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Flag algebras and the stable coefficients of the Jones polynomial

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ABSTRACT

We study the structure of the stable coefficients of the Jones polynomial of an alternating link. We start by identifying the first four stable coefficients with polynomial invariants of a (reduced) Tait graph of the link projection. This leads us to introduce a free polynomial algebra of invariants of graphs whose elements give invariants of alternating links which strictly refine the first four stable coefficients. We conjecture that all stable coefficients are elements of this algebra, and give experimental evidence for the fifth and the sixth stable coefficient. We illustrate our results in tables of all alternating links with at most 10 crossings and all irreducible planar graphs with at most 6 vertices.

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1. Introduction

1.1. The stable coefficients of the Jones polynomial

The paper identifies a quantum knot invariant (the third stable coefficient of the Jones polynomial of an alternating link) with a polynomial of induced graphs countings of a plane graph (a Tait graph of the alternating link). Our input is a q-hypergeometric series $\Phi_G(q) \in \mathbb{Z}[[q]]$ that is associated to a

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plane rooted graph. $\Phi_G(q)$ encodes the stable coefficients of the Jones polynomial of the corresponding alternating link. A combinatorial analysis of the coefficient of q^3 of $\Phi_G(q)$ is the focus of our paper; see Theorem 1.3.

Perhaps more interesting than the explicit formula given in Eq. (5) is the fact that it is a polynomial in induced graph countings of G. This new phenomenon, not observed in the previously known coefficients of q^k of $\Phi_G(q)$ for $k = 0, 1, 2$. This discovery leads on the one hand to atomic knot invariants (discussed after Conjecture 1.5), and on the other hand to the algebra of graph induced countings, an interesting object on its own right.

The aim of our paper is to study this unexpected discovery between the algebra of graph induced countings and the stable coefficients of the Jones polynomial.

Although the results of our paper concern quantum knot invariants, they require no prior knowledge of knot theory nor familiarity with the colored Jones polynomial of a knot or link. As a result, we will not recall the definition of the *Jones polynomial* $J_L(q) \in \mathbb{Z}[q^{\pm 1/2}]$ of a knot or link L in 3-space, which may be found in several texts [8,17,18,9]. A stronger invariant is the *colored Jones polynomial* $J_{L,n}(q) \in \mathbb{Z}[q^{\pm 1/2}]$, where $n \in \mathbb{N}$, which essentially encodes the Jones polynomial of a link and its parallels [10, Corollary 2.15]. When L is an *alternating link*, (i.e., a link with an alternating planar projection [9]) the coefficients of the (shifted) colored Jones polynomial $\hat{J}_{L,n}(q) \in 1 + q\mathbb{Z}[q]$ stabilize, in the following sense: for every $k \in \mathbb{N}$, the coefficient of q^k in $\hat{J}_{L,n}(q) \in \mathbb{Z}[q]$ is independent of n for $n > k$. Those stable coefficients assemble to a formal power series $\Phi_L(q) \in \mathbb{Z}[\![q]\!]$, where $\mathbb{Z}[\![q]\!]$ denotes the ring of formal power series in a variable q with integer coefficients.

The existence of $\Phi_L(q)$ was given in [5,1] and a presentation as a q-hypergeometric series (of Nahm type) which depends only on a plane graph (a Tait graph of L) was given in [5]. For a rooted plane graph G, $\Phi_G(q)$ is given by a q-hypergeometric sum of the form

$$\Phi_G(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod_{(p,v)} (q)_{a_p+b_v}} \tag{1}$$

where the sum is over the set of all admissible states (a, b) of G, (i.e., admissible colorings of the faces and the vertices of G by integers) and the product is over the set of all corners (p, v) of G. Here, $(q)_m = (1-q)(1-q^2)\dots(1-q^m)$ for a natural number m and $(q)_\infty = (1-q)(1-q^2)(1-q^3)\dots$. For a detailed explanation of the notation and terminology, see Section 3.1.

We will denote by $\phi_{G,k}$ (resp., $\phi_{L,k}$) the coefficient of q^k in $\Phi_G(q)$ (resp., $\Phi_L(q)$), and we will often call it the kth stable coefficient of G (resp., L).

In [3] the first three stable coefficients $\phi_k : G \mapsto \phi_{G,k}$ for $k = 0, 1, 2$ were expressed in terms of the number of vertices, edges and 3-cycles of G. The proof used properties of the Kauffman bracket skein module. An independent proof was given in [6]. To express the answer, and to motivate the polynomial algebra $\mathcal{P}$ introduced below, consider the elements $c_1, c_2, c_3 \in \mathcal{P}$ given by

$$(c_1, c_2, c_3) = ([\![\bullet]\!], [\![\bullet\!\!-\!\!\!-\!\!\bullet]\!], [\![\triangle]\!]). \tag{2}$$

c_1, c_2, c_3 count the number of vertices, edges and triangles in a graph G. Then, we have [3]

$$(\phi_0, \phi_1, \phi_2) = \left(1, c_1 - c_2 - 1, \frac{1}{2}\left((c_1 - c_2)^2 - 2c_3 - c_1 + c_2\right)\right). \tag{3}$$

It is natural to ask for a formula for the next coefficient ϕ_3. The answer is given in Theorem 1.3. What is more, Theorem 1.3

(a) motivates us to introduce the algebra $\mathcal{P}$ of polynomial invariants of graphs, in the spirit of flag algebras of [14]. $\mathcal{P}$ turns out to be a free polynomial algebra, see Theorem 1.2.
(b) shows that ϕ_3 is determined by ϕ_k for $k \leq 2$ and $-c_{41} + 2c_{42}$. The latter is an integer linear combination of the refined alternating link invariants c_{41}, c_{42}; see Proposition 2.2
(c) motivates us to write $\Phi(q)$ as an infinite product and conjecture that its exponents are linear forms on the set of irreducible planar graphs, see Conjecture 1.7 and its explicit form, Conjecture 1.5. The latter is verified by explicit computation for all alternating links with at most 10 crossings and all irreducible graphs with at most 7 vertices.

(d) raises the question of how Rozansky's categorification [16] $\varPhi_L(t,q)$ of $\varPhi_L(q) = \varPhi_L(-1,q)$ can further refine Conjecture 1.7. Since this categorification is not yet effectively computable, we cannot make this question more precise.

1.2. *An algebra $\mathcal{P}$ of polynomial invariants of graphs*

Let $\mathcal{G}$ denote the set of simple finite graphs, i.e., non-embedded graphs with no loops and no multiple edges, and unlabeled vertices and edges. For H and G in $\mathcal{G}$, an *embedding* $f : H \to G$ is an injection $f : V(H) \hookrightarrow V(G)$ (where $V(G)$ denotes the set of vertices of G) such that for every $v, v' \in V(H)$ (v, v') is an edge of H if and only if $(f(v), f(v'))$ is an edge of G. Let $i(H,G)$ denote the number of embeddings of H in G, divided by the number of automorphisms of H. Varying G, we get a function $[H] : \mathcal{G} \to \mathbb{N}$ given by $G \in \mathcal{G} \mapsto [H](G) = i(H,G)$. The *degree* of $[H]$ is the number of vertices of H. Let $[\mathcal{G}]$ denote the set $\{[H] \mid H \in \mathcal{G}\}$. Likewise we define $[\mathcal{G}^c]$ where $\mathcal{G}^c$ is the set of connected graphs. $\mathcal{P}$ denotes the $\mathbb{Q}$-vector space on the set $[\mathcal{G}]$.

Proposition 1.1. (a) *$\mathcal{P}$ is a commutative algebra. In fact,*

$$[H_1][H_2] = \sum_H c_H[H] \tag{4}$$

where H is a graph on at most $|V(H_1)| + |V(H_2)|$ vertices and c_H is the number of ordered pairs of induced subgraphs (F_1, F_2) of H (possibly sharing some vertices) such that F_i is isomorphic to H_i for $i = 1, 2$ and moreover $V(F_1) \cup V(F_2) = V(H)$.

