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APPLICATIONS OF THE LANTERN IDENTITY

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ABSTRACT. The purpose of this note is to unify the role of the lantern identity in the proof of several finiteness theorems. In particular, we show that for every nonnegative integer m, the vector space (over the rationals) of type m (resp. T-type m) invariants of integral homology 3-spheres are finite dimensional. These results have already been obtained by [Oh] and [GL2] respectively; our derivation however is simpler, conceptual and relates to several other applications of the lantern identity.

1. THE LANTERN IDENTITY

In his seminal work using the lantern identity, D. Johnson showed that the Torelli group is finitely generated [Jo4] and further determined its abelianization [Jo5].

In the present note we use the lantern identity in order to give a simple proof of the following two *finiteness* theorems:

- The vector space of type m invariants of integral homology 3-spheres is finite dimensional (for every m), due to T. Ohtsuki [Oh].
- The vector space of T-type m invariants of integral homology 3-spheres is finite dimensional (for every m), due to the author and J. Levine, [GL2].

Our proof relates D. Johnson's finiteness theorems to the above mentioned results and explains in a conceptual way the relation of the lantern identity and finiteness theorems. For completeness, in Section 3 we mention two more applications of the lantern identity to finiteness theorems. We also offer exercises and hints for the reader to figure out forms of a *fictional* lantern identity (which would still prove the various finiteness results), hoping to bring together a rather diverse audience and unify the use of the lantern identity in the proofs of finiteness theorems.

We begin by recalling the lantern identity, which is usually written as an identity in the group of *framed pure* 3-*strand braids* P_3 on the plane, or in the group of *framed pure* 4-*strand braids* $P_4(S^2)$ on the 2-sphere S^2 (otherwise known as the mapping class group of a 2-sphere with 4 boundary components). With the notation of Figures 1, 2 we have:

$$\begin{aligned} \tau_{123}\tau_1\tau_2\tau_3 &= \tau_{12}\tau_{13}\tau_{23} && \text{in } P_3 \\ \epsilon_4\epsilon_1\epsilon_2\epsilon_3 &= \alpha\beta\gamma && \text{in } P_4(S^2) \end{aligned}$$

Note that for an unoriented simple closed curve c, a right-handed (resp. left-handed) Dehn twist on c represents an element of the mapping class group denoted by c (resp. c^{-1}), and that we compose maps from left to right (contrary to the usual way which composition of functions is written) and braids from top to bottom. There is a map $P_{n+1} \to P_n(S^2)$ (which places the last strand at infinity) which maps one version of the lantern identity on the other, as is obvious from the left hand side of Figure 2.

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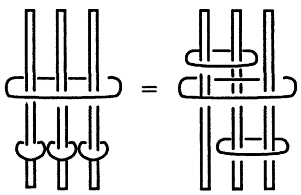

Figure 1. The lantern identity in P_3. Horizontal circles in the picture correspond (by a $+1$, i.e., a right-handed twist) framed pure braids on 3-strands denoted by $\tau_i, \tau_{ij}, \tau_{ijk}$.

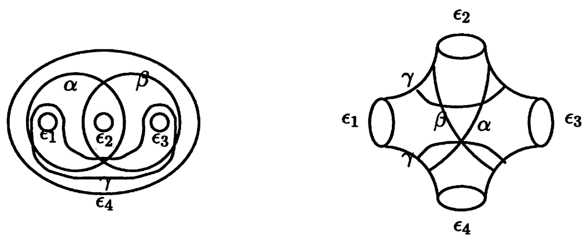

Figure 2. Two views of the curves that appear in the lantern identity on a sphere.

2. The lantern identity and finite type invariantss

2.1. Finite type invariants and the lantern identity. Using the following elementary lemma:

Lemma 2.1. *If $R = \mathbb{Z}\langle a, b\rangle$ is the noncommutative ring on $\{a, b\}$ and $I \overset{def}{=} (\overline{a}, \overline{b})$ the two-sided augmentation ideal (where $\overline{x} \overset{def}{=} 1 - x$), then we have:*

$$\overline{ab} \equiv \overline{a} + \overline{b} \ mod\ I^2 \tag{1}$$

together with the lantern identity in P_3, we deduce that:

Proposition 2.2. [Oh] *For $R = \mathbb{Z}P_3$ and I the augmentation ideal, we have:*

$$\overline{\tau}_{123} \equiv \overline{\tau}_{12} + \overline{\tau}_{13} + \overline{\tau}_{23} - \overline{\tau}_1 - \overline{\tau}_2 - \overline{\tau}_3 \ mod\ I^2 \tag{2}$$

