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CHERN-SIMONS THEORY, ANALYTIC CONTINUATION AND ARITHMETIC

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Abstract. The purpose of the paper is to introduce some conjectures regarding the analytic continuation and the arithmetic properties of quantum invariants of knotted objects. More precisely, we package the perturbative and nonperturbative invariants of knots and 3-manifolds into two power series of type P and NP, convergent in a neighborhood of zero, and we postulate their arithmetic resurgence. By the latter term, we mean analytic continuation as a multivalued analytic function in the complex numbers minus a discrete set of points, with restricted singularities, local and global monodromy. We point out some key features of arithmetic resurgence in connection to various problems of asymptotic expansions of exact and perturbative Chern-Simons theory with compact or complex gauge group. Finally, we discuss theoretical and experimental evidence for our conjecture.

1. Introduction

1.1. **Chern-Simons theory and analytic continuation.** Chern-Simons Quantum Field Theory in 3-dimensions (perturbative, or non-perturbative) produces a plethora of numerical invariants of knotted 3-dimensional objects. We introduce a packaging of these invariants into two power series: one that encodes non-perturbative invariants (model NP), and one that encodes perturbative invariants (model P). The paper is concerned with the analytic continuation, the asymptotic behavior and arithmetic properties of those power series.

Let us begin with a conjecture concerning the structure of nonperturbative quantum invariants. Consider the generating series

$$L^{\mathrm{np}}_{M,G}(z) = \sum_{n=0}^{\infty} Z_{M,G,n} z^n \tag{1}$$

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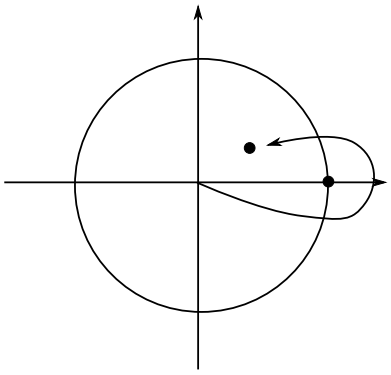
of the *Witten-Reshetikhin-Turaev* invariants $Z_{M,G,n}$ (see Section 2) of a closed, oriented, connected 3-manifold M, using a compact Lie group G and a level $n \in \mathbb{N}$. The power series (1) is known to be convergent inside the unit disk $|z| < 1$ since unitarity implies that $Z_{M,G,n}$ grows at most polynomially with respect to n; see [Ga1].

Conjecture 1. *(Analytic Continuation) For every pair (M, G) as above, the series $L^{\mathrm{np}}_{M,G}(z)$ has analytic continuation as a multivalued function on $\mathbb{C} \setminus e\Lambda_{M,G}$, where $e\Lambda_{M,G} \subset \mathbb{C}$ is a finite set that contains zero and the exponentials of the negative of the critical values of the complexified Chern-Simons action.*

A key observation is that $e\Lambda_{M,G}$ may contain elements inside the unit disk $|z| < 1$ despite the fact that the power series $L^{\mathrm{np}}_{M,G}(z)$ is analytic for z such that $|z| < 1$. One may compare this behavior with the power series $\sum_{n=1}^{\infty} z^n/n^2$ that define the classical dilogarithm, whose analytic continuation is a multivalued analytic function in $\mathbb{C}\setminus\{0,1\}$. Schematically, the analytic continuation of $L^{\mathrm{np}}_{M,G}(z)$ may be depicted as follows:



The above conjecture has the following features:

(a) It can be formulated for pairs $(\mathcal{K}, G)$ where $\mathcal{K}$ denotes a knotted object, i.e., a knot K in 3-space or a closed 3-manifold M and G denotes a compact Lie group.
(b) It implies via elementary complex analysis two well-known Asymptotic Conjectures in Quantum Topology; namely the Volume Conjecture (in the case of knots), and the Witten Conjecture (in the case of 3-manifolds). The complex analysis argument uses the Cauchy formula to write $Z_{M,G,n}$ as a contour integral of $L^{\mathrm{np}}_{M,B}(z)/z^{n+1}$ and then deform the contour around the singularities of the integrand nearest to the origin. For a detailed discussion, see [CG1, Sec.7].
(c) It states a precise relation between exact Chern-Simons theory and its perturbation expansion around a trivial (or not) flat connection. Namely, perturbation theory is simply the expansion of the multivalued function $L^{\mathrm{np}}_{M,G}(z)$ around one of its singularities.

(d) It explains the effect of complexifying a compact gauge group and to the partition function of the corresponding gauge theory. Indeed, analytic continuation captures the critical values of the complexified action; compare also with [GM, Vo].
(e) The Conjecture can be extended to state-sum invariants of sum-product type that generalize $Z_{M,G,n}$ and do not necessarily come from topology.
(f) The Conjecture has been proven for power series of 1-dimensional sum-product type, which includes the case of the 3_1 and 4_1 knots; see [ES] and [CG2].
(g) The Conjecture can, and has been, numerically tested. See Section 7.

1.2. **Chern-Simons theory and Symmetry.** Our next conjecture is a Symmetry Conjecture. Recall that M denotes an oriented 3-manifold; we let τM denote the *orientation reversed* manifold.

Conjecture 2. *(Symmetry) For every pair (M, G) with M an integer homology sphere, we have:*

$$L_{M,G}(z) := L^{\mathrm{np}}_{\tau M,G}(z) - L^{\mathrm{np}}_{M,G}(1/z) \tag{2}$$

has singularities at $z = 0, 1, \infty$.

Let us make some comments regarding the above *paradoxical statement*.

(a) The left (resp. right) hand side is given by a convergent power series for $|z| < 1$ (resp. $|z| > 1$). Thus, the power series never make sense simultaneously, but their analytic continuations do.
(b) Zagier calls a similar statement in [Za1, Eqn.7] a *strange identity* since the the two sides never make sense simultaneously. Our Symmetry Conjecture is closely related to a *modular property*, at least for the series studied by Kontsevich-Zagier; see [Za1, Sec.6].
(c) In physics, Equation (2) is usually called a *duality*.
(d) In algebraic geometry and number theory, one may compare (2) with the following symmetry for the polylogarithm:

$$\mathrm{Li}_k(z) + (-1)^k \mathrm{Li}_k(1/z) = -\frac{(2\pi i)^k}{k!} B_k\left(\frac{\log(z)}{2\pi i}\right), \tag{3}$$

where $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$ is the k-th polylogarithm and $B_k(z)$ is the k-th Bernoulli polynomial; see [Oe, Sec.1.3].
(e) In analysis one may use the above symmetry to deduce the asymptotic behavior (and even more, the asymptotic expansion) of $L^{\mathrm{np}}_{M,G}(z)$ for large $|z|$. In particular, if $L_{M,G}(z) = 0$, it follows that

$$L^{\mathrm{np}}_{M,G}(z) = 1 + O\left(\frac{1}{z}\right) \tag{4}$$

for large $|z|$.
(f) The above symmetry may be explained by the fact that CS changes sign under orientation reversal. Since the level is nonnegative, the path integral formula for $L^{\mathrm{np}}_{M,G}(z)$ formally implies the above symmetry.

(g) If we use the normalized invariants

$$\hat{L^{\mathrm{np}}}_{M,G}(z) = \sum_{n=0}^{\infty} \hat{Z}_{M,G,n} z^n, \qquad \hat{Z}_{M,G,n} = \frac{Z_{M,G,n}}{Z_{S^3,G,n}} \tag{5}$$

and if M is an integer homology sphere, then it is possible that

$$\hat{L^{\mathrm{np}}}_{\tau M,G}(z) - \hat{L^{\mathrm{np}}}_{M,G}(1/z) = 0. \tag{6}$$

(h) When $L_{M,G}(z) = 0$, it follows that the asymptotic expansion of $L^{\mathrm{np}}_{M,G}$ around its singularities *uniquely determines* $L^{\mathrm{np}}_{M,G}$. Indeed, the difference between two determinations is an entire function which is bounded by a constant by the Symmetry Conjecture. Thus, the difference is identically zero.

(i) If $M = M$ is *amphicheiral* (for example, M is given by a connected sum $M = N \# \tau N$ where τ is the orientation reversing involution), then Equation (6) predicts that $\hat{L^{\mathrm{np}}}_{M,G}(z) = \hat{L^{\mathrm{np}}}_{\tau M,G}(1/z)$.

1.3. **Chern-Simons theory and P versus NP.** So far, we considered nonperturbative quantum invariants. Let us now consider perturbative quantum invariants of pairs $(\mathcal{K}, G)$. They can be packaged into a power series $L^{\mathrm{p}}_{M,G}(z)$ which is convergent at $z = 0$. For a detailed definition, see Section 2.

Our next conjecture describes an explicit relation between the perturbative and nonperturbative quantum invariants.

Conjecture 3. (*Exact Implies Perturbative*) *For every integer homology sphere M we have:*

$$\hat{L^{\mathrm{np}}}_{M,G}(1+z) = \log(z) L^{\mathrm{p}}_{M,G}(\log(1+z)) + h(z), \tag{7}$$

where $h(z)$ is a holomorphic function of z at $z = 0$ and $\hat{L^{\mathrm{np}}}_{M,G}(1+z)$ denotes the analytic continuation at $1+z$ along any *path that avoids the singularities.*

As before, we can extend Conjecture 3 to pairs $(K, \mathrm{SU}(2))$, where K is a knot in 3-space.

Remark 1.1. In Écalle's terminology (see [Ec1] and also [Sa, Sec.2.3]), if Δ_z denotes the *alien derivative* in the direction z, Conjecture 3 states that:

$$\Delta_1 \hat{L^{\mathrm{np}}}_{M,G}(z) = L^{\mathrm{p}}_{M,G}(\log(1+z)), \tag{8}$$

$$\Delta_{1-z}\Delta_z \hat{L^{\mathrm{np}}}_{M,G}(z) = 0, \qquad \text{for} \qquad z \in \mathbb{C} \setminus \{0,1\}. \tag{9}$$

Equation (9) is reminiscent of the condition $z \wedge (1-z) \in \wedge^2(\mathbb{C}^*)$ that defines the *Bloch group*; see for example [Ga4].