(b) *It follows that $\mathcal{P}$ is a quotient of the polynomial algebra on $[\mathcal{G}^c]$.*

Eq. (4) shows that the structure constants of the multiplication in $\mathcal{P}$ are natural numbers. For instance we have:

$$\frac{1}{2}([\bullet]^2 - [\bullet]) = [\bullet\!\!-\!\!\!-\!\!\bullet] + [\bullet\ \bullet].$$

This holds since both sides of the above equation evaluated on $G \in \mathcal{G}$ equal to the number of pairs of vertices of G and such a pair is either connected by an edge or not. More generally, if H is a graph on k vertices then

$$[H][\bullet] = k[H] + \sum c_F[F]$$

where the sum is over all graphs F on $k+1$ vertices and c_F is equal to the number of induced subgraphs of F isomorphic to H.

Theorem 1.2. *$\mathcal{P}$ is a free polynomial algebra on the set $[\mathcal{G}^c]$.*

Real valued functions on $\mathcal{G}$ are also called *graph parameters* and linear combinations of graphs are also called *quantum graphs* in the context of graph theory. The algebra $\mathcal{P}$ is reminiscent to the *flag algebras* of graph theory [14].

Since alternating links involve planar graphs only, let $\mathcal{G}^{\mathrm{pl}}$ denote the set of simple planar graphs. For $H \in \mathcal{G}^{\mathrm{pl}}$, we denote by $[\![H]\!]$ the restriction of the function $[H] : \mathcal{G} \to \mathbb{N}$ to $\mathcal{G}^{\mathrm{pl}} \subset \mathcal{G}$, and $\mathcal{P}^{\mathrm{pl}}$ the vector space generated by $[\![H]\!]$ for $H \in \mathcal{G}^{\mathrm{pl}}$. $\mathcal{P}^{\mathrm{pl}}$ is also an algebra. The structure of the algebra $\mathcal{P}^{\mathrm{pl}}$ is an interesting and challenging problem.

1.3. *A formula for ϕ_3*

Let $c_{4,i} = [\![Gv_i^4]\!]$ for $i = 1, 2$ where Gv_i^4 are shown in Fig. 1.

Theorem 1.3. *We have:*

$$\phi_3 = c_{41} - 2c_{42} + \frac{c_2}{6} + c_3c_2 - \frac{c_2^3}{6} - \frac{c_1}{6} - c_3c_1 + \frac{c_2^2c_1}{2} - \frac{c_2c_1^2}{2} + \frac{c_1^3}{6}. \tag{5}$$

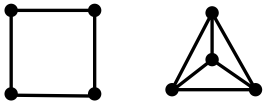


Fig. 1. The irreducible planar graphs Gv_1^4 (left) and Gv_2^4 (right) with 4 vertices.

Eqs. (3) *and* (5) *are equivalent to*

$$\Phi(q) = (1-q)^{1-c_1+c_2}(1-q^2)^{c_3}(1-q^3)^{c_3-c_{41}+2c_{42}} + O(q^4). \tag{6}$$

1.4. A conjecture for ϕ_4, ϕ_5 and ϕ_k

A comparison of Eqs. (5) and (6) suggests us to write $\Phi_G(q)$ as an infinite product

$$\Phi(q) = (1-q)^{1-c_1+c_2}\prod_{k=2}^{\infty}(1-q^k)^{C_k} \tag{7}$$

where $C_k(G) \in \mathbb{Z}$ for all k. This is possible by the following lemma.

Lemma 1.4. *For every sequence of integers* (a_n) *there exists a sequence of integers* (b_n) *such that*

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty}(1-q^n)^{b_n}. \tag{8}$$

Proof. Define b_n inductively by

$$\operatorname{coeff}\left(\left(1+\sum_{m=1}^{\infty} a_m q^m\right)\prod_{k=1}^{n-1}(1-q^k)^{-b_k}, q^n\right) = -b_n.$$

Given b_n as above, by induction on n it follows that for all $n > 0$ we have

$$\left(1+\sum_{m=1}^{\infty} a_m q^m\right)\prod_{k=1}^{n-1}(1-q^k)^{-b_k} \in 1 + q^n\mathbb{Z}[\![q]\!].$$

Letting n go to infinity, it follows that the right hand side of the above equation is 1, hence Eq. (8) follows. □

Theorem 1.3 gives an expression for C_k for $k = 2, 3$. To phrase our conjecture for C_k for $k = 4, 5$, recall the notion of an irreducible planar graph from [6]. The latter is a planar graph which is not a vertex connected sum or an edge connected sum of planar graphs as in Fig. 2. The table of irreducible planar graphs with at most 10 edges is given in Figs. 12–16, and with at most 6 vertices is given in Figs. 1, 10 and 11.

Conjecture 1.5. *We conjecture that*

$$C_4 = c_3 - c_{41} + 5c_{42} + c_{51} - c_{52} - 2c_{53} - 3c_{54} \tag{9}$$

$$\begin{aligned} C_5 = {} & c_3 - c_{41} + 12c_{42} + c_{51} - 4c_{53} - 9c_{54} - c_{61} + c_{62} - 2c_{63} - c_{64} + 2c_{65} + 3c_{66} \\ & + 4c_{68} - 4c_{69} + 2c_{610} + c_{611} - 3c_{612} + 4c_{613} + c_{614} - 5c_{616} - 16c_{618} + c_{619} \end{aligned} \tag{10}$$

where $c_{j,i} = [\![Gv_i^j]\!]$ *and* Gv_i^5 *and* Gv_i^6 *are irreducible planar graphs with* 5 *and* 6 *vertices shown in* Figs. 10 *and* 11.

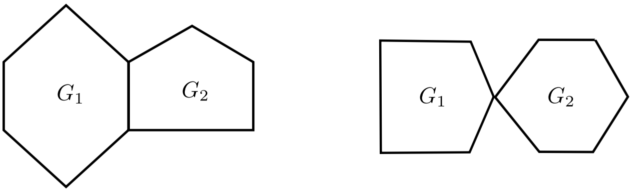


Fig. 2. A vertex connected sum (on the left) and an edge-connected sum on the right.

Independently of the above conjecture, each term that appears in the right hand side of Eqs. (9)–(10) is an alternating link invariant; see Proposition 2.2.

The expression for C_4 and C_5 is the unique linear combination of irreducible planar graphs with 5 and 6 vertices (and this is how it was found) which fits the stable coefficients of the Jones polynomial of all alternating links with at most 10 crossings and all alternating links whose reduced Tait graph has at most 6 vertices. For details, see Appendix A.

The reader may observe that the graph Gv_5^5 is missing from C_4. This motivates the following question.

Question 1.6. Is it true that

$$(\Phi_{G_1^6})^2 = \Phi_{G_1^9}\Phi_{G_0^3}.$$

A direct computation confirms this up to $O(q^{31})$.

Conjecture 1.7. *For all $k \geq 2$, C_k are linear forms with integer coefficients on the set of irreducible planar graphs with at most $k + 1$ vertices.*

The above conjecture has an equivalent formulation.

Conjecture 1.8. *Φ is multiplicative under vertex and edge connected sum, and for every connected irreducible planar graph H there exist $\Psi_H(q) \in 1 + q^{\deg(H)-1}\mathbb{Z}[[q]]$ such that*

$$\Phi(q) = (1-q)^{1-c_1+c_2} \prod_H \Psi_H(q)^{[[H]]} \tag{11}$$

where the product is taken over the set of irreducible planar graphs.

2. The algebra $\mathcal{P}$

2.1. Proof of Proposition 1.1

A *subgraph of a graph G induced by $S \subseteq V(G)$* is a graph $G[S]$ such that $V(G[S]) = S$ and two vertices in S are joined by an edge in $G[S]$ if and only if they are joined by an edge in G. The value $i(H, S)$ can be equivalently defined as the number of sets $S \subseteq V(G)$ such that $G[S]$ is isomorphic to H.

