since $l(p) \geq 3$, we should have $a_p + b_p = 0$ for all faces p. This implies that $a_p + b_v = a_p + b_p + b_v - b_p = b_v - b_p$, and it follows from Corollary 3.2 that $0 = B(a,b) \geq a_p + b_v = b_v - b_p$. This means $b_v - b_p = a_p + b_v = 0$ for all faces p and vertices v of p, so Eq. (4) is equivalent to

$$\sum_{vv' \in p_\infty} b_v b_{v'} = 2\,. \tag{10}$$

If vv' is an edge of G and p is a face that contains vv', then we have $a_p + b_v = 0 = a_p + b_{v'}$, and therefore $b_v = b_{v'}$. So, by Eq. (10), there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 4.1 implies that $B(a,b) \geq 2 + V_\infty > 0$ which is impossible. Therefore, there are no admissible states (a,b) that satisfy $(A(a,b), B(a,b)) = (2,0)$.

Case 2: $(A(a,b), B(a,b)) = (1,1)$. As above, we have that $a_p + b_p = 0$ for all faces p. Since $A(a,b) = 1$, there is either a bounded face p_1 with an edge $v_1 v_1'$ such that $(b_{v_1} - b_{p_1})(b_{v_1'} - b_{p_1}) = 1$ or a boundary edge $v_2 v_2'$ such that $b_{v_2} b_{v_2'} = 1$, and all other terms in Eq. (4) are equal to zero. Let p_2 be the bounded face that contains $v_2 v_2'$ and let $p \neq p_1, p_2$ be a bounded face. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 4.3, we have $A(a',b') = A(a,b)$ and $B(a',b') = B(a,b)$. Continue this process until either $G = p_1$ or $G = p_2$. If $G = p_2$, then $b_{v_2} b_{v_2'} = 1$, and therefore $B(a,b) \geq 2(b_{v_2} + b'_{v_2}) = 4$ which is impossible. If $G = p_1$, then v_1, v_2 are now boundary vertices and so $b_{v_1} b_{v_1'} = 0$ and we can assume that $b_{v_1} = 0$. But this implies that $-b_{p_1}(b_{v_1'} - b_{p_1}) = 1$, and hence $b_{p_1} = -1$. This is impossible since b_{p_1} is a boundary vertex. Thus, there are no admissible states (a,b) that satisfy $(A(a,b), B(a,b)) = (1,1)$.

Case 3: $(A(a,b), B(a,b)) = (0,2)$. Since $A(a,b) = 0$, we should have

- $a_p + b_p = 0$ for all faces p,
- $b_v b_{v'} = 0$ for all boundary edges vv', and
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all bounded faces p and edges $vv' \in p$.

Let p be a bounded face of G. Let G' be the graph obtained from G by deleting the boundary edges of G, and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 4.3, we have $A(a',b') = A(a,b)$ and $B(a',b') = B(a,b) - 2n_p$, where $n_p \in \mathbb{N}$. Since $B(a,b) = 2$, $n_p \leq 1$, and $n_p = 1$ if and only if there exists exactly one boundary vertex $v \in p$ such that $b_v = 1$ and $b'_v = 0$ for any other boundary vertex v' of p. Continuing this process, it is easy to show that an admissible state (a,b) such that $(A(a,b), B(a,b)) = (0,2)$ must satisfy the following:

- $a_p = 0$ for all p, and
- $b_v = 1$ for a vertex v and $b_{v'} = 0$ for any other vertex v' of G.

The contribution of this state to $\Phi_G(q)$ is $\frac{q}{(1-q)^{\deg(v)}} = q + O(q^2)$.

Thus, from Theorem 2.1 and cases 1–3, we have

$$\begin{aligned}\langle \Phi_G^{\mathrm{TQFT}}(q) \rangle_1 &= \left\langle (q)_\infty^{c_2} \left(1 + \sum_v q + O(q^2) \right) \right\rangle_1 \\ &= c_1 - c_2\,.\end{aligned}$$

Therefore,

$$\langle \Phi_G(q) \rangle_1 = \left\langle (1-q)\Phi_G^{\mathrm{TQFT}}(q) \right\rangle_1 = c_1 - c_2 - 1 \,.$$

4.3 The coefficient of q^2 in $\Phi_G(q)$

We need to find the admissible states (a, b) such that $\frac{1}{2}(A(a,b)+B(a,b)) = 2$. Since $A(a,b), B(a,b) \in \mathbb{N}$, we have the following cases:

$A(a,b)$	4	3	2	1	0
$B(a,b)$	0	1	2	3	4

Case 1: $(A(a,b), B(a,b)) = (4,0)$. If there is a face p such that $a_p + b_p > 0$, then by Corollary 3.2, we have $B(a,b) \geq a_p + b_v \geq a_p + b_p > 0$, where v is a vertex of p. Therefore, $a_p + b_p = 0$ for all faces p. Similarly, if there exists a face p and a vertex $v \in p$ such that $b_v - b_p > 0$, then $0 = B(a,b) \geq a_p + b_v = a_p + b_p + b_v - b_p \geq b_v - b_p > 0$. Therefore, $a_p + b_v = b_v - b_p = 0$ for all $v \in p$. Thus, $A(a,b) = 4$ is equivalent to

$$\sum_{vv' \in p_\infty} b_v b_{v'} = 4 \,. \tag{11}$$

If vv' is an edge of G and p is a bounded face that contains vv', then we have $a_p + b_v = 0 = a_p + b_{v'}$, and therefore $b_v = b_{v'}$. So, by Eq. (10), there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 4.1 implies that $B(a,b) \geq 2 + V_\infty > 0$ which is impossible. Therefore, there are no admissible states (a, b) that satisfy $(A(a,b), B(a,b)) = (4,0)$.