Remark 1.2. The series $L^{\mathrm{p}}_{\mathcal{K},G}(z)$ also satisfies a Symmetry Property:

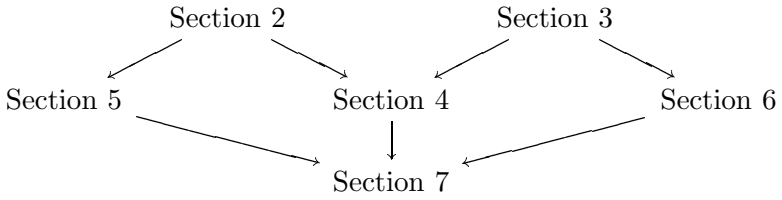
$$L^{\mathrm{p}}_{\tau\mathcal{K},G}(z) = L^{\mathrm{p}}_{\mathcal{K},G}(-z). \tag{10}$$

Unlike the case of Conjecture 2, this is an easy corollary of its very definition.

Remark 1.3. Chern-Simons theory with complex gauge group was studied extensively by Gukov in [Gu]. It is an interesting problem to compare forthcoming work of Gukov-Zagier on modularity properties of the quantum invariants with our conjectures.

1.4. **Chern-Simons theory and arithmetic resurgence.** Based on some partial results of [ES] and [CG2] and stimulating conversations with O. Costin, J. Écalle and D. Zagier, more is actually expected to be true. Namely, we expect arithmetic restrictions on the singularities of the series $L^{\mathrm{np}}_{M,G}(z)$ and of its monodromy, local and global. These restrictions lead us naturally to the notion of arithmetic resurgence, and the Gevrey series of mixed type. In the rest of the paper, we will formulate these expected algebraic/arithmetic aspects of quantum invariants in a precise way and to expose the reader to the wonderful world of resurgence, introduced by Écalle in the eighties for unrelated reasons; [Ec1].

The logical dependence of the sections is the following:



2. Chern-Simons theory and invariants of knotted objects

2.1. **Model NP: Non-perturbative invariants of 3-manifolds.** In this section M will denote a closed 3-manifold and G will denote a simple, compact, simply connected group G. For example, $G = \mathrm{SU}(2)$.

The Witten-Reshetikhin-Turaev invariant is a map:

$$Z_{M,G} : \mathbb{N} \longrightarrow \mathbb{C}. \tag{11}$$

For a definition of $Z_{M,G,n}$ see [RT, Tu2, Wi]. Formally, for $n \in \mathbb{N}$, $Z_{M,G,n}$ is the expectation value of a *path integral* with a topological *Chern-Simons Lagrangian* at level n; see [Wi]. Since the Chern-Simons Lagrangian takes values in $\mathbb{R}/\mathbb{Z}$, it follows that the level n has to be an integer number, which without loss we take it to be nonnegative. We can convert the sequence $(Z_{M,G,n})$ into a generating series as follows:

Definition 2.1. For every M and G as above, we define:

$$L^{\mathrm{np}}_{M,G}(z) = \sum_{n=0}^{\infty} Z_{M,G,n} z^n. \tag{12}$$

Unitarity of the Chern-Simons theory implies that for every M, G the sequence $(Z_{M,G,n})$ grows polynomially with respect to n. In other words, it was shown in

[Ga1] that there exists positive constant C and $m \in \mathbb{N}$ (that depend on M and G) so that

$$|Z_{M,G,n}| < Cn^m$$

for all $n \in \mathbb{N}$. Thus, $L^{\mathrm{np}}_{M,G}(z)$ is analytic inside the unit disk $|z| < 1$.

2.2. **Model P: Perturbative invariants of 3-manifolds.** The path integral interpretation of $Z_{M,G,n}$ formally leads to a perturbation theory along a distinguished critical point of the Chern-Simons action, namely the trivial flat connection. This gives rise to a graph-valued power series invariant, which has been defined by Le-Murakami-Ohtsuki in [LMO]. Additional definitions of this powerful invariant were given by Kuperberg-Thurston; see [KT]. More precisely, LMO define a graph-valued invariant $Z^{\mathrm{LMO}}_M \in \mathcal{A}(\emptyset)$ where $\mathcal{A}(\emptyset)$ is a completed vector space of Jacobi diagrams. A *Jacobi* diagram of degree n is a trivalent graph with $2n$ oriented vertices, considered modulo the AS and IHX relations; see [B-N]. Jacobi diagrams are diagrammatic analogues of tensors on a Lie algebra with an invariant inner product. Indeed, given a simple Lie algebra $\mathfrak{g}$, there is a *weight system map* that replaces a Jacobi diagram of degree n by a rational number times x^{-n}:

$$W_{\mathfrak{g}} : \mathcal{A}(\emptyset) \longrightarrow \mathbb{Q}[[1/x]],$$

see [B-N]. Recall the *Borel transform*:

$$\mathcal{B} : \mathbb{Q}[[1/x]] \longrightarrow \mathbb{Q}[[z]], \qquad \mathcal{B}\left(\sum_{n=0}^{\infty} \frac{a_n}{x^n}\right) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} z^n. \tag{13}$$

Definition 2.2. Let $L^{\mathrm{p}}_{M,\mathfrak{g}}(z)$ denote the Borel transform of $W_{\mathfrak{g}} \circ Z^{\mathrm{LMO}}_M$.

In [GL2] it was proven that if M is a homology sphere and $\mathfrak{g}$ is a simple Lie algebra, then the formal power series $W_{\mathfrak{g}} \circ Z^{\mathrm{LMO}}_M$ is Gevrey-1. In other words, $L^{\mathrm{p}}_{M,\mathfrak{g}}(z)$ is an analytic function in a neighborhood of $z = 0$.

2.3. **The critical values of the Chern-Simons action and the dilogarithm.** Our main Resurgence Conjecture 4 formulated in Section 4 below links the singularities of the analytic continuation of the series $L^{\mathrm{np}}_{\mathcal{K},G}(z)$ and $L^{\mathrm{p}}_{\mathcal{K},G}(z)$ to some classical geometric invariants of 3-manifolds, namely the critical values of the complexified Chern-Simons function. Let us recall those briefly, and refer the reader to [GZ, Wi, Ne, Ga4] for a more detailed discussion.

Let us fix a closed 3-manifold M, and simple, compact simply connected group G, and a trivial bundle $M \times G$ with the trivial connection d. Let $\mathcal{A}$ denote the set of G-connections on $M \times G$. There is a Chern-Simons map:

$$\mathrm{CS} : \mathcal{A} \longrightarrow \mathbb{R}/\mathbb{Z}(2), \tag{14}$$

where, as common in algebraic geometry, we denote

$$\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}. \tag{15}$$

Even though $\mathcal{A}$ is an affine infinite dimensional vector space acted on by an infinite dimensional gauge group, the set $X_G(M)$ of gauge equivalence classes of the critical points of CS is a compact semialgebraic set that consists of flat G-connections. Up to gauge equivalence, the latter are determined by their monodromy. In other words, we may identify:

$$X_G(M) = \mathrm{Hom}(\pi_1(M), G)/G. \tag{16}$$

This gives rise to a map:

$$\mathrm{CS} : X_G(M) \longrightarrow \mathbb{R}/\mathbb{Z}(2), \qquad A \mapsto \mathrm{CS}(A) = \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{17}$$

Stokes's theorem implies that CS is a locally constant map. Since $X_G(M)$ is a compact set, CS takes finitely many values in $\mathbb{R}/\mathbb{Z}(2)$. Let us now *complexify* the action; see also [Vo]. This means that we replace the compact Lie group G by its complexification $G_{\mathbb{C}}$, the moduli space $X_G(M)$ by $X_{G_{\mathbb{C}}}(M)$, and the Chern-Simons action CS by $\mathrm{CS}_{\mathbb{C}}$:

$$\mathrm{CS}_{\mathbb{C}} : X_{G_{\mathbb{C}}}(M) \longrightarrow \mathbb{C}/\mathbb{Z}(2). \tag{18}$$

$\mathrm{CS}_{\mathbb{C}}$ is again a locally constant map, and takes finitely many values in $\mathbb{C}/\mathbb{Z}(2)$. Thus, we may define the following geometric invariants of 3-manifolds.

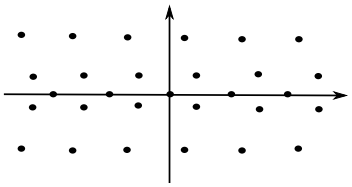
Definition 2.3. For M and G as above, we define

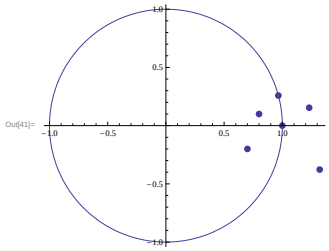
$$\begin{aligned} \Lambda_{M,G} &= \cup_{\rho \in X_{G_{\mathbb{C}}}(M)} (-\mathrm{CS}_{\mathbb{C}}(\rho) + \mathbb{Z}(2)) \subset \mathbb{C}, \\ e\Lambda_{M,G} &= \{0\} \cup \exp\left(\frac{1}{2\pi i} \Lambda_{M,G}\right) \subset \mathbb{C}. \end{aligned} \tag{19}$$

Remark 2.4. Under complex conjugation (but keeping the orientation of the ambient manifold fixed), we have $\mathrm{CS}_{\mathbb{C}}(\bar{\rho}) = \overline{\mathrm{CS}_{\mathbb{C}}(\rho)}$. It follows that $\Lambda_{M,G}$ (resp.$e\Lambda_{M,G}$) is invariant under $\lambda \leftrightarrow \bar{\lambda}$ (resp. $\lambda \leftrightarrow 1/\bar{\lambda}$). The involution $\lambda \leftrightarrow 1/\bar{\lambda}$ preserves the set of rays through zero.