To show that (12) holds we need to show that

$$i(H_1, G)i(H_2, G) = \sum_H c_H i(H, G) \tag{12}$$

for every graph G. Note that $i(H_1, G)i(H_2, G)$ equals the number of pairs (S_1, S_2) of subsets of $V(G)$ such that $G[S_i]$ is isomorphic to H_i for $i = 1, 2$. We claim that for a fixed graph H the number of pairs as above, such that $G[S_1 \cup S_2]$ is isomorphic to H, is equal to $c_H i(H, G)$. The Eq. (12) immediately follows from this claim. The claim holds as the number of sets $S \subseteq V(G)$ such that $G[S]$ is isomorphic to H is equal to $i(H, G)$. Further, for given $S \subseteq V(G)$ the number of pairs (S_1, S_2) defined above with $S = S_1 \cup S_2$ equals c_H, by definition. □

2.2. Proof of Theorem 1.2

The proof of the theorem is derived from the results of [4]. We start by introducing the additional notation, which will allow us to state the necessary results. Let

$$\gamma(H,G)=i(H,G)/\binom{|V(G)|}{|V(H)|}.$$

Let k be a fixed integer and let $H_1, H_2, \ldots, H_m$ be all connected graphs with $|V(H_i)| \leq k$. Given a graph G define a vector

$$\gamma(k,G)=(\gamma(H_1,G),\gamma(H_2,G),\ldots,\gamma(H_m,G)).$$

Let S_k be defined as the set of all vectors $\mathbf{v} \in \mathbb{R}^m$ such that there exists an infinite sequence of graphs $G_1, G_2, \ldots, G_n, \ldots$, such that $|V(G_n)| \to \infty$ and $\gamma(k,G) \to \mathbf{v}$. The following lemma follows immediately from [4, Theorems 1 and 3].

Lemma 2.1. *Let k be a positive integer, let m be the number of connected graphs on at most k vertices and let $S_k \subseteq \mathbb{R}^m$ be as defined above. Then S_k contains an m-dimensional ball of positive radius.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let k be a positive integer and let $H_1, H_2, \ldots, H_m$ be all connected graphs on at most k vertices, as before. It suffices to show that for every $p \in \mathbb{R}[x_1, x_2, \ldots, x_m]$, $p \not\equiv 0$ (i.e., p not identically zero), we have $p([H_1],[H_2],\ldots,[H_m]) \not\equiv 0$. Suppose for a contradiction that for some polynomial $p_0 \in \mathbb{R}[x_1, x_2, \ldots, x_m]$, $p_0 \not\equiv 0$ we have

$$p_0(i(H_1,G),i(H_2,G),\ldots,i(H_m,G))=0$$

for every graph G. As $i(H,G)=\binom{|V(G)|}{|V(H)|}\gamma_H(G)$, there exists an $(m+1)$-variable polynomial $p_1 \in \mathbb{R}[x_1, x_2, \ldots, x_m, y]$, $p_1 \not\equiv 0$ such that for each graph G we have

$$\begin{aligned}&p_0(i(H_1,G),i(H_2,G),\ldots,i(H_m,G))\\&\quad=p_1(\gamma(H_1,G),\gamma(H_2,G),\ldots,\gamma(H_m,G),|V(G)|)(=p_1(\gamma(k,G),|V(G)|)).\end{aligned}$$

Let

$$p_1(x_1,x_2,\ldots,x_m,y)=\sum_{i=1}^{t} r_i(x_1,\ldots,x_m)y^i.$$

Suppose without loss of generality that r_t is not identically zero. We claim that r_t is identically zero on S_k, in contradiction with Lemma 2.1.

To prove the claim, consider $\mathbf{v} \in S_k$ and let $G_1, G_2, \ldots, G_n, \ldots$ be a sequence of graphs such that $|V(G_n)| \to \infty$ and $\gamma(k,G_n) \to \mathbf{v}$, as in the definition of S_k. Let $f(G_n)=p_1(\gamma(k,G_n),|V(G_n)|)/|V(G_n)|^t$. Clearly, $\lim_{n\to\infty} f(G_n)=r_t(\mathbf{v})$. On the other hand, $f(G_n)=0$ for every n by the choice of p_1. It follows that $r_t(\mathbf{v})=0$, as desired. This establishes the claim and the theorem. □

2.3. A subalgebra $\mathcal{P}^{\mathrm{fl}}$ of $\mathcal{P}$

In this section we introduce a subalgebra $\mathcal{P}^{\mathrm{fl}}$ of $\mathcal{P}$ which is motivated by knot theory. Consider a *flype move* on a graph shown in Fig. 3.

The importance of the flype move is Tait's Conjecture proven by Menasco–Thisthlethwaite [13]: every two reduced S^2 projections of an alternating link are connected by a sequence of flype moves. Closely related to a flype move is a *Whitney flip* move [19], illustrated in Fig. 4.

In [5] it was shown that

- a Whitney flip on a planar graph corresponds to a Conway mutation for the corresponding alternating links.
- A flype move can be obtained by two Whitney flip moves.

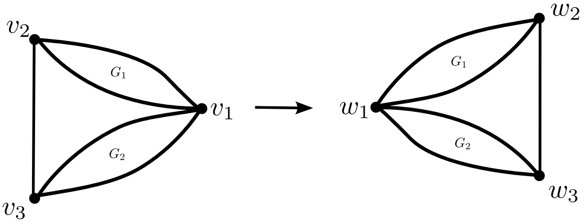


Fig. 3. A flype move on a planar graph.

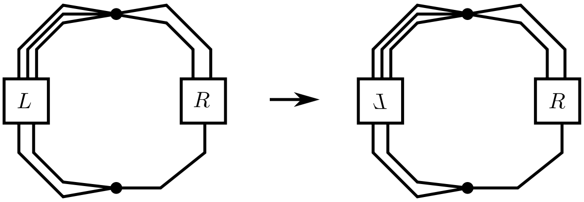


Fig. 4. A Whitney flip on a graph.

Menasco [12] shows that there are two types of Conway mutation, type I (visible in an alternating link projection) and type II (hidden from the link projection). It was pointed out to us by F. Bonahon and J. Greene that a type II mutation can be achieved by two type I mutations. Independent of this fact, in [7, Theorem 1.1] Greene proves that the Tait graph gives a 1–1 correspondence between the set of alternating links, modulo Conway mutation and the set of planar graphs modulo flips. A Conway mutation does not change the colored Jones polynomial, hence $\Phi_G(q)$ does not change under Whitney flips on G.

Let $\mathcal{G}^{\mathrm{fl}}$ denote the set of equivalence classes on $\mathcal{G}$ induced by the Whitney flip equivalence relation. Let $\mathcal{P}^{\mathrm{fl}}$ denote the subalgebra of $\mathcal{P}$ that consists of all polynomials $P : \mathcal{G} \to \mathbb{Q}$ (where $P \in \mathcal{P}$) that satisfy $P(G) = P(G')$ whenever G and G' are related by a Whitney flip.

The above discussion gives rise to a map

$$\mathcal{P}^{\mathrm{fl}} \times \{\text{Alternating links}\}/\{\text{Conway mutation}\} \longrightarrow \mathbb{Q}. \tag{13}$$

Proposition 2.2. (a) *If H is 2-edge-connected and isomorphic to every one of its Whitney flips, then* $[H] \in \mathcal{P}^{\mathrm{fl}}$.

(b) *In particular, $c_{41}, c_{42}, c_n \in \mathcal{P}^{\mathrm{fl}}$ where c_n is the n-cycle and $c_{5,i} \in \mathcal{P}^{\mathrm{fl}}$ for $i = 1, \dots, 5$ and $c_{6,i} \in \mathcal{P}^{\mathrm{fl}}$ for $i = 1, \dots, 19$.*

It follows that each term in the right hand side of Eqs. (9)–(10) is an alternating link invariant.

Proof. For part (a), fix H 2-edge connected and isomorphic to every one of its Whitney flips, and let $G = A \cup B$ be a graph and $G' = A \cup B'$ a Whitney flip of G. If $\phi : H \to G'$ is an embedding, then we can write $H = H_A \cup H_B$ with embeddings $H_A \to A$ and $H_B \to B$. If H_A or H_B is empty, then we can construct an embedding $\psi : H \to G$. Otherwise, we can construct an embedding $\psi : H' \to G$ where H' is a Whitney flip of H. Since H' is isomorphic to H, this gives an embedding $\psi : H \to G$. It is easy to see that the map $\phi \mapsto \psi$ is a bijection, hence the number of embeddings of H in G is equal to the number of embeddings of H in G'.