Case 2: $(A(a,b), B(a,b)) = (3,1)$. If there exists a face p_0 such that $a_{p_0} + b_{p_0} > 0$, then we must have $l(p_0) = 3$ and

- $a_{p_0} + b_{p_0} = 1$, $a_p + b_p = 0$ for any $p \neq p_0$,
- $b_v b_{v'} = 0$ for all boundary edges vv', and
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all bounded faces p and and edges $vv' \in p$.

Let $p \neq p_0$ be a bounded face of G. Let G' be the graph obtained from G by deleting the boundary edges of p, and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 4.3, we have $A(a', b') = A(a,b)$ and $B(a', b') = B(a,b)$. We can continue this process until $G = p_0$. Let v_0, v_0', v_0'' be the vertices of p_0, then $b_{v_0} b_{v_0'} = 0$ so that we can assume that $b_{v_0} = 0$. Since $(b_{v_0} - b_{p_0})(b_{v_0'} - b_{p_0}) = 0$, we have $b_{p_0} = 0$ and, hence, $a_{p_0} = a_{p_0} + b_{p_0} = 1$. Since $1 = B(a,b) = a_{p_0} + 2(b_{v_0} + b_{v_0'} + b_{v_0''})$, it implies that $b_{v_0'} = b_{v_0''} = 0$. This gives us the following set of admissible states (a, b):

- $a_p = 1$ for a triangular face p, $a_{p'} = 0$ for $p' \neq p$, and
- $b_v = 0$ for all vertices v.

The contribution of this state to $\Phi_G(q)$ is $(-1)^1 \frac{q^2}{(1-q)^{l(p)}} = -\frac{q^2}{(1-q)^3} = -q^2 + O(q^3)$.

On the other hand, if $a_p + b_p = 0$ for all p, then we have

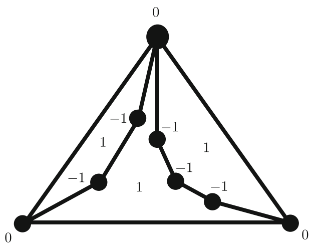
$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 3\,. \tag{12}$$

There are at most three positive terms in the above equation. If a boundary face p has a boundary edge vv' that does not correspond to any positive term, then we have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 0$ which implies that $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 4.3, we have $A(a', b') = A(a, b)$ and $B(a', b') = B(a, b)$. We can continue to do this until all boundary edges of G are $v_i v_i'$, $i = 1, 2, 3$. This only happens if these three edges together form a triangle. Let us denote the triangle's vertices by v, v', v'' and let p, p', p'' be the bounded faces that contain $vv', v'v'', v''v$, respectively. Note that since the positive terms in Eq. (12) correspond to different edges, we must have

$$b_v b_{v'} + (b_v - b_p)(b_{v'} - b_p) = 1$$
$$b_{v'} b_{v''} + (b_{v'} - b_{p'})(b_{v''} - b_{p'}) = 1$$
$$b_{v''} b_v + (b_{v''} - b_{p''})(b_v - b_{p''}) = 1.$$

Case 2.1: If the positive terms are $b_v b_{v'}, b_{v'} b_{v''}, b_{v''} b_v$, then we must have simultaneously $b_v b_{v'} = b_{v'} b_{v''} = b_{v''} b_v = 1$ and $(b_w - b_{\tilde{p}})(b_{w'} - b_{\tilde{p}}) = 0$ for all faces $\tilde{p}$ and edge ww'. The former implies that $b_v = b_{v'} = b_{v''} = 1$. Therefore, from Lemma 4.1, we have $B(a, b) \geq 2 + 3 = 5$ which is impossible.

Case 2.2: If, for instance, $b_v b_{v'} = 0$, then we must also have $(b_v - b_p)(b_{v'} - b_p) = 1$. Thus, we can assume that $b_v = 0$ and so $-b_p(b_{v'} - b_p) = 1$. This implies that $b_p = -1$ and $b_{v'} = 0$. In particular, we have $b_{v'} b_{v''} = 0$, and hence $(b_{v'} - b_{p'})(b_{v''} - b_{p'}) = 1$. Since $b_v b_{v''} = 0$, we also have $(b_{v''} - b_{p''})(b_v - b_{p''}) = 1$. In particular, this implies that $(b_w - b_{\tilde{p}})(b_{w'} - b_{\tilde{p}}) = 0$ for all faces $\tilde{p}$ and edges $ww' \in \tilde{p}$ not on the boundary. Since $B(a, b) = 1$, Lemma 4.2 implies that we must have $b_w = -1$ for all $w \neq v, v', v''$ and $a_p = 1$ for all $p \neq p_\infty$.



This corresponds to the following admissible state of G:

- $a_p = 1$ for all bounded faces p,
- $b_v = b_{v'} = b_{v''} = 0$, where v, v', v'' are the vertices of a 3-cycle in G,
- $b_w = -1$ for all vertices w inside the 3 circle mentioned above, and
- $b_{\tilde{w}} = 0$ for any other vertex w.

The contribution of this state to $\Phi_G(q)$ is

$$(-1)^1 \frac{q^2}{(1-q)^{\deg_\Delta(v)+\deg_\Delta(v')+\deg_\Delta(v'')-3}} = -q^2 + O(q^3),$$

where $\deg_\Delta(v)$ is the degree of v in the triangle $\Delta = vv'v''$.