On the other hand, under orientation reversal, we have $\Lambda_{\tau M,G} = -\Lambda_{M,G}$ and $e\Lambda_{\tau M,G} = \tau e\Lambda_{M,G}$, where $\tau(\lambda) = 1/\lambda$ for $\lambda \neq 0$ and $\tau(0) = 0$. We thank C. Zickert for help in identifying those involutions.

Thus, a typical picture for $\Lambda_{M,G}$ and $e\Lambda_{M,G} \setminus \{0\}$ is the following:





where the horizontal spacing between two dots in any horizontal line is

$$4\pi^2 = 39.4784176044\ldots.$$

Complexification is a key idea, theoretically, as well as computationally. For example, $X_{G_{\mathbb{C}}}(M)$ is an algebraic variety whereas its real part $X_G(M)$ is only a compact set with little structure. The only systematic way (known to us) to give exact formulas for the critical values of CS is to actually compute the critical values of $\mathrm{CS}_{\mathbb{C}}$ and then decide which of these are critical values of CS. For $G = \mathrm{SU}(2)$, there are exact and numerical computer implementations for the critical values of $\mathrm{CS}_{\mathbb{C}}$: see `snap` [Sn] and [Ne, DZ].

Complexification also reveals the arithmetic structure of $\Lambda_{M,G}$: its elements are *periods* of weight 2 (in the sense of Kontsevich-Zagier [KZ]), of a rather special kind. Namely, the critical values of $\mathrm{CS}_{\mathbb{C}}$ are $\mathbb{Q}$-linear combinations of the *Rogers dilogarithm* function evaluated at algebraic numbers. The latter is defined by:

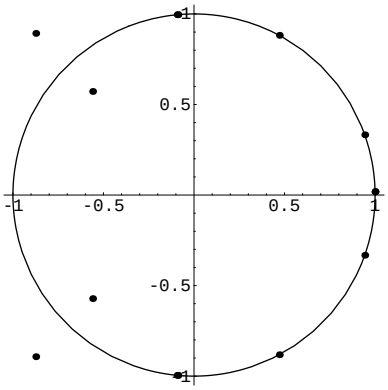
$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) - \frac{\pi^2}{6} \tag{20}$$

for $z \in (0,1)$ and analytically continued as a multivalued analytic function in $\mathbb{C} \setminus \{0,1\}$. Here, $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$ is the classical dilogarithm function. For $G = \mathrm{SU}(2)$, our identification of the complexified Chern-Simons action with [NZ, Ne, GZ] is as follows:

$$\mathrm{CS}_{\mathbb{C}}(\rho) = i\mathrm{Vol}(\rho) + \mathrm{CS}(\rho). \tag{21}$$

For higher rank groups, exact formulas for the critical values of $\mathrm{CS}_{\mathbb{C}}$ may also be given in terms of the Rogers dilogarithm function at algebraic numbers. This will be explained in detail in a separate publication. As an illustration of the above discussion let us give an example.

Example 2.5. If M is obtained by 1/2 surgery on the 4_1 knot, then $e\Lambda \setminus \{0\}$ consists of 13 points plotted as follows:



In this picture, a higher resolution reveals that the points nearest to the vertical axis consist of *two* distinct but close pairs. We thank C. Zickert for providing us with an exact and numerical computation of the critical values of the complexified Chern-Simons map.

Let us end this section with a problem:

Problem 1. Give an direct relation between the cubic polynomial action (14) and the Rogers dilogarithm (20).

A transcendental relation between the Chern-Simons action and the Rogers dilogarithm was given in [Ga4, Sec.6.2], using the third algebraic K-theory group $K_3^{\mathrm{ind}}(\mathbb{C})$.

2.4. **Extension to knots in 3-space.** So far, the discussion involved closed 3-manifolds. Let us now consider knots K in the 3-sphere. For simplicity, we will assume that $G = \mathrm{SU}(2)$ (so $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$) in this section.

Let us fix a knot K in 3-sphere and a nonnegative integer n. Let us denote by $Z_{K,\mathrm{SU}(2),n}$ the *Kashaev invariant* of K (see [Ka]), which is also identified by [MM] with the value of the Jones polynomial colored by the n-th dimensional irreducible representation of $\mathfrak{sl}_2$ evaluated at $q = e^{2\pi i/n}$ (and normalized to be 1 at the unknot). Thus, we may define:

$$L^{\mathrm{np}}_{K,\mathrm{SU}(2)}(z) = \sum_{n=0}^{\infty} Z_{K,\mathrm{SU}(2),n} z^n. \tag{22}$$

We define the perturbative invariant $L^{\mathrm{p}}_{K,\mathrm{SU}(2)}(z)$ as follows:

(a) Take the sequence $J_{K,n}(q) \in \mathbb{Z}[q^{\pm}]$ of the Jones polynomials of n, colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$, and normalized by $J_{\mathrm{unknot},n}(q) = 1$. See for example, [Tu1, Tu2].

(b) It turns out that there exists a power series $J_K(u,q) \in \mathbb{Q}(u)[[q-1]]$ so that $J_K(q^n,q) = J_{K,n}(q) \in \mathbb{Q}[[q-1]]$ for all $n \in \mathbb{N}$; see for example [Ga2, GL3].

(c) Consider the power series $J_K(1, e^{1/x}) \in \mathbb{Q}[[1/x]]$.
(d) Define $L^{\mathrm{p}}_{K,\mathrm{SU}(2)}(z) = \mathcal{B}(J_K(1, e^{1/x})) \in \mathbb{Q}[[z]]$.

In [GL3] (resp. [GL2]) it was shown that $L^{\mathrm{np}}_{K,\mathrm{SU}(2)}(z)$ (resp. $L^{\mathrm{p}}_{K,\mathrm{SU}(2)}(z)$) is analytic for z in a neighborhood of 0. Regarding the critical values of the Chern-Simons action, we will consider only *parabolic* $\mathrm{SL}(2,\mathbb{C})$ representations; i.e., those representations so that the trace of every peripheral element is ± 2. As in the case of closed 3-manifolds, we may identify the moduli space of *parabolic* flat $\mathrm{SL}(2,\mathbb{C})$-connections on the knot complement with $X^{\mathrm{par}}_{\mathrm{SL}(2,\mathbb{C})}(K)$:

$$X^{\mathrm{par}}_{\mathrm{SL}(2,\mathbb{C})}(K) = \mathrm{Hom}^{\mathrm{par}}(\pi_1(S^3 \setminus K), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C}). \tag{23}$$

In addition, we have a map, described in detail in [GZ]:

$$\mathrm{CS}_{\mathbb{C}} : X^{\mathrm{par}}_{\mathrm{SL}(2,\mathbb{C})}(M) \longrightarrow \mathbb{C}/\mathbb{Z}(2). \tag{24}$$

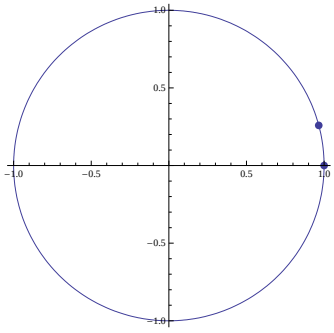
$\mathrm{CS}_{\mathbb{C}}$ is again a locally constant map, and takes finitely many values in $\mathbb{C}/\mathbb{Z}(2)$, and allows us to define the sets $\Lambda_{K,\mathrm{SU}(2)}$ and $e\Lambda_{K,\mathrm{SU}(2)}$.

Let us end this section with an example of the simplest knot 3_1 and the simplest hyperbolic knot 4_1.

Example 2.6. If $K = 3_1$ is the right hand trefoil knot 3_1, then

$$e\Lambda = \{e^{\pi i/12}, 1, 0\} \tag{25}$$

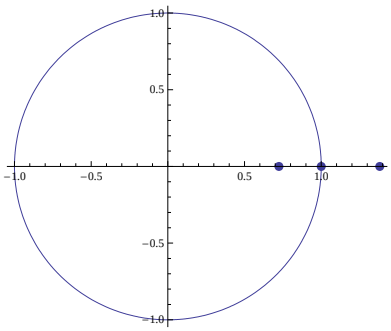
can be plotted as follows:



Example 2.7. If $K = 4_1$ is the simplest hyperbolic knot, then

$$e\Lambda = \{e^{-\mathrm{Vol}(4_1)/(2\pi)}, 1, e^{\mathrm{Vol}(4_1)/(2\pi)}, 0\} \tag{26}$$

can be plotted as follows:



where

$$\mathrm{Vol}(4_1) = -i\mathrm{Li}_2(e^{2\pi i/6}) + i\mathrm{Li}_2(e^{-2\pi i/6}) = 2.02988321281930725004240510855\ldots$$

is the *Volume* of 4_1; see [Th], and numerically,

$$\begin{aligned} e^{-\mathrm{Vol}(4_1)/(2\pi)} &= 0.72392611187952434703122933736\ldots \\ e^{\mathrm{Vol}(4_1)/(2\pi)} &= 1.38135644451849779337146695685\ldots. \end{aligned}$$

3. Arithmetic resurgent functions

3.1. **Resurgent functions.** The arithmetic nature of $\Lambda_{M,G}$ is only the beginning. It turns out that

(a) the *Ray-Singer torsion invariants* associated to a $G_{\mathbb{C}}$-representation of $\pi_1(M)$ are also algebraic numbers (this is proven and discussed in detail in [DG]),
(b) in case M is hyperbolic and $G = \mathrm{SU}(2)$ the geometric representation is defined over a number field; see [MR],
(c) the perturbative expansions of the quantum invariants $Z_{M,G,n}$ around an $G_{\mathbb{C}}$-representation of $\pi_1(M)$ are conjectured to be algebraic numbers; see for example [GM] and [CG1].

The need to formulate these algebricity properties in a uniform way, as well as some results in some key cases, lead us to the notion of an arithmetic resurgent series, which is the focus of this section.

Along the way, we will also discuss the auxiliary notion of a Gevrey series of mixed type, perhaps of interest on its own.