For part (b), since Whitney flips preserve the number of vertices, by Proposition 2.2 it suffices to show that no two of the graphs Gv_i^5 (and similarly Gv_j^6) differ by Whitney flips. In [5, Section 13.2] it was shown that if two planar graphs differ by Whitney flips, the corresponding alternating links are Conway mutant, and hence they have equal colored Jones polynomial, hence equal $\Phi(q)$ invariant. Inspection shows that the 5 irreducible graphs with 5 vertices shown in Fig. 10 and the 19 irreducible graphs with 6 vertices shown in Fig. 11 all have different Jones polynomial. Therefore, no two graphs are flip equivalent. □

Let

$$(\gamma,\delta)=([\overset{\bullet}{\bullet\;\;\bullet}],[\bullet\!-\!\bullet\!-\!\bullet]).$$

Lemma 2.3. (a) $\gamma-\delta=\tfrac{1}{6}[\bullet]^3+2[\triangle]-[\bullet\!-\!\bullet][\bullet]+2[\bullet\!-\!\bullet]-\tfrac{1}{2}[\bullet]^2+\tfrac{1}{3}[\bullet]$.
(b) $\gamma-\delta\in\mathcal{P}^{fl}$ *is an invariant of alternating links, polynomially determined by* c_1, c_2, c_3.

Proof. (a) By the multiplication formula (4) we have

$$[\bullet][\bullet]=2[\bullet\;\bullet]+2[\bullet\!-\!\bullet]+[\bullet] \tag{14}$$

$$[\bullet\;\bullet][\bullet]=2[\bullet\;\bullet]+[\bullet\!-\!\bullet\!-\!\bullet]+2[\bullet\!-\!\bullet\;\;\bullet]+3[\overset{\bullet}{\bullet\;\;\bullet}] \tag{15}$$

$$[\bullet\!-\!\bullet][\bullet]=2[\bullet\!-\!\bullet]+3[\triangle]+2[\bullet\!-\!\bullet\!-\!\bullet]+[\bullet\!-\!\bullet\;\;\bullet]. \tag{16}$$

It follows that

$$\begin{aligned}[\bullet]^3 &= 2[\bullet\;\bullet][\bullet]+2[\bullet\!-\!\bullet][\bullet]+[\bullet][\bullet]\\ &= 2\left(2[\bullet\;\bullet]+[\bullet\!-\!\bullet\!-\!\bullet]+2[\bullet\!-\!\bullet\;\;\bullet]+3[\overset{\bullet}{\bullet\;\;\bullet}]\right)\\ &\quad +2\left(2[\bullet\!-\!\bullet]+3[\triangle]+2[\bullet\!-\!\bullet\!-\!\bullet]+[\bullet\!-\!\bullet\;\;\bullet]\right)+2[\bullet\;\bullet]+2[\bullet\!-\!\bullet]+[\bullet]\\ &= 6[\triangle]+6[\bullet\!-\!\bullet\;\;\bullet]+6[\overset{\bullet}{\bullet\;\;\bullet}]+6[\bullet\!-\!\bullet\!-\!\bullet]+6[\bullet\;\bullet]+6[\bullet\!-\!\bullet]+[\bullet].\end{aligned}$$

Therefore

$$6[\overset{\bullet}{\bullet\;\;\bullet}]=[\bullet]^3-6[\triangle]-6[\bullet\!-\!\bullet\;\;\bullet]-6[\bullet\!-\!\bullet\!-\!\bullet]-6[\bullet\;\bullet]-6[\bullet\!-\!\bullet]-[\bullet]. \tag{17}$$

On the other hand, from Eq. (14) we have

$$[\bullet\;\bullet]=\frac{1}{2}[\bullet]^2-[\bullet\!-\!\bullet]-\frac{1}{2}[\bullet] \tag{18}$$

and from Eq. (16)

$$[\bullet\!-\!\bullet\;\;\bullet]=[\bullet\!-\!\bullet][\bullet]-2[\bullet\!-\!\bullet]-3[\triangle]-2[\bullet\!-\!\bullet\!-\!\bullet]. \tag{19}$$

Eqs. (17)–(19) give

$$\begin{aligned}6[\overset{\bullet}{\bullet\;\;\bullet}] &= [\bullet]^3-6[\triangle]-6([\bullet\!-\!\bullet][\bullet]-2[\bullet\!-\!\bullet]-3[\triangle]-2[\bullet\!-\!\bullet\!-\!\bullet])-6[\bullet\!-\!\bullet\!-\!\bullet]\\ &\qquad -6\left(\frac{1}{2}[\bullet]^2-[\bullet\!-\!\bullet]-\frac{1}{2}[\bullet]\right)-6[\bullet\!-\!\bullet]-[\bullet]\\ &\quad = [\bullet]^3+12[\triangle]-6[\bullet\!-\!\bullet][\bullet]+12[\bullet\!-\!\bullet]-3[\bullet]^2+2[\bullet]+6[\bullet\!-\!\bullet\!-\!\bullet].\end{aligned}$$

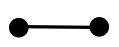
So

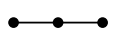
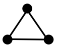
$$[\overset{\bullet}{\bullet\;\;\bullet}]-[\bullet\!-\!\bullet\!-\!\bullet]=\frac{1}{6}[\bullet]^3+2[\triangle]-[\bullet\!-\!\bullet][\bullet]+2[\bullet\!-\!\bullet]-\frac{1}{2}[\bullet]^2+\frac{1}{3}[\bullet].$$

(b) This follows from (a) and Proposition 2.2. □

Define the *k*th moment of a graph to be the sum of the *k*-powers of the degrees (i.e., valencies) of the vertices of *G*. The next lemma shows that the second moment is a polynomial of an induced graph counting problem. This holds for all moments, though we do not need this more general statement.

Lemma 2.4. *We have:*

$\displaystyle\sum_v \deg(v) = 2[$  $]$ (20)

$\displaystyle\sum_v \binom{\deg(v)}{2} = [$  $] + 3[$  $].$ (21)

Proof. The equalities follow by a simple counting argument. For the first equation every edge has two vertices. For the second equation, given a vertex v of G and an unordered pair of two distinct neighboring vertices w, w' of v, either ww' is an edge of G (hence vww' is an induced triangle and contributes three times on the second moment) or not (and contributes δ to the second moment). □

3. A review of the q-series $\Phi_G(q)$

3.1. *The q-series $\Phi_G(q)$*

In this section we will review the definition of the q-series $\Phi_G(q)$ of [5] following our earlier work [6]. Fix a rooted plane multigraph G, i.e., a planar multigraph (possibly with loops and multiple edges) together with a drawing on the plane together with a vertex v_∞ of its unbounded face p_∞. A *corner* (p, v) of G is a face p of G and a vertex v of p. An *admissible state* (a, b) of G is an integer assignment a_p for each face p and b_v for each vertex v of p such that

- $a_p + b_v \geq 0$ for all corners (p, v) of G.
- For the unbounded face p_∞ we have $a_\infty = 0$.
- For the vertex v_∞ of G we have $b_{v_\infty} = 0$.

In the formulas below, v, w will denote vertices of G and p a face of G. We also write $vw \in p$ if v, w are vertices and vw is an edge of p.

For an admissible state (a, b) and a face p of G with $l(p)$ edges, we define

$$\gamma(p) = l(p)a_p^2 + 2a_p(b_1 + b_2 + \cdots + b_{l(p)})$$

where $b_1, \ldots, b_{l(p)}$ are the values of the state on the vertices of p in counterclockwise order. Let

$$A(a, b) = \left(\sum_p \gamma(p)\right) + 2\left(\sum_{e=(v_iv_j)} b_{v_i}b_{v_j}\right) \tag{22}$$

where the first sum is over the set of all faces p of G (including the unbounded one) the second sum is over the set of edges of G. Let

$$B(a, b) = 2\sum_v b_v + \sum_p (l(p) - 2)a_p \tag{23}$$

where the v-summation is over the set of vertices of G and the p-summation is over the set of all faces of G. This explains the notation of Eq. (1).

For an admissible state (a, b) and a face p of G, let $b_p = \min\{b_v : v \in p\}$.

Theorem 3.1 ([6]).

(a) *We have*

$$\begin{aligned} A(a, b) = \sum_p \Bigg(& l(p)(a_p + b_p)^2 + 2(a_p + b_p)\left(\sum_{v\in p}(b_v - b_p)\right) \\ & + \sum_{vv'\in p}(b_v - b_p)(b_{v'} - b_p)\Bigg) + \sum_{vv'\in p_\infty} b_v b_{v'}, \end{aligned} \tag{24}$$

where the p-summation is over the set of all faces of G. Each term in the above sum is manifestly nonnegative.