Case 3: We consider the two cases $(A(a,b), B(a,b)) = (2,2)$ and $(A(a,b), B(a,b)) = (1,3)$ together. Since $A(a,b) \leq 2$, we should have $a_p + b_p = 0$ for all faces p, and $A(a,b) = 2$ is equivalent to

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2.$$

There are at most two positive terms in the above equation. If a boundary face p has a boundary edge vv' that does not correspond to any positive term, then we have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 0$ which implies that $a_p = 0$. By part (d) of Lemma 4.3, it follows that if w is a boundary vertex of p, then $B(a,b) \geq 2b_w$ and since $B(a,b) \leq 3$, we have $b_w = 0$ or 1. Therefore, by parts (b,c) of Lemma 4.3, we can remove the boundary edges of p to obtain a new graph G' that satisfies $A(a,b) = A'(a,b)$ and $B(a,b) = B'(a,b)$ or $B(a,b) = B'(a,b) + 1$ where $A'(a,b)$, $B'(a,b)$ are the restrictions of $A(a,b)$ and $B(a,b)$ on G'. By continuing this process until $G = \emptyset$, it is easy to see that we must have $A(a,b) = 0$, $B(a,b) \leq 1$, and $B(a,b) = 1$ if and only if there exists a unique boundary vertex w of p such that $b_w = 1$. Thus, there are no admissible states that satisfy $(A(a,b), B(a,b)) = (2,2)$ or $(A(a,b), B(a,b)) = (1,3)$.

Case 4: $(A(a,b), B(a,b)) = (0,4)$. Since $A(a,b) = 0$, we should have

$$a_p + b_p = 0 \text{ for all faces } p, \tag{13}$$

$$(b_v - b_p)(b_{v'} - b_p) = 0 \text{ for all faces } p \text{ and edges } vv' \in p, \textit{ and} \tag{14}$$

$$b_v b_{v'} = 0 \text{ for all edges } vv' \in p. \tag{15}$$

Let p be a boundary face of G, and $vv' \in p$ be a boundary edge. Eqs. (14) and (15) imply that $b_p = 0$ and so $a_p = 0$ by Eq. (13). Let G' be the graph obtained from G by deleting the boundary edges of G, and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 4.3, we have $A(a', b') = A(a,b)$, $B(a', b') = B(a,b) - 2n_p$ where $n_p \in \mathbb{N}$. Since $B(a,b) = 4$, we have $n_p \leq 2$ and

- $n_p = 2$ if and only if there exist either exactly two boundary vertices $v, w \in p$ that are not connected by an edge such that $b_v = b_{v'} = 1$ or exactly one boundary vertex $v \in p$ such that $b_v = 2$ and $b_{v'} = 0$ for all other boundary vertices $v' \in p$, and
- $n_p = 1$ if and only if there exists exactly one boundary vertex $v \in p$ such that $b_v = 1$ and $b'_v = 0$ for any other boundary vertex v' of p.

Similarly, by continuing this process, it is easy to show that an admissible state (a,b) such that $(A(a,b), B(a,b)) = (0,4)$ must satisfy one of the following.

- $b_v = b_{v'} = 1$ for a pair of vertices that are not connected by an edge of G, $b_w = 0$ for any other vertex w, and
- $a_p = 0$ for all faces p.

The contribution of this state to $\Phi_G(q)$ is $\frac{q^2}{(1-q)^{\deg(v)+\deg(v')}} = -q^2 + O(q^3)$.

- $b_v = 2$ for a vertex v, $b_w = 0$ for any other vertex w, and
- $a_p = 0$ for all faces p.

The contribution of this state to $\Phi_G(q)$ is $\frac{q^2}{(1-q)_2^{\deg(v)}} = -q^2 + O(q^3)$.

It follows from Theorem 2.1, Sect. 4.2, and cases 1–4 that

$$\begin{aligned}
\left\langle \Phi_G^{\mathrm{TQFT}}(q) \right\rangle_2 &= \left\langle (q)_\infty^{c_2} \left(1 + \sum_v \frac{q}{(1-q)^{\deg(v)}} + \left(-c_3 + c_1 + \frac{c_1(c_1-1)}{2} - c_2 \right) \right) q^2 \right\rangle_2 \\
&= \left\langle (q)_\infty^{c_2} \left(1 + q(c_1 + 2c_2 q) + \left(\frac{c_1(c_1+1)}{2} - c_2 - c_3 \right) \right) q^2 \right\rangle_2 \\
&= \left\langle \left(1 - c_2 q + \frac{c_2(c_2-3)}{2} q^2 \right) \left(1 + c_1 q + \left(\frac{c_1(c_1+1)}{2} + c_2 - c_3 \right) \right) q^2 \right\rangle_2 \\
&= \frac{(c_1-c_2)^2}{2} - c_3 + \frac{c_1 - c_2}{2} .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \Phi_G(q) \rangle_2 &= \langle (1-q) \Phi_G^{\mathrm{TQFT}}(q) \rangle_2 \\
&= \left\langle (1-q) \left(1 + (c_1 - c_2) q + \left(\frac{(c_1-c_2)^2}{2} - c_3 + \frac{c_1-c_2}{2} \right) q^2 \right) \right\rangle_2 \\
&= \frac{1}{2} \left((c_1 - c_2)^2 - 2c_3 - c_1 + c_2 \right) .
\end{aligned}$$