Resurgence was coined by Écalle in his study of analytic continuation of formal and actual solutions of differential equations, linear or not; see [Ec1]. An earlier term used by Écalle was the notion of *endless analytic continuation*. The concept of resurgence has influenced our thinking deeply. Unfortunately, it is not easy to find an accepted definition of resurgence, or a reference for it in the literature. On the other hand, there are several expositions of instances of resurgence, covering special cases of this rather general notion. The curious reader may consult [Co1, Co2, Dl, Ml, Sa] for a detailed discussion in addition to the original work [Ec1].

Given the gap in the literature, we will do our best to give a working (and exact) definition of resurgence, which features some properties which are arithmetically important, and analytically rare. Let us begin by recalling the monodromy of multivalued germs of interest in this paper. We refer the reader to [Ml] for further details. The next definition is motivated by the types of singularities that appear in algebraic geometry; see [Kz].

Definition 3.1. A *multivalued analytic germ* $f(z)$ at $z = 0$ is called *quasi-unipotent* if its monodromy T around 0 satisfies the condition:

$$(T^r - 1)^s = 0 \tag{27}$$

for some nonzero natural numbers r and s.

It is easy to see that a quasi-unipotent germ $f(z)$ can be written as a finite sum of germs of the form:

$$\sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha (\log(z))^\beta h_{\alpha,\beta}(z), \tag{28}$$

where $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, and $h_{\alpha,\beta}(z) \in \mathbb{C}\{z\}_0$, where $\mathbb{C}\{z\}_0$ is the ring of power series convergent at $z = 0$ (identified with the ring of germs of functions analytic at $z = 0$). See for example, [Ml]. Series of the form (28) are often known in the literature as series of the *Nilsson class*; see [Ni1, Ni2].

The rationality of the exponents $\{\alpha\}$ above is an important feature that always appears in algebraic geometry and arithmetic and rarely appears in analysis. For a further discussion; see [Ga5].

Definition 3.2. We say that $G(z) = \sum_{n=0}^\infty a_n z^n$ is an *resurgent series* (and write $G(z) \in \mathrm{RES}$) if

(a) $G(z)$ is convergent at $z = 0$.
(b) $G(z)$ has analytic continuation as a multivalued function in $\mathbb{C} \setminus \Lambda$, where $\Lambda \subset \mathbb{C}$ is a discrete subset of $\mathbb{C}$.
(c) The local monodromy is quasi-unipotent.

In what follows, we will make little distinction between a germ, its analytic continuation, and the corresponding function. So, we will speak about the algebra of resurgent functions.

3.2. **Gevrey series of mixed type.** The following definition is motivated by the properties of some power series that are associated to knotted 3-dimensional objects.

Definition 3.3. We say that series $G(z) = \sum_{n=0}^\infty a_n z^n$ is a *Gevrey series of mixed type* (r, s) if

(a) $r, s \in \mathbb{Q}$ and the coefficients a_n lie in a number field K, and
(b) there exists a constant $C > 0$ so that for every $n \in \mathbb{N}$ the absolute value of every Galois conjugate of a_n is less than or equal to $C^n n!^r$, and
(c) the common denominator of $a_0/0!^s, \dots, a_n/n!^s$ is less than or equal to C^n.

Remark 3.4. If $G(z)$ is Gevrey of mixed type (r, s) and $r' \geqslant r$, $s' \leqslant s$, then $G(z)$ is also Gevrey of mixed type (r', s').

Let $\mathbb{C}^{\mathrm{GM}}\{z\}$ (resp. $\mathbb{C}^{\mathrm{GM}}\{z\}_{r,s}$) denote the $\overline{\mathbb{Q}}[z]$-algebra of Gevrey series of mixed type (resp. mixed type (r, s)).

Remark 3.5. The Gevrey series of mixed type (r, r) are precisely the important class of *arithmetic Gevrey series of type* r, introduced and studied by André; see [An].

Remark 3.6. A *G-function* $G(z)$ in the sense of Siegel is a Gevrey series of mixed type $(0, 0)$ which is *holonomic*, i.e., it satisfies a linear ODE with coefficients in $\overline{\mathbb{Q}}[z]$; see [An, Bo, DGS].

3.3. **Arithmetic resurgent functions.** Restricting the functions $h_{\alpha,\beta}(z)$ in (28) to be Gevrey series of type $(0, s)$, we arrive at the notion of an arithmetic quasi-unipotent germ.

Definition 3.7. We say that a multivalued analytic germ $f(z)$ at $z = 0$ is *arithmetic quasi-unipotent* if it can be written as a finite sum of germs of the form (28) where $h_{\alpha,\beta}(z) \in \mathbb{C}^{\mathrm{GM}}\{z\}$.

Combining this definition with the notion of a resurgent function, we arrive at the notion of an arithmetic resurgent function. Let $\mathcal{P} \subset \mathbb{C}$ denote the countable set of *periods* in the sense of Kontsevich-Zagier; see [KZ].

Definition 3.8. We say that $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is an *arithmetic resurgent series* (and write $G(z) \in \mathrm{ARES}$) if

(a) $G(z)$ is a resurgent series.
(b) The singularities Λ of $G(z)$ is a discrete subset of $\mathcal{P}$, where $\mathcal{P}$ denotes the set of *periods* as defined by Kontsevich-Zagier; see [KZ].
(c) The local monodromy is arithmetic quasi-unipotent.
(d) The global monodromy is defined over $\bar{\mathbb{Q}}$.

3.4. **The Taylor series of an arithmetic resurgent function.** In a separate publication we will give the proof of the following proposition which shows that the coefficients of the Taylor series at the origin of an arithmetic resurgent function have asymptotic expansions themselves. The transseries conclusion of the proposition below (without any claims on the mixed Gevrey type) follows from the resurgence hypothesis on $G(z)$ alone, and are studied systematically in the upcoming book of Costin [Co2], as well as in [Co3].

Proposition 3.9. *If* $G(z) = \sum_{n=0}^{\infty} a_n z^n$ *is arithmetic resurgent, then*

$$a_n \sim \sum_{\lambda} \lambda^{-n} f_\lambda\left(\frac{1}{n}\right),$$

where the sum is over the finite set of singularities of $G(z)$ *nearest to the origin, and* $f_\lambda(z)$ *is a finite sum of series of the form* (28) *where* $h_{\alpha,\beta,\lambda}(z)$ *are Gevrey of mixed type* $(1, s)$ *(for some* s*).*

3.5. **Arithmetic invariants of arithmetic resurgent functions.** Obvious arithmetic invariants of an arithmetic resurgent function $G(z)$ are:

(a) The set of singularities $\Lambda \subset \mathcal{P}$.
(b) The local quasi-unipotent monodromy, and its field of definition.
(c) The global monodromy, defined over $\bar{\mathbb{Q}}$.

3.6. **G-functions are arithmetic resurgent.** This section is logically independent of the rest of the paper and can be skipped at first reading, although it provides some useful examples of arithmetic resurgent functions. A main example of arithmetic resurgent functions comes from the following theorem of André.

Theorem 1. *G-functions are arithmetic resurgent with singularities a finite set of algebraic numbers.*

G-functions arise naturally in three contexts:

(a) From geometry, related to the regularity of the Gauss-Manin connection.
(b) From arithmetic.
(c) From enumerative combinatorics.

For a geometric construction of resurgent functions, let us recall the following result from [Kz]; see also [De, Br]. Let $\overline{S}/\mathbb{C}$ be a projective non-singular connected curve and $S = \overline{S} \setminus \{p_1, \dots, p_r\}$ the complement of a finite set of points. Suppose that

$$\pi : X \longrightarrow S$$

is a proper and smooth morphism. For every i, the algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X/S)$ is an algebraic differential equation on S, and the local system $H^i(X_s, \mathbb{C})$ (for $s \in S$) is the local system of germs of solutions of that equation. Let T denote the local monodromy around a point p_i. Then, we have the following theorem.

Theorem 2. [Ka] *The algebraic differential equation is regular singular and the local monodromy T is quasi-unipotent.*

The G-functions obtained by Theorems 2 and 1 are closely related. The main conjecture is that all G-functions come from geometry. For a discussion of this topic, and for a precise formulation of the Bombieri-Dwork Conjecture, see the survey papers of [Bo, Ka] and also [To, p.8].

Let us discuss a third source of resurgent functions, which was discovered recently by the author in [Ga5].

Definition 3.10. A *hypergeometric term* $\mathfrak{t}_{n,k}$ (in short, *general term*) in variables (n, k) where $k = (k_1, \dots, k_r)$ is an expression of the form:

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{k_i} \prod_{j=1}^{J} A_j(n,k)!^{\epsilon_j}, \tag{29}$$

where $C_i \in \overline{\mathbb{Q}}$ for $i = 0, \ldots, r$, $\epsilon_j = \pm 1$ for $j = 1, \ldots, J$, and A_j are integral linear forms in the variables (n, k). We assume that for every $n \in \mathbb{N}$, the set

$$\{k \in \mathbb{Z}^r \mid A_j(n,k) \geqslant 0,\ j = 1, \ldots, J\} \tag{30}$$

is finite. We will call a general term *balanced* if in addition it satisfies the *balance condition*:

$$\sum_{j=1}^{J} \epsilon_j A_j = 0. \tag{31}$$

Given a balanced term $\mathfrak{t}$, consider the corresponding sequence $(a_{\mathfrak{t},n})$ defined by

$$a_{\mathfrak{t},n} = \sum_k \mathfrak{t}_{n,k}, \tag{32}$$

where the summation index lies in the finite set (30), and the corresponding generating series:

$$G_{\mathfrak{t}}(z) = \sum_{n=0}^{\infty} a_{\mathfrak{t},n} z^n \in \mathbb{Q}[[z]]. \tag{33}$$

We will call sequences of the form $(a_{\mathfrak{t},n})$ *balanced multisum sequences*.