(b) *$B(a, b)$ can also be written as a finite sum of manifestly nonnegative linear forms on (a, b).*

(c) *If $\frac{1}{2}(A(a, b) + B(a, b)) \le N$ for some natural number N, then for every i and every j there exist c_i, c_i' and c_j, c_j' (computed effectively from G) such that*

$$c_i N \le b_i \le c_i' N, \qquad c_j' \sqrt{N} \le a_j \le c_j N + c_j' \sqrt{N}.$$

3.2. Some properties of $\Phi_G(q)$

In this section we summarize some properties of $\Phi_G(q)$.

Lemma 3.2 ([1,5]).

(a) *The series $\Phi_G(q)$ depends only on the abstract planar graph G and not on its plane embedding, nor on the choice of vertex of the unbounded face.*

(b) *If $G = G_1 \sqcup G_2$ is disconnected, then*

$$(1 - q)\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q).$$

(c) *If G has a separating vertex v and $G \setminus \{v\} = G_1 \sqcup G_2$, then*

$$\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q).$$

(d) *If G is a planar graph (possibly with multiple edges and loops) and G^{red} denotes the corresponding simple graph obtained by removing all loops and replacing all edges of multiplicity more than with edges of multiplicity one, then*

$$\Phi_G(q) = \Phi_{G^{\mathrm{red}}}(q).$$

Note that we use the normalization

$$\Phi_{\bullet}(q) = \Phi_{\bullet\!\!-\!\!\!-\!\!\bullet}(q) = 1. \tag{25}$$

In view of the above lemma, in the rest of the paper G will denote a simple, 2-edge-connected rooted plane graph.

3.3. Some lemmas from [6]

In this section we review the statements of some lemmas from [6] which we use for the proof of Theorem 1.3.

Lemma 3.3 ([6, Corollary 3.2]). *For a pair (p, v) a 2-edge-connected graph G where p is a face and v is a vertex of p we have $B(a, b) \ge a_p + b_v$.*

The proofs of the three lemmas below can be found in [6, Section 4].

Lemma 3.4. *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

(1) *$b_v = b_{v'} = 1$ where vv' is an edge of p_∞,*
(2) *$a_p + b_p = 0$ for any face p of G,*
(3) *$(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ for any face p of G and edge $v_1 v_2$ of p,*

then $b_v \ge 1$ for all vertices v, $a_p = -1$ for all faces $p \ne p_\infty$ and $B(a, b) \ge 2 + V_\infty$.

Lemma 3.5. *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

(1) *$b_v = b_{v'} = 0$ and $(b_v - b_p)(b_{v'} - b_p) = 1$ where p is a boundary face and vv' is a boundary edge that belongs to p,*
(2) *$a_p + b_p = 0$ for any face p of G,*
(3) *$(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ for any face p of G and edge v_1v_2 not on the boundary of p.*

Then $b_w \geq -1$ for all vertices w, $a_p = 1$ for all faces $p \neq p_\infty$ and $B(a, b) \geq V_\infty - 2$. Furthermore $B(a, b) = V_\infty - 2$ if and only if

- *$b_v = 0$ for all boundary vertices v and $b_w = -1$ for all other vertices w.*
- *$a_p = 1$ for all faces p.*

Lemma 3.6. *Let G be a 2-connected planar graph, p_0 be a boundary face and (a, b) be an admissible state such that*

(1) *$a_{p_0} + b_{p_0} = 0$,*
(2) *There exists a boundary edge vv' of p_0 such that $b_vb_{v'} = 0$ and $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$.*

Let G_0 be the graph obtained from G by deleting the boundary edges of p_0 and let (a_0, b_0) be the restriction of the admissible state (a, b) on G_0. Then,

(a) *(a_0, b_0) is an admissible state for G_0,*
(b) *$A(a_0, b_0) = A(a, b) - \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_vb_{v'}$,*
(c) *$B(a_0, b_0) = B(a, b) - 2\sum_{v \in V_0} b_v$, where V_0 is the set of boundary vertices of p_0 that do not belong to any other bounded face,*
(d) *$B(a, b) \geq 2\sum_{v \in V_0} b_v$,*
(e) *If furthermore $B(a, b) \leq 1$ then $A(a, b) = A(a_0, b_0)$, $B(a, b) = B(a_0, b_0)$.*

4. The coefficient q^3 in $\Phi_G(q)$

4.1. Analysis of admissible states

In this section we find the admissible states (a, b) such that $\frac{1}{2}(A(a, b) + B(a, b)) = 3$. Since $A(a, b), B(a, b) \in \mathbb{N}$ we have the following cases:

$A(a, b)$	6	5	4	3	2	1	0
$B(a, b)$	0	1	2	3	4	5	6

Case 1: $(A(a, b), B(a, b)) = (6, 0)$. By Lemma 3.3 we have $B(a, b) \geq a_p + b_p \geq 0$ and so $a_p + b_p = 0$ for all faces p. Similarly since $B(a, b) \geq a_p + b_v = b_v - b_p \geq 0$ we have $a_p + b_v = b_v - b_p = 0$ for all $v \in p$. Thus $A(a, b) = 6$ is equivalent to

$$\sum_{vv' \in p_\infty} b_vb_{v'} = 6. \tag{26}$$

If vv' is an edge of G and p is a face that contains vv' then we have $b_v = b_p = b_{v'}$. So by Eq. (26) there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 3.4 implies that $B(a, b) \geq 2 + V_\infty > 0$ which is impossible. Therefore there are no admissible states (a, b) that satisfy $(A(a, b), B(a, b)) = (6, 0)$.

Case 2: $(A(a, b), B(a, b)) = (5, 1)$. Since $l(p) \geq 3$ we have $a_p + b_p \leq 1$ for all p.

Case 2.1: There exists a face p_0 such that $a_{p_0} + b_{p_0} = 1$, which implies that $a_p + b_p = 0$ for all $p \neq p_0$.

Case 2.1.1: $l(p_0) = 4$ or 5. We have $B(a, b) \geq (a_{p_0} + b_{v_1}) + (a_{p_0} + b_{v_2}) = 2(a_{p_0} + b_{p_0}) = 2$ which is impossible, here v_1, v_2 are two vertices of p_0.

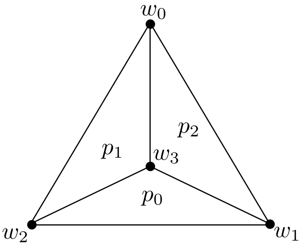
Case 2.1.2: $l(p_0) = 3$. We have

$$5 = A(a, b) = 3 + \sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_vb_{v'}$$

and therefore

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2. \tag{27}$$

There are at most two positive terms in Eq. (27). Let $v_i v_i' \in p_i,\ 1 \le i \le 2$ be the edges and bounded faces that appear in these terms. If a bounded face p contains a boundary edge $vv' \neq v_i v_i',\ i = 1, 2$ then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 3.6 we have $A(a', b') = A(a, b), B(a', b') = B(a, b)$. Continue this way until G does not have any face p with a boundary edge $vv' \neq v_i v_i',\ i = 1, 2$. It is easy to see that the only possibility for this to happen is when $G = p_0 \cup p_1 \cup p_2$, where say $v_i v_i' \in p_i,\ i = 1, 2$. Since p_1, p_2 do not contain any boundary edge other than $v_i v_i',\ i = 1, 2$, G should be isomorphic to the graph in the following figure.



where $w_0 w_1 = v_1 v_1',\ w_0 w_2 = v_2 v_2'$. It follows that $b_{w_1} b_{w_2} = 0$ and let us assume that $b_{w_2} = 0$, and so $b_{w_2} b_{w_0} = 0$. This forces $(b_{w_2} - b_{p_1})(b_{w_0} - b_{p_1}) = 1$ since the edge $w_0 w_2$ corresponds to a positive term in Eq. (27), which must equal 1. It follows from the latter that $b_{w_0} = b_{w_2} = 0$ and therefore from Eq. (27), $2 = \sum_{vv' \in p_\infty} b_v b_{v'} = b_{w_0} b_{w_1} + b_{w_1} b_{w_2} + b_{w_2} b_{w_0} = 0$ which is impossible.