This completes the proof of Theorem 1.4. □

4.4 Proof of Lemma 1.5

Fix a planar graph G and consider $G \cdot P_r$, where P_r is a polygon with r sides and vertices $b_1, \dots, b_r$ as in the following figure:

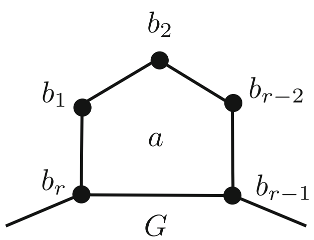
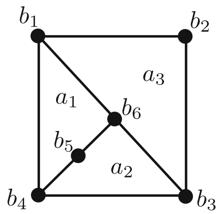


Fig. 4 The planar graph of the link $L8a7$



Consider the corresponding portion $S(b_{r-1}, b_r)$ of the formula of $\Phi_{G \cdot P_r}(q)$

$$S(b_{r-1}, b_r) = \sum_{a, b_1, \dots, b_{r-2}} (-1)^{ra} \frac{q^{\frac{r}{2}a^2 + a(b_1 + \dots b_r) + \sum_{i=1}^{r-2} b_i b_{i+1} + b_1 b_r + \sum_{i=1}^{r-2} b_i + \frac{r-2}{2}a}}{(q)_{b_1}(q)_{b_2} \dots (q)_{b_{r-2}}(q)_{b_1+a}(q)_{b_2+a} \dots (q)_{b_r+a}} \tag{16}$$

for fixed $b_{r-1}, b_r \geq 0$. Armond-Dasbach [1, Theorem 3.7] and Andrews [2] prove that

$$S(b_{r-1}, 0) = (q)_\infty^{-r+1} h_r(q),$$

for all $b_{r-1} \geq 0$. Summing over the remaining variables in the formula for $\Phi_{G \cdot P_r}(q)$ concludes the proof of the Lemma. □

5 The computation of $\Phi_G(q)$

5.1 The computation of $\Phi_{L8a7}(q)$ in detail

In this section, we explain in detail the computation of $\Phi_{L8a7}(q)$. Consider the planar graph of the alternating link $L8a7$ shown in Fig. 4, with the marking of its vertices by b_i for $i = 1, \dots, 6$ and its bounded faces by a_j for $j = 1, 2, 3$.

Consider the minimum values of the b-variables at each bounded face:

$$\begin{aligned}
\bar{b}_1 &= \min\{b_1, b_4, b_5, b_6\} \\
\bar{b}_2 &= \min\{b_3, b_4, b_5, b_6\} \\
\bar{b}_3 &= \min\{b_1, b_2, b_3, b_6\}.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{1}{2}A(a, b) = &2(a_1 + \bar{b}_1)^2 + (a_1 + \bar{b}_1)(b_1 + b_4 + b_5 + b_6 - 4\bar{b}_1) \\
&+ 2(a_2 + \bar{b}_2)^2 + (a_1 + \bar{b}_2)(b_3 + b_4 + b_5 + b_6 - 4\bar{b}_2) \\
&+ 2(a_3 + \bar{b}_3)^2 + (a_3 + \bar{b}_3)(b_1 + b_2 + b_3 + b_6 - 4\bar{b}_3) \\
&+ \frac{1}{2}\big((b_1 - \bar{b}_1)(b_6 - \bar{b}_1) + (b_6 - \bar{b}_1)(b_5 - \bar{b}_1) + (b_5 - \bar{b}_1)(b_4 - \bar{b}_1)
\end{aligned}$$

$$+(b_4-\bar{b}_1)(b_1-\bar{b}_1)\big)$$
$$+\frac{1}{2}\big((b_3-\bar{b}_2)(b_4-\bar{b}_2)+(b_4-\bar{b}_2)(b_5-\bar{b}_2)+(b_5-\bar{b}_2)(b_6-\bar{b}_2)$$
$$+(b_6-\bar{b}_2)(b_3-\bar{b}_2)\big)$$
$$+\frac{1}{2}\big((b_1-\bar{b}_3)(b_2-\bar{b}_3)+(b_2-\bar{b}_3)(b_3-\bar{b}_3)+(b_3-\bar{b}_3)(b_6-\bar{b}_3)$$
$$+(b_6-\bar{b}_3)(b_1-\bar{b}_3)\big)$$
$$+\frac{1}{2}(b_1b_2+b_2b_3+b_3b_4+b_4b_1)$$
$$=C(a_1,a_2,a_3,b_1,b_2,b_3,b_4,b_5,b_6)+D(b_1,b_2,b_3,b_4,b_5,b_6) \tag{17}$$

and

$$\frac{1}{2}B(a,b)=a_1+a_2+a_3+b_1+b_2+b_3+b_4+b_5+b_6$$
$$=\frac{a_1+b_1}{2}+\frac{a_1+b_5}{2}+\frac{a_2+b_5}{2}+\frac{a_2+b_6}{2}+\frac{a_3+b_1}{2}+\frac{a_3+b_6}{2}$$
$$+b_2+b_3+b_4\,. \tag{18}$$

If $\frac{1}{2}(A(a,b)+B(a,b))\leq N$, then $\frac{1}{2}B(a,b)\leq N$, so

$$0\leq b_2\leq N \tag{19}$$
$$0\leq b_3\leq N-b_2 \tag{20}$$
$$0\leq b_4\leq N-b_2-b_3\,. \tag{21}$$