Theorem 3. [Ga3] *For every balanced term* $\mathfrak{t}$*, the generating series* $G_{\mathfrak{t}}(z)$ *is a G-function.*

Let us point out that the proof of Theorem 3 in general offers no help of locating the singularities of the function $G_{\mathfrak{t}}(z)$. To fill this gap, the author developed an efficient ansatz for the location of the singularities of $G_{\mathfrak{t}}(z)$; see [Ga3].

4. An arithmetic resurgence conjecture

A *knotted object* $\mathcal{K}$ denotes either a closed 3-manifold M or a knot K in 3-space. A *pair* $(\mathcal{K}, G)$ denotes either a closed 3-manifold M and a compact Lie group G, or a knot $\mathcal{K} = K$ in 3-space and $G = \mathrm{SU}(2)$. If $G = \mathrm{SU}(2)$ and M or K is hyperbolic, let

$$\Lambda^{\mathrm{geom}}_{\mathcal{K},G} = \cup_{\rho}(-\mathrm{CS}_{\mathbb{C}}(\rho) + \mathbb{Z}(2)) \tag{34}$$

denote the critical values of the Galois conjugates ρ of the geometric $\mathrm{SL}(2,\mathbb{C})$-representation.

We now have all the ingredients to formulate our Arithmetic Resurgence Conjecture, which is a refinement of the Analytic Continuation Conjecture 1.

Conjecture 4. *(Arithmetic Resurgence) For every pair* $(\mathcal{K}, G)$*,* $L^{\mathrm{np}}_{\mathcal{K},G}(e^{z/(2\pi i)})$ *and* $L^{\mathrm{p}}_{\mathcal{K},G}(z)$ *are arithmetic resurgent with possible singularities in the set* $\Lambda_{\mathcal{K},G}$*. If* $\mathcal{K}$ *is hyperbolic and* $G = \mathrm{SU}(2)$*, then the singularities of the above functions include* $\Lambda^{\mathrm{geom}}_{\mathcal{K},G}$*.*

Conjecture 4 implies the following corollaries.

Corollary 4.1. *For every pair $(\mathcal{K}, G)$, the power series $L^{\mathrm{np}}_{\mathcal{K},G}(z)$ has analytic continuation as a multivalued function on the complement $\mathbb{C} \setminus e\Lambda_{\mathcal{K},G}$ of the finite set $e\Lambda_{\mathcal{K},G}$.*

Corollary 4.2. *If M is a closed hyperbolic 3-manifold, then the Witten-Reshetikhin-Turaev invariants determine the Volume of M by:r*

$$e^{-\mathrm{Vol}(M)/(2\pi)} = \min\{|\lambda| \mid L^{\mathrm{np}}_{M,\mathrm{SU}(2)}(z) \text{ is singular at } z = \lambda \neq 0\}. \tag{35}$$

This follows from the fact that $L^{\mathrm{np}}_{M,\mathrm{SU}(2)}(z)$ has a singularity at

$$e^{-(\mathrm{CS}_{\mathbb{C}}(\rho)+\mathbb{Z}(2))/(2\pi i)} = e^{-\mathrm{Vol}(\rho)/(2\pi)+i\theta_\rho}$$

and $\mathrm{Vol}(\rho) \leqslant \mathrm{Vol}(\rho_M) = \mathrm{Vol}(M)$ where ρ_M is a discrete faithful representation.

Corollary 4.3. *Witten's conjecture (formulated in* [Wi]*) regarding the asymptotic expansion of the Witten-Reshetikhin-Turaev invariants holds.*

Corollary 4.4. *For every hyperbolic knot K in 3-space, the Kashaev invariants determine the Volume of K by:*

$$e^{-\mathrm{Vol}(K)/(2\pi)} = \min\{|\lambda| \mid L^{\mathrm{np}}_{K,\mathrm{SU}(2)}(z) \text{ is singular at } z = \lambda \neq 0\}. \tag{36}$$

Moreover, there is an asymptotic expansion of the Kashaev invariants in powers of $1/n$ using Proposition 3.9.

5. The Habiro ring, and P versus NP

In this section we we describe an arithmetic relation, due to Habiro, between the perturbative $L^{\mathrm{p}}_{\mathcal{K},G}(z)$ and the non-perturbative $L^{\mathrm{np}}_{\mathcal{K},G}(z)$ invariants of knotted objects. This section is independent of our conjecture. However, Habiro's results

(a) are a good complement of our conjecture,
(b) are important and interesting on their own right,
(c) point to a different arithmetic origin for the invariants of knotted objects. This point of view has been studied by Gukov-Zagier [GZa].

In this section, a knotted object $\mathcal{K}$ denotes either a homology sphere M or a knot K. For simplicity, we will assume that $G = \mathrm{SU}(2)$. A theorem of Habiro implies that $L^{\mathrm{np}}_{\mathcal{K},\mathrm{SU}(2)}(z)$ determines $L^{\mathrm{p}}_{\mathcal{K},\mathrm{SU}(2)}(z)$ and vice-versa; [Ha1, Ha2]. Let us explain more about Habiro's key results. In [Ha2] Habiro introduces the ring

$$\widehat{\mathbb{Z}[q^{\pm}]} = \lim_{\leftarrow n} \mathbb{Z}[q^{\pm}]/((q)_n), \tag{37}$$

where $(q)_n$ is the quantum n-factorial defined by:

$$(q)_n = \prod_{k=1}^{n} (1 - q^k) \tag{38}$$

with $(q)_0 = 1$. In a sense, one may think of elements of the Habiro ring as complex-valued *analytic functions* with domain Ω, the set of complex roots of

unity. This way of thinking is motivated by the following features of the Habiro ring, shown in [Ha1, Ha2]:

(a) It is easy to see that every element $f(q) \in \widehat{\mathbb{Z}[q^{\pm}]}$ can be written (nonuniquely) in the form:

$$f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n, \qquad f_n(q) \in \mathbb{Z}[q^{\pm}], \quad \text{for} \quad n \in \mathbb{N}. \tag{39}$$

Note however that the above form is not unique, since for example:

$$1 = \sum_{n=0}^{\infty} q^{n+1}(q)_n.$$

Nevertheless the form (39) can be used to generate easily elements of the Habiro ring.

(b) Elements of the Habiro ring can be evaluated at complex roots of unity. In other words, there is a map:

$$\widehat{\mathbb{Z}[q^{\pm}]} \longrightarrow \mathbb{C}^{\Omega}, \qquad f(q) \longrightarrow (f : \Omega \longrightarrow \mathbb{C}, \qquad \omega \mapsto f(\omega)). \tag{40}$$

In particular, we can associate a map:

$$\widehat{\mathbb{Z}[q^{\pm}]} \longrightarrow \mathbb{C}[[z]], \qquad f(q) \longrightarrow L_f^{\mathrm{np}}(z) = 1 + \sum_{n=1}^{\infty} f(e^{2\pi i/n}) z^n. \tag{41}$$

(c) Elements of the Habiro ring have Taylor series expansions around $q = 1$ (and also around every complex root of unity). In other words, we can define a map:

$$\widehat{\mathbb{Z}[q^{\pm}]} \longrightarrow \mathbb{Q}[[z]], \qquad f(q) \longrightarrow L_f^{\mathrm{p}}(z) = \mathcal{B}(f(e^{1/x})). \tag{42}$$

(d) As in the case of analytic functions, the maps (41) and (42) are 1-1. Thus, $L_f^{\mathrm{np}}(z)$ determines $L_f^{\mathrm{p}}(z)$ and vice-versa. However, we need all the coefficients of the power series $L_f^{\mathrm{np}}(z)$ to determine a single (eg. the third) coefficient of $L_f^{\mathrm{p}}(z)$.

(e) Given a homology sphere M, there exists an element $f_{M,\mathrm{SU}(2)}(q) \in \widehat{\mathbb{Z}[q^{\pm}]}$ such that its image under the maps (41) and (42) coincide with the non-perturbative and perturbative invariants of M discussed in Section 2.1 and 2.2. This was a main motivation for Habiro, and was extended to knots in 3-space by Huynh-Le in [HL].

One may ask for an extension of Conjecture 4 for the series $L_f^{\mathrm{np}}(z)$ and $L_f^{\mathrm{p}}(z)$ that come from the Habiro ring. Unfortunately, the Habiro ring is uncountable (whereas all quantum invariants of knotted objects lie in a countable subring) and it has little structure as such. Thus, it is unlikely that the series $L_f^{\mathrm{np}}(z)$ associated to a random sequence of Laurent polynomials $(f_n(q))$ (as in (39)) will be resurgent. Concretely, we can pose the following problem with overwhelming numerical evidence:

Problem 2. Show that $L_f^{\mathrm{np}}(z)$ is not resurgent when

$$f(q) = \sum_{n=0}^{\infty} q^{2^n} (q)_n. \tag{43}$$

In [GL4], Le and the author introduced a countable subring $\widehat{\mathbb{Z}[q^{\pm}]}^{\mathrm{hol}}$ that consists of elements of the form:

$$f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n, \qquad f_n(q) \in \mathbb{Z}[q^{\pm}], \quad (f_n(q)) \quad \text{is } q\text{-holonomic} \tag{44}$$

where q-holonomic means that $(f_n(q))$ satisfies a linear q-difference equation with coefficients in $\mathbb{Q}[q^{\pm}, q^{\pm n}]$; see [WZ]. In [GL4] it was shown that the elements $f_{M,\mathrm{SU}(2)}(q)$ and $f_{K,\mathrm{SU}(2)}(q)$ of the Habiro ring actually lie in the countable subring $\widehat{\mathbb{Z}[q^{\pm}]}^{\mathrm{hol}}$.

Problem 3. Show that for every $f \in \widehat{\mathbb{Z}[q^{\pm}]}^{\mathrm{hol}}$, the series $L_f^{\mathrm{np}}(z)$ and $L_f^{\mathrm{np}}(z)$ are arithmetic resurgent.

In the next section, we will discuss formulate a resurgence conjecture for some special elements of the Habiro ring that do not always come from topology.