Case 2.2: $a_p + b_p = 0$ for all p. Then we have

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 5. \tag{28}$$

There are at most 5 positive terms in Eq. (28). Let $v_i v_i' \in p_i,\ 1 \le i \le 5$ be the edges and bounded faces that appear in these terms. If a bounded face p contains a boundary edge $vv' \neq v_i v_i',\ 1 \le i \le 5$ then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 3.6 we have $A(a', b') = A(a, b), B(a', b') = B(a, b)$. We can continue this way until all the boundary edges of G are among the $v_i v_i'$. This means we can assume that G has m boundary edges where $3 \le m \le 5$. Let us relabel the boundary vertices by $v_1, v_2, .., v_m$.

Case 2.2.1: All the positive terms in Eq. (28) correspond to boundary edges. If the positive terms are $b_{v_1} b_{v_2}, \ldots, b_{v_m} b_{v_1}$ then since $b_{v_1} b_{v_2} + \cdots + b_{v_m} b_{v_1} = 5$,

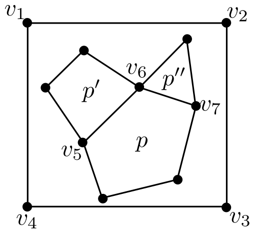
- there exists $1 \le i \le m$ such that $b_{v_i} b_{v_{i+1}} = 1$,
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all faces p and edges vv' of G.

It follows from 3.4 that $B(a, b) \ge V_\infty + 2 \ge 5$ which is impossible. On the other hand, if for instance $b_{v_1} b_{v_2} = 0$ then we can assume that $b_{v_1} = 0$. Since the edge $v_1 v_2$ corresponds to a positive term, we have

$$(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = k \tag{29}$$

where $1 \le k \le 3$ and p_1 is the bounded face that contains $v_1 v_2$. Here $k \neq 4, 5$ since we are assuming that all positive terms correspond to boundary edges and there are at least 3 edges. We claim that $k = 1$. Indeed, let us assume to the contradiction that $k \ge 2$. Eq. (29) implies that either $b_{p_1} = -k$ and $b_{v_2} - b_{p_1} = 1$ or $b_{p_1} = -1$ and $b_{v_2} - b_{p_1} = k$. The former is impossible since $b_{v_2} \ge 0$. From the later we have $b_{v_2} = k - 1$ and since $a_{p_1} + b_{p_1} = 0$ we also have $a_{p_1} = 1$. So by Lemma 3.3 we have $B(a, b) \geq a_{p_1} + b_{v_2} = k \geq 2$ which is impossible and the claim is proven. Therefore $k = 1$ and hence $b_{v_1} = b_{v_2} = 0$, $b_{p_1} = -1$. It follows that $b_{v_2}b_{v_3} = 0$ which means $(b_{v_2} - b_{p_2})(b_{v_3} - b_{p_2}) = k'$, $1 \leq k' \leq 3$, because the edge v_2v_3 corresponds to a positive term. By a similar argument we can show that $k' = 1$ and $b_{p_2} = -1$, $b_{v_3} = 0$. Similarly we can prove that $b_{v_i} = 0$ and $b_{p_i} = -1$ for all $1 \leq i \leq 5$ for all $1 \leq i \leq m$ where p_i is the boundary face that contains v_iv_{i+1}. In particular, this implies that $m = 5$ and $(b_{v_i} - b_{p_i})(b_{v_{i+1}} - b_{p_i}) = 1$ for $1 \leq i \leq 5$ and therefore $(b_v - b_p)(b_{v'} - b_p) = 0$ for all $(p, vv') \neq (p_i, v_iv_{i+1})$ for all i. So by Lemma 3.5 we have $B(a, b) \geq V_\infty - 2 = 3$ which is impossible.

Case 2.2.2: There are 1 or 2 positive terms in Eq. (28) that do not correspond to the boundary edges. By a similar argument as the above, we can reduce this to the case where the unbounded face of G has 3 or 4 vertices. Let us consider the case where G has 4 boundary edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 that correspond to 4 of the 5 positive terms and the other positive term corresponds to an edge v_5v_6 inside of G as in the figure below. The other cases are completely similar.



If the positive terms that correspond to the boundary edges are $b_{v_1}b_{v_2}, \dots, b_{v_4}b_{v_1}$ then since $b_{v_1}b_{v_2} + \cdots + b_{v_4}b_{v_1} = 4$. This means that each of the terms $b_{v_i}b_{v_{i+1}}$ is equal to 1 and by an argument similar to the one of Case 2.2.1 we can conclude that $B(a, b) \geq V_\infty + 2 = 6$, which is impossible. If, say $b_{v_1}b_{v_2} = 0$ then $(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = k > 0$ since the edge v_1v_2 corresponds to a positive term, here p_1 is the bounded face that contains v_1v_2. Since we have 4 positive terms and 4 boundary edges, each positive term is equal to 1, hence $k = 1$. Similar to the argument in Case 2.2.1, we can show that $b_p = -1$ for all faces p. Let p be the face that appears in the positive term that contains v_5v_6 and p' be the other face that contains v_5v_6. It follows from $(b_{v_5} - b_p)(b_{v_6} - b_p) = 1$ that $b_{v_5} = b_{v_6} = b_p + 1 = 0$. Since $(b_{v_5} - b_{p'})(b_{v_6} - b_{p'}) = 0$ we have $b_{p'} = 0$ which is impossible.

Case 3: $(A(a, b), B(a, b)) = (4, 2)$.

Case 3.1: There exists a face p_0 such that $a_{p_0} + b_{p_0} = 1$, which implies that $a_p + b_p = 0$ for all $p \neq p_0$. Since $A(a, b) = 4$ we have $l(p_0) \leq 4$.

Case 3.1.1: $l(p_0) = 4$. By a similar argument to the case 2 of Section 4.3 in [6] we can show that this gives us the following set of admissible states (a, b):

- $a_{p_0} = 1$ for a square face p_0, $a_p = 0$ for $p \neq p_0$,
- $b_v = 0$ for all vertices v.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{l(p_0)}} = \frac{q^3}{(1-q)^4} = q^3 + O(q^4).$$

Case 3.1.2: $l(p_0) = 3$. We have

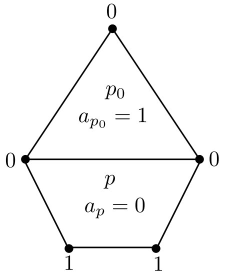
$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 1. \tag{30}$$

There is exactly one positive term in Eq. (30). Let $vv' \in p$ be the edge and bounded face that appears in this term. If a bounded face p' contains a boundary edge $ww' \neq vv'$ then we should have $b_w b_{w'} = (b_w - b_{p'})(b_{w'} - b_{p'}) = 0$. This implies that $b_{p'} = 0$ and hence $a_{p'} = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p' and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 3.6 we have $A(a', b') = A(a, b)$, $B(a', b') = B(a, b) - 2k$, $k \in \{0, 1\}$.

Here

- $k = 0$ if and only if $b_v = 0$ for all removed vertices v,
- $k = 1$ if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.