Let us set

$$b_1=0. \tag{22}$$

Equation (18) implies that $0\leq\frac{a_1+b_1}{2}\leq N-b_2-b_3-b_4$ which implies that $0\leq a_1\leq 2(N-b_2-b_3-b_4)$. It follows from $0\leq\frac{a_1+b_5}{2}\leq N$ that

$$-2(N-b_2-b_3-b_4)\leq b_5\leq 2(N-b_2-b_3-b_4)\,. \tag{23}$$

Since $0\leq\frac{a_2+b_5}{2}\leq N-b_2-b_3-b_4$ from (23), we have $-2(N-b_2-b_3-b_4)\leq a_2\leq 4(N-b_2-b_3-b_4)$. Therefore, since $0\leq a_2\leq\frac{a_2+b_6}{2}$, we have

$$-4(N-b_2-b_3-b_4)\leq b_6\leq 4(N-b_2-b_3-b_4)\,. \tag{24}$$

Equations (19)–(24) in particular bound b_2, b_3, b_4, b_5, and b_6 from above and from below by linear forms in N. But even better, Eqs. (19)–(24) allow for an iterated summation for the b_i variables which improve the computation of the $\Phi_{L8a7}(q)$ series.

To bound a_1, a_2, and a_3, we will use the auxiliary function

$$u(c,d) = \left[\frac{-c+\sqrt{c^2+2d}}{2}\right],$$

where the integer part $[x]$ of a real number x is the biggest integer less than or equal to x. The argument of $u(c,d)$ inside the integer part is one of the solutions to the equation $2x^2+cx-d=0$. Let

$$\begin{aligned}
\tilde{b}_1 &= b_1+b_4+b_5+b_6-4\bar{b}_1\\
\tilde{b}_2 &= b_3+b_4+b_5+b_6-4\bar{b}_2\\
\tilde{b}_3 &= b_1+b_2+b_3+b_6-4\bar{b}_3\\
\tilde{D} &= D(b_1,b_2,b_3,b_4,b_5,b_6)+b_2+b_3+b_4.
\end{aligned}$$

Since

$$2(a_1+\bar{b}_1)^2+(a_1+\bar{b}_1)\tilde{b}_1 \le N-\tilde{D},$$

we have

$$-\bar{b}_1 \le a_1 \le -\bar{b}_1+u(\tilde{b}_1, N-\tilde{D}), \tag{25}$$

where the left inequality follows from the fact that $a_1 \ge -b_i$, $i=1,4,5,6$. Similarly, we have

$$-\bar{b}_2 \le a_2 \le -\bar{b}_2+u(\tilde{b}_1, N-\tilde{D}-2(a_1+\bar{b}_1)^2-(a_1+\bar{b}_1)\tilde{b}_1) \tag{26}$$

and

$$\begin{aligned}
&-\bar{b}_3 \le a_3 \le -\bar{b}_3+u(\tilde{b}_1, N-\tilde{D}-2(a_1+\bar{b}_1)^2\\
&-(a_1+\bar{b}_1)\tilde{b}_1-2(a_2+\bar{b}_2)^2-(a_2+\bar{b}_2)\tilde{b}_2).
\end{aligned} \tag{27}$$

Note that Eqs. (25)–(27) allow for an iterated summation in the a_i variables, and in particular imply that the span of the a_i variables is bounded by a linear form of $\sqrt{N}$.

It follows that

$$\begin{aligned}
&\Phi_{L8a7}(q)+O(q)^{N+1}\\
&=(q)_\infty^8 \sum_{(a,b)} \frac{q^{\frac{1}{2}(A(a,b)+B(a,b))}}{(q)_{a_1+b_1}(q)_{a_1+b_4}(q)_{a_1+b_5}(q)_{a_1+b_6}(q)_{a_2+b_3}(q)_{a_2+b_4}(q)_{a_2+b_5}(q)_{a_2+b_6}}\\
&\cdot \frac{1}{(q)_{a_3+b_1}(q)_{a_3+b_2}(q)_{a_3+b_3}(q)_{a_3+b_6}(q)_{b_1}(q)_{b_2}(q)_{b_3}(q)_{b_4}}+O(q)^{N+1},
\end{aligned}$$

where $(a,b)=(a_1,a_2,a_3,b_1,b_2,b_3,b_4,b_5,b_6)$ satisfy the inequalities (19)–(24) and (25)–(27). We give the first 21 terms of this series in Fig. 12.

5.2 The computation of $\Phi_G(q)$ by iterated summation

Our method of computation requires not only the planar graph with its vertices and faces (which is relatively easy to automate), but also the inequalities for the b_i and a_j variables which lead to an iterated summation formula for $\Phi_G(q)$. Although Theorem 1.2 implies the existence of an iterated summation formula for every planar graph, we did not implement this algorithm in general.

Instead, for each of the 11 graphs that appear in Figs. 6, 7, and 13, we computed the corresponding inequalities for the iterated summation by hand. These inequalities are too long to present them here, but we have them available. A consistency check of our computation is obtained by Eq. (7), where the shifted colored Jones polynomial of an alternating link is available from [5] for several values. Our data matche those values.

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Appendix 1: Tables

In this section, we give various tables of graphs, and their corresponding alternating knots (following Rolfsen's notation [18]) and links (following Thistlethwaite's notation [5]) and several terms of $\Phi_G(q)$. In view of an expected positive answer to Question 1.6, we will list *irreducible* graphs, i.e., simple planar 2-connected graphs which are not of the form $G_1 \cdot G_2$ (for the operation $\cdot$ defined in Sect. 1.3).