6. Series of sum-product type

Conjecture 4 and Problem 3 ask for proving that certain power series are arithmetic resurgent. However, they do not explain the source of resurgence. Usually, resurgence is associated with a differential equation, linear or not; see for example [Ec1] and also [Co1, Sa].

In this section we will give another construction of powers series $L^{\mathrm{np}}(z)$ and $L^{\mathrm{p}}(z)$ which aims to explain the origin of arithmetic resurgence. This section was motivated by conversations with O. Costin and J. Écalle whom we thank for their generous sharing of their ideas.

Let us first introduce the notion of *series of Sum-Product type*.

Definition 6.1. Consider function F analytic in $[0,1]$ with $F(0) = 0$ and the corresponding sequence and series of sum-product type:

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \prod_{j=1}^{k} F\left(\frac{j}{n}\right) \\ &= F\left(\frac{1}{n}\right) + F\left(\frac{1}{n}\right) F\left(\frac{2}{n}\right) + \cdots + F\left(\frac{1}{n}\right) F\left(\frac{2}{n}\right) \dots F\left(\frac{n}{n}\right) \end{aligned} \tag{45}$$

and the corresponding series

$$L^{\mathrm{np}}(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathbb{C}[[z]]. \tag{46}$$

Since $F(0) = 0$, it follows that the formal power series

$$\Sigma\Pi(x) := \sum_{n=1}^{\infty} \prod_{j=1}^{n} F\left(\frac{j}{x}\right) \in \mathbb{C}[[\frac{1}{x}]] \tag{47}$$

is also well-defined. Let $L^{\mathrm{p}}(z)$ denote the Borel transform:

$$L^{\mathrm{p}}(z) = \mathcal{B}\left(\Sigma\Pi(x)\right) \in \mathbb{C}[[z]]. \tag{48}$$

Let us now give a flavor of some results from [CG2] and [ES]. In the rest of the section, let us consider F of the following trigonometric type:

$$F(x) = \phi(e^{2\pi i x}), \tag{49}$$

where

$$\phi(q) = \epsilon q^{c\frac{n(n+1)}{2}} \prod_{r=1}^{\infty} (1-q^r)^{c_r}, \tag{50}$$

where $c \in \mathbb{Z}$, $\epsilon = \pm 1$, $c_r \in \mathbb{N}$ for all r, and $c_r = 0$ for all but finitely many r. In [Ga4] we construct elements of the extended Bloch group, given by solutions of the algebraic equations:

$$\phi(q) = 1 \qquad \text{or} \qquad \phi(q) = 0. \tag{51}$$

The values of these elements under the Rogers dilogarithm defines a set $\Lambda \subset \mathbb{C}$, and its exponentiated cousin $e\Lambda = \exp(\Lambda/(2\pi i)) \cup \{0\}$.

Theorem 4. [CG2, ES] *$L^{\mathrm{np}}(z)$ and $L^{\mathrm{p}}(z)$ are arithmetic resurgent with singularities included in Λ.*

Remark 6.2. Notice that $L^{\mathrm{np}}(z) = L^{\mathrm{np}}_f(z)$ and $L^{\mathrm{p}}(z) = L^{\mathrm{p}}_f(z)$, where

$$f(q) = \sum_{n=0}^{\infty} \epsilon^n q^{cn} \prod_{r=1}^{\infty} (q^r)_n^{c_r}$$

is an element of the countable Habiro subring $\widehat{\mathbb{Z}[q^{\pm}]}^{\mathrm{hol}}$, where

$$(q^r)_n = \prod_{k=1}^{n} (1-q^{kr}).$$

Thus, Theorem 4 is a special case of Problem 2.

Remark 6.3. Equations (51) appear in the *dilogarithm ladders* of Lewin and others, whose aim is to produce interesting elements of algebraic K-theory. For a detailed discussion, see [Le] and [Za2].

Remark 6.4. For the simplest knot 3_1 and the simplest hyperbolic knot 4_1, we have:

$$L^{\mathrm{np}}_{3_1,\mathrm{SU}(2)}(z) = L^{\mathrm{np}}_{f_{3_1}}(z), \qquad L^{\mathrm{np}}_{4_1,\mathrm{SU}(2)}(z) = L^{\mathrm{np}}_{f_{4_1}}(z)$$

(and likewise, equality for the L^{p}-series), where

$$\begin{aligned} f_{3_1}(q) &= \sum_{n=0}^{\infty} (q)_n \\ f_{4_1}(q) &= \sum_{n=0}^{\infty} (-1)^n q^{-\frac{n(n+1)}{2}} (q)_n^2 \end{aligned}$$

are both covered by Theorem 4.

Remark 6.5. The resurgence conclusion of Theorem 4 is valid for very general entire functions F, with some mild hypothesis. For a detailed discussion, see [ES] and [CG2].

7. Evidence

7.1. **Some results.** Let us summarize what is known about Conjecture 4. Conjecture 4 is known

(a) for all 3-manifolds M of the form $\Sigma \times S^1$ where Σ is a closed surface and all compact groups G. Indeed, this follows from the fact that $Z_{M,G,n}$ is a polynomial in n. Thus, $L^{\mathrm{np}}_{M,G}(z)$ is a rational function of z with denominator a power of $1-z$. On the other hand, $e\Lambda_{M,G} = \{0,1\}$.
(b) for $L^{\mathrm{p}}_{M,\mathrm{SU}(2)}$ where M is the *Poincare homology sphere*, or small Seifert fibered 3-manifolds, see [CG1]. In this lucky case, one uses explicit formulas for the coefficients of $L^{\mathrm{p}}_{M,\mathrm{SU}(2)}(z)$ given Zagier (see [Za1]) which allow one show resurgence relatively easily.
(c) for the simplest knot 3_1 (and also for $(2,p)$ torus knots); and for the simplest hyperbolic knot 4_1; see [CG1] and [CG2]. See Remark 6.4.

Our sample calculations below show the importance of the fractional polylogarithms and their analytic continuation, studied in detail in [CG3].

7.2. **Conjecture 4 for S^3.** Let us confirm Conjecture 4 for S^3. For simplicity, we will choose $G = \mathrm{SU}(2)$. The case of other Lie groups is similar. The Witten-Reshetikhin-Turaev invariant is given by [Wi, Eqn.2.26]:

$$Z_{S^3,\mathrm{SU}(2),n} = \sqrt{\frac{2}{n+2}} \sin\left(\frac{\pi}{n+2}\right). \tag{52}$$

Expanding the above as a convergent power series in $1/n$:

$$Z_{S^3,\mathrm{SU}(2),n} = \sqrt{2} \sum_{k=0}^{\infty} \frac{\pi^{2k+1}(-1)^k}{(2k+1)!} \frac{1}{(n+2)^{2k+3/2}}$$

and using the *fractional polylogarithm* $L_\alpha(z)$ defined for $\alpha \in \mathbb{C}$ and $|z| < 1$ by the convergent series:

$$\mathrm{Li}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha} \tag{53}$$

it follows that

$$L^{\mathrm{np}}_{S^3,\mathrm{SU}(2)}(z) = \frac{\sqrt{2}}{z^2} \sum_{k=0}^{\infty} \frac{\pi^{2k+1}(-1)^k}{(2k+1)!} \left(\mathrm{Li}_{2k+3/2}(z) - \zeta(2k+3/2)\right). \tag{54}$$

Since $\mathrm{Li}_\alpha(z)$ has analytic continuation as a multivalued function in $\mathbb{C} \setminus \{0,1\}$ (see [CG3]), it follows that $L^{\mathrm{np}}_{S^3,\mathrm{SU}(2)}(z)$ has analytic continuation on $\mathbb{C} \setminus \{0,1\}$. We can further compute the monodromy around $z = 0$ and $z = 1$ using [CG3].

7.3. **Conjecture 4 for $S^1 \times \Sigma_g$.** Let us confirm Conjecture 4 for 3-manifolds of the form $S^1 \times \Sigma_g$. For simplicity, we will choose $G = \mathrm{SU}(2)$. The case of other Lie groups is similar. The Witten-Reshetikhin-Turaev invariant is given by the famous *Verlinde formula* [Wi, Sz]:

$$Z_{S^1\times\Sigma_g,\mathrm{SU}(2),n} = \sum_{j=1}^{n+1} \left(\frac{n+2}{2\sin^2 \frac{\pi j}{n+2}} \right)^{g-1}. \tag{55}$$

Altough not a priori obvious, it it true that the right hand side of the above expression is a polynomial in n of degree $3g-3$. In fact, we have [Sz, Sec.3]:

$$Z_{S^1\times\Sigma_g,\mathrm{SU}(2),n} = -(2n+2)^{g-1} \mathrm{Res}\left(\frac{(n+2)\cot((n+2)x)}{(2\sin x)^{2g-2}}, x = 0 \right) \tag{56}$$

when $g \geqslant 2$. For example, we have,

$$\begin{aligned} Z_{S^1\times\Sigma_0,\mathrm{SU}(2),n} &= 1, \\ Z_{S^1\times\Sigma_1,\mathrm{SU}(2),n} &= n+1, \\ Z_{S^1\times\Sigma_2,\mathrm{SU}(2),n} &= \frac{n^3}{6} + n^2 + \frac{11n}{6} + 1. \end{aligned}$$

It follows that $L^{\mathrm{np}}_{S^1\times\Sigma_g,\mathrm{SU}(2)}(z)$ is a rational function with denominator a power of $z-1$. For example, we have:

$$\begin{aligned} L^{\mathrm{np}}_{S^1\times\Sigma_0,\mathrm{SU}(2)}(z) &= \frac{1}{1-z}, \\ L^{\mathrm{np}}_{S^1\times\Sigma_1,\mathrm{SU}(2)}(z) &= \frac{1}{(1-z)^2}, \\ L^{\mathrm{np}}_{S^1\times\Sigma_2,\mathrm{SU}(2)}(z) &= \frac{1}{(1-z)^4}. \end{aligned}$$

Since $X_G(S^1 \times \Sigma_g)$ is connected for all g and G, it follows that $e\Lambda_{S^1\times\Sigma_g,G} = \{0,1\}$ confirming Conjecture 4.