Together with the moves of [Oh, Figure 2.1], Ohtsuki used the above equation to show that the vector space of finite type m invariants of integral homology 3-spheres is finite dimensional for every nonnegative integer m, [Oh]. It was shown in [L] (see also [GL1, Remark 2.9]) that the moves of [Oh, Figure 2.1] generate the surgery equivalence relation of algebraically split links (i.e., links with linking number zero); however a conceptual understanding of equation (2) was missing. The lantern identity provides such an explanation.

A few questions are in order:

Question 1. Can we improve Lemma 2.1 and Corollary 2.2 mod I^m for any $m > 2$?

The answer is positive, as follows.

Lemma 2.3. *If $R = \mathbb{Z}\langle a, b, b^{-1}\rangle$, then in the completion $\hat{R}_I$ with respect to the augmentation ideal, we have:*

$$1 - \overline{ab^{-1}} = (1 - \overline{a})(1 - \overline{b})^{-1}$$

Proof. Since $a = 1 - \overline{a}$ and $b^{-1} = 1/b = (1-\overline{b})^{-1}$, it follows that $1 - \overline{ab^{-1}} = ab^{-1} = (1-\overline{a})(1-\overline{b})^{-1}$. □

On the other hand, the lantern identity implies that for any number $n \geq 3$ of strands we have in P_n:

$$\tau_{12\ldots n}\prod_{i=1}^{n}\tau_i^{n-2} = \prod_{1\leq i<j\leq n}\tau_{ij}$$

(where τ_i commute with τ_{ij}, and the product is taken lexicographically). Thus, we deduce that in the group ring $\widehat{\mathbb{Z}P_n}$ (completed with respect to the augmentation ideal) we have:

$$1 - \overline{\tau}_{12\ldots n} = \frac{\prod_{1\leq i<j\leq n}(1-\overline{\tau}_{ij})}{\prod_{i=1}^{n}(1-\overline{\tau}_i)^{n-2}}$$

(where the product is taken lexicographically). This was first obtained in [GL1, Theorem 4]. The reader is invited to be convinced that the above identity is indeed a generalization of Proposition 2.2.

Question 2. Do we really need the lantern identity in order to show that the space of finite type m invariants of integral homology 3-spheres is finite dimensional?

Ohtsuki's proof uses only the fact that we can express $\overline{\tau}_{123}$ of equation (2) in terms of $\overline{\tau}_i, \overline{\tau}_{ij}$. Therefore any fictional identity on P_3 that expressed a nonzero power of τ_{123} in terms of τ_i, τ_{ij} would suffice.

2.2. **Finite T-type invariants and the lantern identity.** We begin by recalling a few definitions and notation from [GL2]. All manifolds will be oriented and all maps will be orientation preserving. Let $\mathcal{M}$ denote the vector space (over $\mathbb{Q}$) on the set of (oriented) integral homology 3-spheres. Given an embedding $\Sigma_g \to M$ of a closed genus g surface in an integral homology 3-sphere M, let $\mathcal{T}_g$ denote the Torelli group of Σ_g (i.e., the group of (orientation preserving) diffeomorphism classes of surface diffeomorphisms that act trivially in the homology), let $\mathbb{Q}\mathcal{T}_g$ denote the group-ring and I its augmentation ideal. The process of cutting M across Σ_g, twisting by an element of $\mathcal{T}_g$ and gluing back, defines (by linear extension) a map: $\Phi_f : \mathbb{Q}\mathcal{T}_g \to \mathcal{M}$. In [GL2, Definition 1.1] we considered the decreasing filtration $\mathcal{F}_*^T\mathcal{M}$ on $\mathcal{M}$ defined by $\mathcal{F}_m^T\mathcal{M} = \cup_{\text{all } f}\Phi_f(I^m)$. We call a map $\lambda : \mathcal{M} \to \mathbb{Q}$ an invariant of integral homology 3-spheres of *T-type m* if $\lambda(\mathcal{F}_{m+1}^T\mathcal{M}) = 0$. As an application of the lantern identity we show that:

Proposition 2.4. *The vector space of T-type m invariants of integral homology 3-spheres is finite dimensional for every nonnegative integer m. Furthermore, the graded vector space is zero dimensional for odd m.*