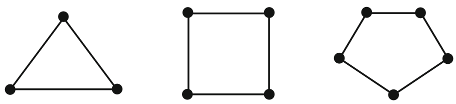


Fig. 5 The irreducible planar graphs G_0^3, G_0^4, and G_0^5 with 3, 4, and 5 edges

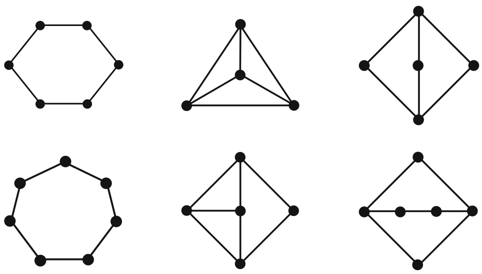


Fig. 6 The irreducible planar graphs with 6 and 7 edges: G_0^6, G_1^6, and G_2^6 on the *top* and G_0^7, G_1^7, and G_2^7 on the *bottom*

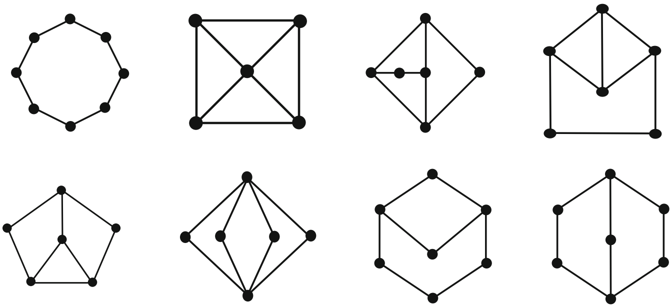


Fig. 7 The irreducible planar graphs with 8 edges: $G_0^8, \dots, G_3^8$ on the *top* (from *left* to *right*) and $G_4^8, \dots, G_7^8$ on the *bottom*

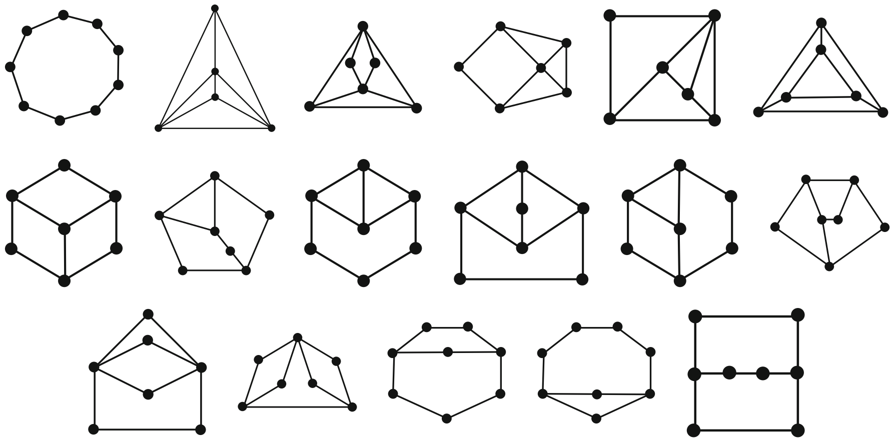


Fig. 8 The irreducible planar graphs with 9 edges: $G_0^9, \dots, G_5^9$ on the *top*, $G_6^9, \dots, G_{11}^9$ on the *middle* and $G_{12}^9, \dots, G_{16}^9$ on the *bottom*

K	G	$-G$	K	G	$-G$	K	G	$-G$	K	G	$-G$
0_1	P_2	P_2	7_2	P_6	P_3	8_4	P_3	$P_4 \cdot P_5$	8_{13}	$P_3 \cdot P_3 \cdot P_4$	$P_3 \cdot P_3$
3_1	P_3	P_2	7_3	P_5	P_4	8_5	G_7^8	P_3	8_{14}	$P_3 \cdot P_4$	$P_3 \cdot P_3 \cdot P_3$
4_1	P_3	P_3	7_4	$P_4 \cdot P_4$	P_3	8_6	$P_3 \cdot P_4$	P_5	8_{15}	$P_3 \cdot P_3 \cdot P_3$	G_2^6
5_1	P_5	P_2	7_5	$P_3 \cdot P_4$	P_4	8_7	$P_3 \cdot P_5$	P_4	8_{16}	G_4^8	G_1^6
5_2	P_4	P_3	7_6	$P_3 \cdot P_4$	$P_3 \cdot P_3$	8_8	$P_3 \cdot P_5$	$P_3 \cdot P_3$	8_{17}	G_1^7	G_1^7
6_1	P_5	P_3	7_7	$P_3 \cdot P_3 \cdot P_3$	$P_3 \cdot P_3$	8_9	$P_3 \cdot P_4$	$P_3 \cdot P_4$	8_{18}	G_1^8	G_1^8
6_2	$P_3 \cdot P_4$	P_3	8_1	P_7	P_3	8_{10}	G_2^7	$P_3 \cdot P_3$			
6_3	$P_3 \cdot P_3$	$P_3 \cdot P_3$	8_2	$P_3 \cdot P_6$	P_3	8_{11}	$P_3 \cdot P_4$	$P_3 \cdot P_4$			
7_1	P_7	P_2	8_3	P_5	P_5	8_{12}	$P_3 \cdot P_4$	$P_3 \cdot P_4$			

Fig. 9 The reduced Tait graphs of the alternating knots with at most 8 crossings

- The first table gives number of alternating links with at most 10 crossings and the number of irreducible graphs with at most 10 edges