We can continue this way until $G = p_0$ if $p = p_0$ or $G = p \cup p_0$ if $p \neq p_0$. Let us consider first the case where $p = p_0$. Let the three vertices of p_0 be v, v', v'' and $b_{p_0} = b_v$. We have $2 \geq B(a,b) = a_{p_0} + 2(b_v + b_{v'} + b_{v''}) = (a_{p_0} + b_{p_0}) + b_{p_0} + 2(b_{v'} + b_{v''}) = 1 + b_{p_0} + 2(b_{v'} + b_{v''})$. It follows that $1 \geq b_{p_0} + 2(b_{v'} + b_{v''})$ and hence $b_{p_0} = b_{v'} = b_{v''} = 0$ since they are all non-negative. This implies that $a_{p_0} = 1$ and so $A(a,b) = 3a_{p_0}^2 + 2a_{p_0}(b_v + b_{v'} + b_{v''}) = 3$ which is impossible. If $p \neq p_0$ then there should exist an edge v_0v_0' of p_0 that does not correspond to a positive term and hence $b_{v_0}b_{v_0'} = 0$. It follows that $b_{p_0} = 0$ and so $a_{p_0} = 1$. This forces $b_v = 0$ for all $v \in p_0$ since otherwise $B(a,b) = a_{p_0} + 2\sum_{v\in p} b_v \geq 3$ which is impossible. Similarly there should exist an edge ww' of p such that $b_w b_{w'} = 0$ which implies that $a_p = 0$ and hence $b_p = 0$. If p and p_0 are disjoint then we have $2 = B(a,b) = B^p(a,b) + B^{p_0}(a,b) = B^p(a,b) + 1$ where $B^p(a,b)$ denotes the restriction of $B(a,b)$ on p. It follows that $B^p(a,b) = 1$ and the argument in Lemma 3.6 implies that $b_v = 0$ for all $v \in p$. This is impossible since it gives $B(a,b) = a_p + 2\sum_{v\in p} b_v = 0$. So p and p_0 are not disjoint. If v is a vertex of both p and p_0 then $b_v = 0$ and therefore $b_p = 0$ which implies that $a_p = 0$ since $a_p + b_p = 0$.



As before, the argument in Lemma 3.6 implies that $b_v = 0$ for all $v \in p$ and so $B(a,b) = B^p(a,b) + B^{p_0}(a,b) = 1$ which is impossible.

Case 3.2: $a_p + b_p = 0$ for all p. Then we have

$$\sum_p \sum_{vv'\in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv'\in p_\infty} b_v b_{v'} = 4. \tag{31}$$

There are at most 4 positive terms in Eq. (31). If an edge $vv' \in p$ does not correspond to a positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 3.6 we have $A(a', b') = A(a, b), B(a', b') = B(a, b) - 2k,\ k \in \{0, 1\}$. Here

- $k = 0$ if and only if $b_v = 0$ for all removed vertices v,
- $k = 1$ if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.

We can continue to do this until the boundary of G has at most 4 edges all of which correspond to positive terms.

Case 3.2.1: All of the positive terms in Eq. (31) correspond to boundary edges.

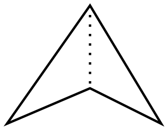
Case 3.2.1.1 G has 3 vertices on the boundary, say v_1, v_2, v_3. If all the positive terms are equal to 1 then there must exist a boundary edge, for instance, v_1v_2 of G such that $b_{v_1}b_{v_2} = (b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = 1$ where p_1 is the bounded face that contains v_1v_2. This implies that $b_{p_1} = 0$ and hence $a_{p_1} = 0$. Let $vv' \notin \{v_1v_2, v_2v_3, v_3v_1\}$ be another edge of p_1 and let p be the other bounded face that contains vv'. Since vv' does not correspond to a positive term, we have $(b_v - b_{p_1})(b_{v'} - b_{p_1}) = 0$ and so $b_v b_{v'} = 0$. We also have $(b_v - b_p)(b_{v'} - b_p) = 0$ which means $b_p = \min\{b_v, b_{v'}\} = 0$ and hence

$a_p = 0$. Similarly we can show that $b_{p'} = a_{p'} = 0$ for all faces p' and in particular $b_w \geq 0$ for all w. It follows that $B(a, b) \geq 2(b_{v_1} + b_{v_2}) = 4$ which is impossible.

If one of the positive terms is equal to 2 then the other two are equal to 1. Without loss of generality we can assume that the edge v_1v_2 corresponds to this term, so either $b_{v_1}b_{v_2} = 2$ or $(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_2}) = 2$. For the former we can assume that $b_{v_1} = 1$ and $b_{v_2} = 2$. This implies that $b_{v_3} = 0$ since otherwise $A(a, b) \geq b_{v_1}b_{v_2} + b_{v_2}b_{v_3} + b_{v_3}b_{v_1} \geq 2 + 1 + 2 = 5$ which is impossible. Since $b_{v_2}b_{v_3} = 0$ which means $(b_{v_2} - b_{p_2})(b_{v_3} - b_{p_2}) = 1$ and this leads to $-b_{p_2}(2 - b_{p_2}) = 1$ which is impossible.

Case 3.2.1.2 G has 4 vertices on the boundary, say v_1, v_2, v_3, v_4. By a similar argument to the case 2.2 of Section 4.3 in [6], this corresponds to the following admissible state of G:

- $a_p = 1$ for all bounded faces p,
- $b_{v_1} = b_{v_2} = b_{v_3} = b_{v_4} = 0$ where v_1, v_2, v_3, v_4 are the vertices of a square G_0 that does not have any diagonal in its interior. We will write $c_{40} = [G_0](G)$.

$G_0 =$ 

where the dotted line means G_0 does not contain an internal diagonal,
- $b_w = -1$ for all vertices w inside the 4-circle mentioned above,
- $b_{\tilde{w}} = 0$ for any other vertex w.

The contribution of this state to $\Phi_G(q)$ is

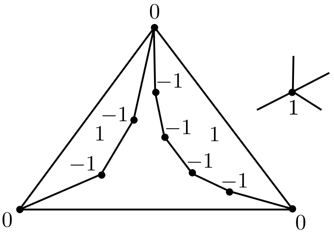
$$\frac{q^3}{(1-q)^{\deg_\square(v_1)+\deg_\square(v_2)+\deg_\square(v_3)+\deg_\square(v_4)-4}} = q^3 + O(q^4)$$

where $\deg_\square(v)$ is the degree of v in the square $\square = v_1v_2v_3v_4$.

Case 3.2.2: One of the positive terms in Eq. (31) does not correspond to any boundary edge. By a similar argument to the Case 2.2.2 we can show that there are no admissible states here.

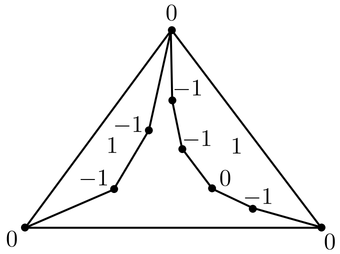
Case 4: $(A(a, b), B(a, b)) = (3, 3)$. By a similar argument to the case 2 of Section 4.3 in [6] we can show that the admissible states for this case are

- $a_p = 1$ for all faces p.
- $b_{v_1} = b_{v_2} = b_{v_3} = 0$ where v_1, v_2, v_3 are the vertices of a 3-cycle in G.
- $b_v = -1$ for all v inside the 3-cycle mentioned above.
- $b_{v_0} = 1$ for a fixed vertex w outside of the 3-cycle.
- $b_w = 0$ for all other vertices w.



and

- $a_p = 1$ for all faces p.
- $b_{v_1} = b_{v_2} = b_{v_3} = 0$ where v_1, v_2, v_3 are the vertices of a 3-cycle in G.
- $b_{v_0} = 0$ for a fixed vertex v_0 inside the 3-cycle that is not adjacent to any of the vertices v_1, v_2, v_3 and $b_v = -1$ for all other v also inside the cycle.
- $b_w = 0$ for all other vertices w.



The contribution of both types of states above to $\Phi_G(q)$ is

$$(-1)^3 \frac{q^3}{(1-q)^{\deg_\Delta(v_1)+\deg_\Delta(v_2)+\deg_\Delta(v_3)+\deg_\Delta(v_0)-3}} = -q^3 + O(q^4)$$

where $\deg_\Delta(v)$ is the degree of v in the triangle $\Delta = v_1v_2v_3$.

Case 5: $(A(a,b), B(a,b)) = (2,4)$. Since $A(a,b) = 2$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2. \tag{32}$$

There are at most 2 positive terms in Eq. (32). If a boundary face p contains a boundary edge vv' that does not correspond to a positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 3.6 we have $A(a', b') = A(a, b) - 2i, B(a', b') = B(a, b) - 2k,\ k \in \{0, 1, 2\}$. Here

- $i = 0$ and $k = 0$ if and only if $b_v = 0$ for all removed vertices.
- $i = 0$ and $k = 1$ if and only if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.
- $i = 0$ and $k = 2$ if and only if there exist two removed vertices v, v' which are not connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.
- $i = 1$ and $k = 2$ if and only if there exist two removed vertices v, v' which are connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.