7.4. **Conjecture 4 for 3_1.** Let us give some details about how the work of [Za1] and [CG1] verify Conjecture 4 for the simplest 3_1 knot. An independent verification of the Conjecture, valid for series of sum-product type, can be obtained by [ES].

Equation (36) of [Za1] and [CG1] imply that we can write:

$$Z_{3_1,\mathrm{SU}(2),n} = \zeta_{24}^{-n+3} n^{3/2} + 1 + \int_0^\infty e^{-np} G(p) dp,$$

where $\zeta_c = e^{2\pi i c}$ and $G(z)$ is a multivalued analytic function (analytic at $z=0$):

$$G(z) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{\chi(n) n}{(-z + n^2\pi^2/6)^{5/2}}, \tag{57}$$

where $\chi(\cdot)$ denotes the *unique primitive character of conductor* 12:

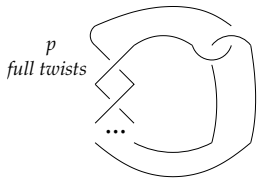
$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \bmod 12, \\ -1 & \text{if } n \equiv 5, 7 \bmod 12, \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

Together with Proposition 3.9, this implies that the singularities of $L^{\mathrm{np}}_{3_1,\mathrm{SU}(2)}(z)$ are $\{0, 1, e^{\pi i/12}\}$. Moreover, the local expansion of $L^{\mathrm{np}}_{3_1,\mathrm{SU}(2)}(z)$ around $z = e^{\pi i/12}$ is given by:

$$\begin{aligned} L^{\mathrm{np}}_{3_1,\mathrm{SU}(2)}(z) &= \sum_{n=1}^{\infty} \zeta_{24}^{-n+3} n^{3/2} z^n + h(z) \\ &= \zeta_8 \, \mathrm{Li}_{-3/2}(\zeta_{-24} z) + h(z), \end{aligned}$$

where $h(z)$ is a function holomorphic at $z=0$. Since $\mathrm{Li}_{-3/2}(z)$ is multivalued analytic at $\mathbb{C} \setminus \{0,1\}$ (see [CG3]), this confirms Conjecture 4 for 3_1. Using a Mittag-Leffler type decomposition for the fractional polylogarithm from [CG3, Eqn.13], we can also verify the Symmetry Conjecture for 3_1.

7.5. **Numerical evidence for the nearest singularity.** Conjecture 4 gives an exact formula for the singularity of $L^{\mathrm{p}}_{K,\mathrm{SU}(2)}(z)$ which is nearest to the origin. Notice that this singularity does not in general coincide with the critical value of $\mathrm{CS}_{\mathbb{C}}$ corresponding to the discrete faithful representation. This was indeed observed numerically for several *twist knots*. Let K_p denote the twist knot with negative clasp and p full twists, where $p \in \mathbb{Z}$:



In particular, we have:
(59)
$K_1 = 3_1$, $K_2 = 5_2$, $K_3 = 7_2$, $K_4 = 9_2$, $K_{-1} = 4_1$, $K_{-2} = 6_1$, $K_{-3} = 8_1$, $K_{-4} = 10_1$.

The invariant trace field of K_p is of type $[1, p-1]$ for $p > 1$ and $[0, |p|]$ for $p < 0$. It follows that $\Lambda_{K_p,\mathrm{SU}(2)}$ is a subset of a union of $2(p-1)$ (resp. $2|p|$) horizontal lines, symmetric with respect to the z-axis, and a superposition of 2 (resp. 1) copies of the z-axis for $p > 1$ (resp. $p < 0$). This set can be computed exactly and numerically using the methods of [GZ].

The corresponding element of the Habiro ring is given by:

$$f_{K_p,\mathrm{SU}(2)}(q) = \sum_{n=0}^{\infty} C_{K_p,n}(q)(q)_n(q^{-1})_n, \tag{60}$$

where $C_{K_p,n}(q) \in \mathbb{Z}[q^{\pm}]$ denote the n-cyclotomic polynomial of K_p. The latter may be computed inductively with respect to n for each fixed p, see [GS].

Using this formula, one can compute 500 coefficients of

$$L^{\mathrm{p}}_{K_p,\mathrm{SU}(2)}(z) = \mathcal{B}(f_{K_p,\mathrm{SU}(2)}(e^{1/x})),$$

and then numerically compute the singularity of the series $L^{\mathrm{p}}_{K_p,\mathrm{SU}(2)}(z)$ nearest to the origin. The numerical method used was the following:

- Fix a truncated power series
$$L(z) = \sum_{n=0}^{N} a_n z^n,$$
where N is a sufficiently large integer (eg $N = 500$).
- Using the root test, one can compute approximately the radius of convergence r_0 of the series $L(z)$
- Plot $|L(r\exp(2\pi i t)|$ for r near the inverse of the radius of convergence. The plot reveals a blow-up at certain values t_0 of t.
- This suggests singularities at $r_0 e^{2\pi i t_0}$, and an asymptotic expansion of a_n with a term of the form:
$$r_0^{-n} e^{-2\pi i t_0 n} n^{\alpha} \left(c_0 + c_1 \frac{1}{n} + \dots \right).$$
In general, we have a finite sum (over t_0) of terms of the above form.
- One can numerically compute the constants α and c_0 by fitting data. A fitting method (also used by Zagier in [Za1, p.953]) can improve the rate of convergence to $O(1/n^d)$ for any d. We used $d = 100$.

This was done for the twist knots of Equation (59). If $s(K_p)$ denote the inverse of the radius of convergence of the series $L^{\mathrm{p}}_{K_p,\mathrm{SU}(2)}(z)$ then we obtain numerically that:

$$s(K_1) = \frac{\pi^2}{6} = 1.644\ldots \qquad s(K_2) = 1.119\ldots \qquad s(K_3) = 0.882\ldots$$
$$s(K_4) = 0.745\ldots \qquad s(K_5) = 0.745\ldots$$

These numbers agree with the absolute value of $e^{\mathrm{CS}_{\mathbb{C}}(\rho)/(2\pi i)}$ for ρ some Galois conjugate of the discrete faithful representation. We thank N. Dunfield, S. Shumakovitch and C. Zickert for their help in the numerical computations.

Additional numerical evidence for K_1 and K_{-1} (and for many series of 1-dimensional sum-product type) was obtained by [ES].

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Appendix A. A formal relation among $L^{\mathrm{np}}(z)$ and $L^{\mathrm{p}}(z)$ for series of sum-product type

In this section we will give a formal proof of the relation among $L^{\mathrm{np}}(z)$ and $L^{\mathrm{p}}(z)$ for series of sum-product type. We will use the notation from Section 6.

Suppose that $F(x) < 1$ for all $x \in (0,1]$. Then, it is easy to see that $L^{\mathrm{np}}(z)$ is convergent inside the unit disk $|z| < 1$. The next theorem describes an explicit relation between $L^{\mathrm{np}}(z)$ and $L^{\mathrm{p}}(z)$.

Theorem 5. *We have*

$$L^{\mathrm{np}}(1+z) = \log(z) L^{\mathrm{p}}(\log(1+z)) + h(z), \tag{61}$$

where $h(z)$ is analytic at $z = 0$.

Proof. We will give only the formal calculation, leaving the analytic details to the reader. Below, $h(z)$ will denote a germ of an analytic function at $z = 0$. For $n \in \mathbb{N}$, let c_n denote the coefficient of $1/x^n$ in $\Sigma\Pi(x)$ given in (47). Let us fix $N \in \mathbb{N}$ and consider n large enough. Then, we have:

$$a_n = \sum_{k=1}^{N} \prod_{j=1}^{k} F\left(\frac{j}{n}\right) + O\left(\frac{1}{n^{N+1}}\right) = \sum_{k=1}^{N} \frac{c_k}{n^k} + O\left(\frac{1}{n^{N+1}}\right).$$

Thus,

$$L^{\mathrm{np}}(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{N} \frac{c_k}{n^k} + O\left(\frac{1}{n^{N+1}}\right) \right) z^n.$$

Ignore the $O(\cdot)$ terms, and interchange summation and integration. We obtain that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{N} \frac{c_k}{n^k} z^n = \sum_{k=1}^{N} c_k \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \sum_{k=1}^{N} c_k \mathrm{Li}_k(z),$$

where

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

is the k-polylogarithm. The latter is a multivalued analytic function on $\mathbb{C} \setminus \{0, 1\}$ with an asymptotic expansion at $z = 1$ of the form:

$$\mathrm{Li}_k(z) = \log(z-1) \frac{\log(z)^{k-1}}{(k-1)!} + h(z),$$

where $h(z)$ is an analytic function at $z = 0$; see for example, [Oe, Eqn.6]. Thus,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{N} \frac{c_k}{n^k} z^n \quad = \quad \log(z-1) \sum_{k=1}^{N} c_k \frac{\log^{k-1}(z)}{(k-1)!} + h(z).$$

Letting $N \to \infty$, and replacing z by $z+1$ it follows that:

$$\begin{aligned} L^{\mathrm{np}}(1+z) &= \log(z) \sum_{k=1}^{\infty} c_k \frac{\log^{k-1}(1+z)}{(k-1)!} + h(z) \\ &= \log(z) L^{\mathrm{p}}(\log(1+z)) + h(z). \end{aligned}$$

This concludes the formal calculation. □

Appendix B. A path integral formula for $L^{\mathrm{np}}(z)$

In this section we will give a path integral formula for $L^{\mathrm{np}}_{M,G}(z)$ using as input the famous Chern-Simons path integral studied in the seminal paper of Witten; see [Wi]. With the notation of [Wi] and with our normalization we have:

$$Z_{M,G,n} = \int_{\mathcal{A}} e^{\frac{n}{2\pi i} \mathrm{CS}(A)} \mathcal{D}A,$$

where $\mathcal{A}$ is the affine space $\mathcal{A}$ of G-connections on the trivial bundle $M \times G$ over M. Since CS takes values in $\mathbb{C}/\mathbb{Z}(2)$, we require that the level n (which plays the role of the inverse Planck's constant) be integer. Without loss of generality, we assume that $n \in \mathbb{N}$. Formally separating the $n = 0$ contribution in (12) and interchanging summation and integration in (12), it follows that

$$\begin{aligned} L^{\mathrm{np}}_{M,G}(z) &= 1+\int_{\mathcal{A}}\sum_{n=1}^{\infty} e^{\frac{n}{2\pi i}\mathrm{CS}(A)} z^n dA \\ &= 1+z\int_{\mathcal{A}} \frac{1}{e^{-\frac{n}{2\pi i}\mathrm{CS}(A)}-z}\mathcal{D}A. \end{aligned}$$

The above formula is an infinite dimensional analogue of a *Riemann-Hilbert problem*, and was obtained during a conversation with Kontsevich in the fall of 2006. For a detailed discussion on the Riemann-Hilbert problem, see [Df]. Finite dimensional analogues of the Riemann-Hilbert problem are discussed in [CG2].