Crossings = edges	3	4	5	6	7	8	9	10
Alternating links	1	2	3	8	14	39	96	297
Irreducible graphs	1	1	1	3	3	8	17	41

(28)

To list planar graphs, observe that they are *sparse*: if G is a planar graph which is not a tree, with V vertices and E edges, then

$$V \leq E \leq 3V - 6\,.$$

L	G	$-G$	L	G	$-G$	L	G	$-G$	L	G	$-G$
$2a1$	P_2	P_2	$7a2$	$P_3{\cdot}P_3$	G^6_2	$8a4$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3{\cdot}P_3$	$8a13$	$P_4{\cdot}P_4$	P_4
$4a1$	P_4	P_2	$7a3$	G^7_2	P_3	$8a5$	P_4	$P_3{\cdot}P_3{\cdot}P_4$	$8a14$	P_8	P_2
$5a1$	$P_3{\cdot}P_3$	P_3	$7a4$	P_5	$P_3{\cdot}P_3$	$8a6$	P_6	$P_3{\cdot}P_3$	$8a15$	P_5	$P_3{\cdot}P_3{\cdot}P_3$
$6a1$	P_4	$P_3{\cdot}P_3$	$7a5$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3$	$8a7$	G^8_2	G^6_1	$8a16$	G^8_3	G^6_1
$6a2$	P_4	P_4	$7a6$	$P_3{\cdot}P_5$	P_3	$8a8$	$P_3{\cdot}P_4{\cdot}P_3$	$P_3{\cdot}P_3$	$8a17$	$P_3{\cdot}P_4$	G^6_2
$6a3$	P_6	P_2	$7a7$	P_4	G^6_2	$8a9$	$P_3{\cdot}P_3{\cdot}P_3$	$P_3{\cdot}P_3{\cdot}P_3$	$8a18$	G^8_6	P_3
$6a4$	G^6_1	G^6_1	$8a1$	G^7_1	$P_3{\cdot}G^6_1$	$8a10$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3$	$8a19$	G^7_1	G^7_1
$6a5$	P_3	G^6_2	$8a2$	$P_3{\cdot}P_3$	$P_3{\cdot}G^6_2$	$8a11$	$P_3{\cdot}P_5$	P_4	$8a20$	G^6_2	G^6_2
$7a1$	G^7_1	G^6_1	$8a3$	G^7_2	$P_3{\cdot}P_3$	$8a12$	P_6	P_4	$8a21$	P_4	G^8_5

Fig. 10 The reduced Tait graphs of the alternating links with at most 8 crossings

G^6_1	$L6a4$ $-L6a4$ $-L7a1$ $-L8a7$ -8_{16} $-L8a16$
G^6_2	$-L6a5$ $-L7a2$ $-L7a7$ $-L8a17$ -8_{15} $L8a20$ $-L8a20$
G^7_1	$L7a1$ $L8a1$ 8_{17} -8_{17} $L8a19$ $-L8a19$
G^7_2	8_{10} $L7a3$ $L8a3$
G^8_1	8_{18} -8_{18}
G^8_2	$L8a7$
G^8_3	$L8a16$
G^8_4	8_{16}
G^8_5	$-L8a21$
G^8_6	$L8a18$
G^8_7	8_5

Fig. 11 The irreducible planar graphs with at most 8 edges and the corresponding alternating links

- The next table gives the number of planar 2-connected irreducible graphs with at most 9 vertices

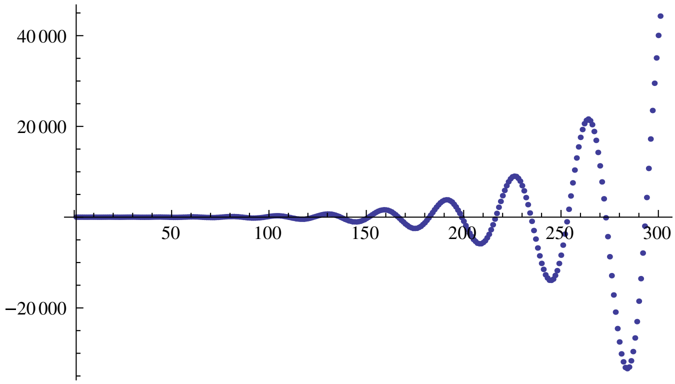
Vertices	3	4	5	6	7	8	9
Graphs	1	2	5	19	106	897	10160

(29)

- Figures 5, 6, 7 and 8 give the list of irreducible graphs with at most 9 edges. These tables were constructed by listing all graphs with $n \leq 9$ vertices, selecting those which are planar, and further selecting those that are irreducible. Note that if G is a planar graph with $E \leq 9$ edges, V vertices, and F faces, then $E - V = F - 2 \geq 0$, and hence $V \leq E \leq 9$.
- Figures 9 and 10 give the reduced Tait graphs of all alternating knots and links (and their mirrors) with at most 8 crossings. Here, P_r is the planar polygon with r sides,