Proof. It suffices to show that the graded vector space $\mathcal{G}_m^T\mathcal{M} \stackrel{\text{def}}{=} \mathcal{F}_m^T\mathcal{M}/\mathcal{F}_{m+1}^T\mathcal{M}$ is finite (resp. zero) dimensional for every nonnegative (resp. odd) integer m. A geometric argument of [GL2, Proposition 1.6] implies that $\mathcal{F}_m^T\mathcal{M}$ is the union over all Heegaard surfaces $h : \Sigma_g \to M$ in integral homology 3-spheres M, and thus $\mathcal{G}_m^T\mathcal{M} = \cup_h\Phi_h(I^m/I^{m+1})$. For the convenience of the reader, we sketch this argument here. Given $\Sigma \hookrightarrow M$, it separates M in two components M^+, M^-. Though M^+, M^- need not be handelbodies, after we drill in tubes in them (thus increasing the genus of their boundary) we can assume that they are. Any finite set of elements of the Torelli group of Σ can be extended to elements of the Torelli group of the extended surface.

The action of the mapping class group Γ_g on $\mathcal{T}_g$ by conjugation implies that I^m/I^{m+1} is a $Sp(H)$ (and thus, a $GL(L^+)$) module, where $H = H_1(\Sigma_g, \mathbb{Z})$, $L^{\pm} = \text{Ker}(h : H \to H_1(M^{\pm}, \mathbb{Z}))$ and $M = M^+ \cup_h M^-$. Furthermore, another geometric argument of [GL3, Lemmas 3.1-3.3] implies that $\cup_h \Phi_h(I^m/I^{m+1}) = \cup_{h'} \Phi_{h'}((I^m/I^{m+1})^{GL(L^+)})$, for all h' standard Heegaard splittings of S^3 (of arbitrary genus).

For the convenience of the reader, we briefly sketch this argument here. First, we show that given a handelbody Q and an automorphism α of $H_1(\partial Q, \mathbb{Z})$ preserving as a set the Lagrangian $L = \mathrm{Ker}(H_1(\partial Q, \mathbb{Z}) \to H_1(Q, \mathbb{Z}))$, then it can be geometrically realized by a homeomorphism of Q. Using that, we show that given two Heegaard splittings $f_1, f_2 : \Sigma \hookrightarrow M$ such that $f_1(\Sigma) = f_2(\Sigma)$ and such that the two Lagrangians in $H_1(\Sigma, \mathbb{Z})$ with respect to f_1 equal to those with respect to f_2, then $\Phi_{f_1}(I^m/I^{m+1}) = \Phi_{f_2}(I^m/I^{m+1})$. Next, we show that given two Heegaard splittings $f_1, f_2 : \Sigma \hookrightarrow M$ such that the two Lagrangians L^+, L^- in $H_1(\Sigma, \mathbb{Z})$ with respect to f_1 equal to those with respect to f_2, then $\Phi_{f_1}(I^m/I^{m+1}) = \Phi_{f_2}(I^m/I^{m+1})$. This, together with the fact that $\Phi_{f_1}(I^m/I^{m+1})$ is $GL(L^+)$ invariant, implies the conclusion.

Next, we recall that D. Johnson [Jo2] introduced a group homomorphism (well known as the Johnson homomorphism) $\mathcal{T}_g \to U \overset{\text{def}}{=} \Lambda^3 H/H$. Using the lantern identity, he showed in [Jo5] that the Johnson homomorphism coincides (rationally) with the abelianization of the Torelli group $\mathcal{T}_g$ as a $Sp(H)$ module.

Furthermore, both the Johnson homomorphism and the map $\Phi_{h'}$ above are *stable* with respect to the inclusion of a surface to another, in the following sense: for a surface Σ with one boundary component included in a closed surface Σ', let $\hat{\Sigma}$ denote the surface obtained by closing Σ by the addition of a disc. Then, we have a canonical inclusion $H_1(\hat{\Sigma}, \mathbb{Z}) \simeq H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma', \mathbb{Z})$, and both the Johnson homomorphism and the map $\Phi_{h'}$ respect this inclusion.