References

[An] Y. André, Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité, *Ann. of Math.* **151** (2) (2000), 705–740.

[B-N] D. Bar-Natan, On the Vassiliev knot invariants, *Topology* **34** (1995), 423–472.

[Bo] E. Bombieri, On G-functions, in *Recent progress in analytic number theory*, Academic Press Vol. 2 (1981), 1–67.

[Br] E. Brieskorn, Die Monodromie der isolierten Singularitäten von Hyperflächen, *Manuscripta Math.* **2** (1970), 103–161.

[Co1] O. Costin, On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations, *Duke Math. J.* **93** (1998), 289–344.

[Co2] O. Costin, *Asymptotics and Borel summability*, Taylor and Francis, 2008.

[Co3] O. Costin, Global reconstruction of analytic functions from local expansions, *preprint 2007.*

[CG1] O. Costin and S. Garoufalidis, Resurgence of the Kontsevich-Zagier power series, *Annales de l' Institut Fourier*, *in press.*

[CG2] O. Costin and S. Garoufalidis, Resurgence of series of 1-dimensional sum-product type, *preprint 2007.*

[CG3] O. Costin and S. Garoufalidis, Resurgence of the fractional polylogarithms, *Mathematical Research Letters*, *in press.*

[Dl] E. Delabaere, Introduction to the Écalle theory, in *Computer algebra and differential equations*, London Math. Soc. Lecture Note Ser. **193** (1994), 59–101.

[De] P. Deligne, Équations différentielles à points singuliers réguliers, *Lecture Notes in Mathematics*, **163** Springer-Verlag, 1970.

[Df] P. A. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach, *Courant Lecture Notes in Mathematics* **3** 1999.

[DG] J. Dubois and S. Garoufalidis, The τ-polynomial of a knot, *preprint 2007.*

[DZ] J. Dupont and C. Zickert, A dilogarithmic formula for the Cheeger-Chern-Simons class, *Geom. Topol.* **10** (2006), 1347–1372.

[DGS] B. Dwork, G. Gerotto and F.J. Sullivan, An introduction to G-functions, *Annals of Mathematics Studies*, Princeton University Press **133** 1994.

[Ec1] J. Écalle, *Resurgent Functions*, Vol. I-III Mathematical Publications of Orsay **81-05** 1981, **81-06** 1981 and **85-05** 1985.

[ES] J. Écalle and S. Sharma, *in preparation.*

[Ga1] S. Garoufalidis, Applications of TQFT invariants to low dimensional topology, *Topology* **37** (1998), 219–224.

[Ga2] S. Garoufalidis, Periodicity of Goussarov-Vassiliev knot invariants, *Geom. and Topology Monographs* **4** (2002), 43–54.

[Ga3] S. Garoufalidis, An ansatz for the asymptotics of hypergeometric multisums, *Advances in Applied Mathematics* **41** (2008), 423–451.

[Ga4] S. Garoufalidis, q-terms, singularities and the extended Bloch group, *preprint 2007* `arXiv:0708.0018`.

[Ga5] S. Garoufalidis, G-functions and multisum versus holonomic sequences, *Advances in Math. in press.*
[GL1] S. Garoufalidis and T.T.Q. Le, The colored Jones function is q-holonomic, *Geom. and Topology* **9** (2005), 1253–1293.
[GL2] S. Garoufalidis and T.T.Q. Le, Gevrey asymptotics in quantum topology, *J. Reine Angew. Math.* **618** (2008), 169–195.
[GL3] S. Garoufalidis and T.T.Q. Le, Asymptotics of the colored Jones function of a knot, *preprint 2005*, math.GT/0508100.
[GL4] S. Garoufalidis and T.T.Q. Le, Is the Jones polynomial of a knot really a polynomial?, *Journal of Knot Theory and its Ramifications*, **15** (2006), 1–18.
[GS] S. Garoufalidis and X. Sun, The C-polynomial of a knot, *Alg. Geom. Topology* **6** (2006), 1001–1031.
[GZ] S. Goette and C. Zickert, The extended Bloch group and the Cheeger-Chern-Simons class, *Geom. Topol.* **11** (2007), 1623–1635.
[Gu] S. Gukov, Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial, *Comm. Math. Phys.* **255** (3) (2005), 577–627.
[GM] S. Gukov and H. Murakami, $SL(2,\mathbb{C})$ Chern-Simons theory and the asymptotic behavior of the colored Jones polynomial, *preprint 2006*, math/0608324.
[GZa] S. Gukov and D. Zagier, *to appear.*
[Ha1] K. Habiro, On the quantum $\mathfrak{sl}_2$ invariants of knots and integral homology spheres, *Geom. Topol. Monogr.* **4** (2002), 55–68.
[Ha2] K. Habiro, Cyclotomic completions of polynomial rings, *Publ. Res. Inst. Math. Sci.* **40** (2004), 1127–1146.
[HL] V. Huynh and T.T.Q. Le, On the Colored Jones Polynomial and the Kashaev invariant, *Fundam. Prikl. Mat.* **11** (2005), 57–78.
[Ju] R. Jungen, Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence, *Comment. Math. Helv.* **3** (1931), 266–306.
[Ka] R. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm, *Modern Phys. Lett. A* **39** (1997), 269–275.
[Kz] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, *IHES Publ. Math.* **39** (1970), 175–232.
[KT] G. Kuperberg and D. P. Thurston, Finite-type 3-manifold invariants via cut-and-paste topology, *preprint 1999* `math.GT/9912167`.
[Le] L. Lewin, The inner structure of the dilogarithm in algebraic fields, *J. Number Theory* **19** (1984), 345–373.
[Ml] B. Malgrange, Introduction aux travaux de J. Écalle, *Enseign. Math.* **31** (1985), 261–282.
[Ko] M. Kontsevich, Problems proposed to the CTQM, Aarhus University, 2006.
[KZ] M. Kontsevich and D. Zagier, *Periods*, in Mathematics unlimited—2001 and beyond, Springer, Berlin 2001, 771–808.
[LMO] T.T.Q. Le, J. Murakami and T. Ohtsuki, A universal quantum invariant of 3-manifolds, *Topology* **37** (1998), 539–574.
[MR] C. Maclachlan and A.W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics **219** Springer-Verlag, 2003.
[MM] H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, *Acta Math.* **186** (2001), 85–104.
[NZ] W. D. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, *Topology* **24** (1985), 307–332.
[Ne] W. D. Neumann and D. Zagier, Extended Bloch group and the Cheeger-Chern-Simons class, *Geom. Topol.* **8** (2004), 413–474.
[Ni1] N. Nilsson, Some growth and ramification properties of certain integrals on algebraic manifolds, *Ark. Mat.* **5** (1965), 463–476.
[Ni2] N. Nilsson, Monodromy and asymptotic properties of certain multiple integrals, *Ark. Mat.* **18** (1980), 181–198.
[Oe] J. Oesterlé, Polylogarithmes, Séminaire Bourbaki, Vol. 1992/93. Astérisque No. 216 (1993), Exp. No. 762 49–67.

[RT] N. Reshetikhin, V. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Commun. Math. Phys.* **127** (1990), 1–26.
[Sa] D. Sauzin, Resurgent functions and splitting problems, *preprint 2006*.
[Sn] Snap, software by O. Goodman, available at: http://sourceforge.net/projects/snap-pari/
[Sz] A. Szenes, The combinatorics of the Verlinde formulas, in *Vector bundles in algebraic geometry*, London Math. Soc. Lecture Note Ser. **208** (1995), 241–253.
[Th] W. Thurston, *The geometry and topology of 3-manifolds*, 1979 notes, available from MSRI.
[To] B. Totaro, Euler and algebraic geometry, *Bulletin AMS,* **44** (2007), 541–559.
[Tu1] V. Turaev, The Yang-Baxter equation and invariants of links, *Invent. Math.* **92** (1988), 527–553.
[Tu2] V. Turaev, *Quantum Invariants of Knots and 3-manifolds*, de Gruyter Studies in Mathematics **18**, Walter de Gruyter, Berlin New York, 1994.
[Vo] A. Voros, The return of the quartic oscillator: the complex WKB method, *Ann. Inst. H. Poincaré Sect. A* **39** (1983), 211–338.
[Wi] E.Witten, Quantum field theory and the Jones polynomial, *Commun. Math. Physics.* **121** (1989), 360–376.
[WZ] H. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities, *Inventiones Math.* **108** (1992), 575–633.
[Za1] D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, *Topology* **40** (2001), 945–960.
[Za2] D. Zagier, The dilogarithm function, in *Frontiers in number theory, physics, and geometry*, II Springer (2007), 3–65.
[Ze] D. Zeilberger, A holonomic systems approach to special functions identities, *J. Comput. Appl. Math.* **32** (1990), 321–368.
[Zi] C. Zickert, The Chern-Simons invariant of a representation, *preprint 2007*, arXiv:0710.2049.

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