G	$\Phi_G(q) + O(q)^{21}$
G_1^6	$1 - 3q - q^2 + 5q^3 + 3q^4 + 3q^5 - 7q^6 - 5q^7 - 8q^8 - 6q^9 + 6q^{10}$ $+7q^{11} + 12q^{12} + 15q^{13} + 16q^{14} - 3q^{15} - q^{16} - 15q^{17} - 21q^{18} - 31q^{19} - 30q^{20}$
G_2^6	$1 - 2q + q^2 + 3q^3 - 2q^4 - 2q^5 - 3q^6 + 3q^7 + 4q^8 + q^9 + 3q^{10}$ $-6q^{11} - 5q^{12} - 3q^{13} + q^{15} + 7q^{16} + 9q^{17} + 3q^{18} - 6q^{20}$
G_1^7	$1 - 3q + q^2 + 5q^3 - 3q^4 - 3q^5 - 6q^6 + 6q^7 + 8q^8 + 3q^9 + 6q^{10}$ $-13q^{11} - 14q^{12} - 9q^{13} - q^{14} + 3q^{15} + 21q^{16} + 27q^{17} + 14q^{18} + 3q^{19} - 17q^{20}$
G_2^7	$1 - 2q + q^2 + q^3 - 3q^4 + q^5 + q^6 + 3q^7 - 2q^8 - 4q^9 + q^{10}$ $+4q^{12} + 5q^{13} - 2q^{14} - 5q^{15} - 4q^{16} - 2q^{17} - 2q^{18} + 5q^{19} + 8q^{20}$
G_1^8	$1 - 4q + 2q^2 + 9q^3 - 5q^4 - 8q^5 - 14q^6 + 10q^7 + 21q^8 + 14q^9 + 19q^{10}$ $-29q^{11} - 42q^{12} - 42q^{13} - 20q^{14} + 3q^{15} + 64q^{16} + 104q^{17} + 88q^{18} + 55q^{19} - 25q^{20}$
G_2^8	$1 - 3q + 3q^2 + 4q^3 - 8q^4 - 2q^5 + 2q^6 + 12q^7 + 3q^8 - 15q^9 - 4q^{10}$ $-14q^{11} + 10q^{12} + 25q^{13} + 15q^{14} - 18q^{16} - 22q^{17} - 39q^{18} - 12q^{19} + 19q^{20}$
G_3^8	$1 - 3q + q^2 + 3q^3 - 3q^4 + 3q^5 + 4q^7 - 6q^8 - 10q^9 + q^{10}$ $-q^{11} + 9q^{12} + 13q^{13} + 3q^{14} - 9q^{15} - 3q^{16} - 6q^{17} - 4q^{18} + 5q^{19} + 13q^{20}$
G_4^8	$1 - 3q + 2q^2 + 3q^3 - 6q^4 + q^5 + 2q^6 + 8q^7 - 3q^8 - 13q^9$ $-3q^{11} + 13q^{12} + 19q^{13} + q^{14} - 15q^{15} - 20q^{16} - 16q^{17} - 13q^{18} + 15q^{19} + 37q^{20}$
G_5^8	$1 - 3q + 3q^2 + 5q^3 - 8q^4 - 5q^5 - q^6 + 15q^7 + 12q^8 - 8q^9 - 7q^{10}$ $-31q^{11} - 11q^{12} + 14q^{13} + 30q^{14} + 35q^{15} + 27q^{16} + 8q^{17} - 48q^{18} - 66q^{19} - 72q^{20}$
G_6^8	$1 - 2q + q^2 + q^3 - q^4 + 2q^5 - 2q^6 - q^7 - 2q^8 + 2q^9 + 5q^{10}$ $-q^{11} - q^{12} - 3q^{13} - 2q^{14} + 5q^{16} - 2q^{18} - q^{19} - q^{20}$
G_7^8	$1 - 2q + q^2 - 2q^4 + 3q^5 - 3q^8 + q^9 + 4q^{10}$ $-q^{11} - 2q^{12} - 2q^{13} - 3q^{14} + 3q^{15} + 7q^{16} + 2q^{17} - 4q^{18} - 4q^{19} - 4q^{20}$

Fig. 12 The first 21 terms of $\Phi_G(q)$ for the irreducible planar graphs with at most 8 edges



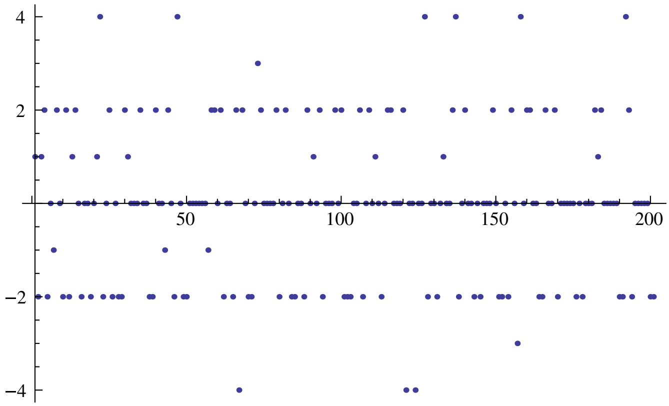


Fig. 13 Plot of the coefficients of $\Phi_{G_2^6}(q)$ on the *top* and $h_4(q)^2$ (keeping in mind that G_2^6 has two bounded square faces) on the *bottom*

and $-K$ denotes the mirror of K. Moreover, the notation $G = G_1 \cdot G_2 \cdot G_3$ indicates that $\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)\Phi_{G_3}(q)$ by Lemma 1.5.

- Figure 11 gives the alternating knots and links with at most 8 crossings for the irreducible graphs with at most 8 edges.
- Figure 12 gives the first 21 terms of of $\Phi_G(q)$ for all irreducible graphs with at most 8 edges (Fig. 13). Many more terms are available from http://www.math.gatech.edu/~stavros/publications/phi0.graphs.data/

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