Since I/I^2 can also be (rationally) identified with the abelianization of $\mathcal{T}_g$, we deduce that $I/I^2 \simeq U$ as a $Sp(H)$ module. Thus, for a fixed m, I^m/I^{m+1} is a quotient of $\otimes^m U$, and hence of $\Lambda^3 H$ and of $\mathrm{T}^3 H$, the third tensor power of H. Note that $\mathrm{T}^3 H = \mathrm{T}^3 L + \mathrm{T}^2 L \otimes L^\star + \mathrm{T}^2 L^\star \otimes L + \mathrm{T}^3 L^\star$ (where $L = L^+$ and $L^\star$ is the dual of L) is a $GL(L)$ representation.

We now need to recall some facts from classical invariant theory, which we refer the reader to [W] for a complete exposition. The contraction $L \otimes L^\star \to \mathbb{Q}$ of the indices of L with the indices of $L^\star$ is a $GL(L)$ invariant map. It follows from the first fundamental theorem of classical invariant theory that for all nonegative integers a, b $(\mathrm{T}^a L \otimes \mathrm{T}^b L^\star)^{GL(L)}$ is generated by all ways of contracting *all* indices of $\mathrm{T}^a L$ with all indices of $\mathrm{T}^b L^\star$; in particular $(\mathrm{T}^a L \otimes \mathrm{T}^b L^\star)^{GL(L)}$ is zero dimensional unless $a = b$. The second fundamental theorem of classical invariant theory implies that if the dimension of L is greater than $a + b$ then the above mentioned contractions are linearly independent and thus form a basis of $(\mathrm{T}^a L \otimes \mathrm{T}^b L^\star)^{GL(L)}$ independent of L. Thus, the dimension of the vector space $(\mathrm{T}^m \mathrm{T}^3(L + L^\star))^{GL(L)}$ is independent of L as long as $\dim(L) > 3m$. and zero for odd m.

The above discussion together with the stability of the Johnson homomorphism with respect to g imply that the vector spaces $(I^m/I^{m+1})^{GL(L^+)}$ are naturally isomorphic, finite dimensional for genus $g \geq 3m$ and in addition zero dimensional for odd m. Furthermore, by the stability of Φ_h we have that $\mathcal{G}^T_m \mathcal{M} = \Phi_{h_{3m}}((I^m/I^{m+1})^{GL(L^+)})$ where h_{3m} is the standard genus $3m$ Heegaard splitting of S^3. This finishes the proof of the proposition. □

Remark 2.5. In [GL2, Theorem 4] we have shown among other things that T-type $2m$ invariants of integral homology 3-spheres coincide with T-type $2m - 1$ and with type $3m$ invariants of integral homology 3-spheres. Thus, the above proposition follows from the fact type m invariants of integral homology 3-spheres form a finite dimensional vector space. Compared to the rather involved proof of [GL2, Theorem 4], the above proof is new and shorter; furthermore it reduces the finiteness result of the proposition to the fact that the abelianization of the Torelli group is finitely generated.

It seems natural to ask the following:

Question 3. Do we really need the lantern identity in order to show that the space of T-type m invariants of integral homology 3-spheres is finite dimensional?

The above proof shows that as long as (rationally) the abelianization of the Torelli is finitely generated, T-type m invariants of integral homology 3-spheres form a finite dimensional space. In addition, as long as the abelianization of the Torelli group as a $Sp(H)$ module is included in an odd tensor power of H, the graded vector space of finite T-type m invariants is zero dimensional for odd m. The ambitious reader may figure out a form of a fictional lantern identity compatible with Johnson's arguments which show that the Torelli group (and thus its abelianization) is finitely generated [Jo4] or with Johnson's arguments that determine the abelianization of the Torelli group [Jo5].

3. Two additional applications

In this section we mention two more applications of the lantern identity to the following finiteness theorems:

- The abelianization of the mapping class group of a closed genus $g > 2$ surface is trivial, due to J. Powell [Po] and independently due to J. Harer [Ha].
- The conformal dimensions and the central charge of an arbitrary topological quantum field theory in 3 dimensions are rational numbers, due to C. Vafa [Va].

We summarize Harer's argument here. Since the mapping class group is generated by Dehn twists on simple closed curves, the lantern identity implies that it is generated by twists on nonseparating curves. Since any two nonseparating simple closed curves are conjugate, the lantern identity (applied so that the Dehn twists on all seven participating curves are nonseparating and right-handed) implies the abelianization of the the mapping class group is trivial.

For a reproduction of Vafa's argument as well as a friendly and complete definition of the terms and concepts involved, see [BK].

The ambitious reader may figure out a form of a fictional lantern identity which would still make J. Harer's and C. Vafa's arguments work.

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