It is easy to see that only the last item gives admissible states (a, b) with $(A(a, b), B(a, b)) = (2, 4)$. To summarize, the admissible states in this case are those (a, b) that satisfy

- $a_p = 0$ for all faces p.
- There exist two vertices v, v' which are connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)+\deg(v')}} = q^3 + O(q^4).$$

Case 6: $(A(a,b), B(a,b)) = (1,5)$. Since $A(a,b) = 1$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 1. \tag{33}$$

There is exactly 1 positive term in Eq. (33). If the pair $(vv', p),\ vv' \in p$ does not correspond to this positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Similarly we can show that $a_{p'} = 0$ for all other faces p'. This implies that $5 = B(a, b) = 2\sum_v b_v$ which is impossible. So there are no admissible states in this case.

Case 7: $(A(a,b), B(a,b)) = (0,6)$. Since $A(a,b) = 0$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 0. \tag{34}$$

Let $vv' \in p$ where p is a boundary face then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 3.6 we have $A(a', b') = A(a, b), B(a', b') = B(a, b) - 2k,\ k \in \{0, 1, 2, 3\}$. Here

- $k = 0$ if and only if $b_v = 0$ for all removed vertices.
- $k = 1$ if and only if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.
- $k = 2$ if and only if there exists a removed vertex v such that $b_v = 2$ or if and only if there exist two removed vertices v, v' which are not connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.
- $k = 3$ if and only if there exists a removed vertex v such that $b_v = 3$ or if and only if there exist two removed vertices v, v' which are not connected by an edge such that $b_v = 1,\ b_{v'} = 2$ or if and only if there exist three removed vertices v, v', v'' none of which are connected by an edge such that $b_v = b_{v'} = b_{v''} = 1$ and $b_w = 0$ for all other removed vertices w.

The above possible values of k lead to the following admissible states (a, b):

- $a_p = 0$ for all faces p.
- There exists a vertex v such that $b_v = 3$ and $b_w = 0$ for all $w \neq v$.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)_3^{\deg(v)}} = q^3 + O(q^4)$$

- $a_p = 0$ for all faces p.
- There exist two vertices v, v' which are not connected by an edge such that $b_v = 1,\ b_{v'} = 2$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)}(1-q)_2^{\deg(v')}} = q^3 + O(q^4)$$

- $a_p = 0$ for all faces p.
- There exist three vertices v, v', v'' none of which are connected by an edge such that $b_v = b_{v'} = b_{v''} = 1$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)+\deg(v')+\deg(v'')}} = q^3 + O(q^4).$$

4.2. *Proof of Theorem 1.3*

We now give a proof of Theorem 1.3 based on cases 1–7 of Section 4.1. We write

$$\begin{aligned}\Phi_G(q) &= (1-q)(q)_\infty^{c_2}(1 + a_1 q + a_2 q^2 + a_3 q^3 + O(q^4)) \\ &= (1-q)(1 + b_1 q + b_2 q^2 + b_3 q^3)(1 + a_1 q + a_2 q^2 + a_3 q^3) + O(q^4)\end{aligned}$$

where from [6, Section 4.2] we have

$$\begin{aligned}a_1 &= c_1 \\ a_2 &= \frac{c_1(c_1+1)}{2} + c_2 - c_3\end{aligned}$$

and a_3 receives contributions from

- States (a, b) such that $\frac{1}{2}(A+B) = 3$. These are discussed in Section 4.1.
- States (a, b) such that $\frac{1}{2}(A+B) \leq 2$ which are discussed in [6, Section 4.2].

By expanding the factor $(q)_\infty^{c_2}$ we have

$$\begin{aligned} b_1 &= -c_2 \\ b_2 &= \frac{c_2(c_2-3)}{2} \\ b_3 &= \frac{-c_2^3+9c_2^2-8c_2}{6}. \end{aligned}$$

The total contribution of the admissible states found in cases 1–7 to $a_3q^3 + O(q^4)$ is

$$\left(c_{40} + c_1 + c_2 + 2\left(\frac{c_1(c_1-1)}{2} - c_2\right) + \gamma - \sum_{C_3 = vv'v''} (c_1 - \alpha(C_3)) \right) q^3 + O(q^4) \tag{35}$$

where $\frac{c_1(c_1-1)}{2} - c_2$ is the number of pair of vertices in G that are not connected by an edge. The last term is a summation over 3-cycles $C_3 = vv'v''$ of G and $\alpha(C_3)$ is 3 plus the number of vertices contained in C_3 that are adjacent to either v, v' or v''. The admissible states in Sections 4.2 and 4.3 in [6] gives the following contribution to $a_3q^3 + O(q^4)$:

$$\begin{aligned} &1 + \sum_v q(1+q+q^2)^{\deg(v)} - q^2 \sum_{C_3 = vv'v''} (1+q)^{\deg_{C3}(v)+\deg_{C3}(v')+\deg_{C3}(v'')-3} \\ &\quad + q^2 \sum_{(vv')\neq e} (1+q)^{\deg(v)+\deg(v')} + \sum_v q^2(1+q)^{\deg(v)} + O(q^4) \end{aligned} \tag{36}$$

where by $(vv') \neq e$ we mean a pair of vertices $v,\ v'$ that are not connected by an edge and $\deg_{C3}(v)$ denotes the degree of v in the subgraph of G that is contained in C_3. Summing up (35) and (36) we get

$$\begin{aligned} a_3 &= c_{40} + c_1 + c_2 + 2\left(\frac{c_1(c_1-1)}{2} - c_2\right) + \gamma + \delta + c_2 + \sum_{(vv')\neq e} (\deg(v) + \deg(v')) \\ &\quad + 2c_2 - \sum_{C_3 = vv'v''} (c_1 + \deg_{C3}(v) + \deg_{C3}(v') + \deg_{C3}(v'') - 3 - \alpha(C_3)). \end{aligned} \tag{37}$$

Note that

$$\begin{aligned} \sum_{(vv')\neq e} (\deg(v) + \deg(v')) &= \sum_v \deg(v)(c_1 - 1 - \deg(v)) \\ &= 2c_2(c_1-1) - \sum_v (\deg(v))^2 \\ &= 2c_2(c_1-1) - 2\delta - 6c_3 - 2c_2 \end{aligned}$$

where the last equality follows from Lemma 2.4. Let us define

$$d_3 = \deg_{C3}(v) + \deg_{C3}(v') + \deg_{C3}(v'') - 3 - \alpha(C_3)$$

and $c'_{40} = [\![$  $]\!]$, $c_{41} = [\![$  $]\!]$.

Lemma 4.1. *We have*

(a) $d_3 = c'_{40} + 2c_{42}$.
(b) $c_{40} - c'_{40} = c_{41}$.

crossings = edges	3	4	5	6	7	8	9	10
alternating links	1	2	3	8	14	39	96	297
irreducible graphs	1	1	1	3	3	8	17	41

Fig. 5. The number of alternating links with at most 10 crossings and the number of irreducible graphs with at most 10 edges.

G	c	C	L	$\Phi_L(q)+O(q^6)$
G_0^3	$3,3,1,0,0$	$1,1,1,1,1$	3_1	$1-q-q^2+q^5$
G_0^4	$4,4,0,1,0$	$1,0,-1,-1,-1$	4_1^2	$1-q+q^3$
G_0^5	$5,5,0,0,0$	$1,0,0,1,1$	5_1	$1-q-q^4$
G_0^6	$6,6,0,0,0$	$1,0,0,0,-1$	6_1^2	$1-q+q^5$
G_1^6	$4,6,4,0,1$	$3,4,6,9,16$	6_2^3	$1-3q-q^2+5q^3+3q^4+3q^5$
G_2^6	$5,6,0,3,0$	$2,0,-3,-4,-3$	6_1^3	$1-2q+q^2+3q^3-2q^4-2q^5$
G_0^7	$7,7,0,0,0$	$1,0,0,0,0$	7_1	$1-q$
G_1^7	$5,7,2,2,0$	$3,2,0,-2,-4$	7_6^2	$1-3q+q^2+5q^3-3q^4-3q^5$
G_2^7	$6,7,0,1,0$	$2,0,-1,1,2$	7_4^2	$1-2q+q^2+q^3-3q^4+q^5$

Fig. 6. The irreducible graphs G with at most 10 edges, the 6-tuple of polynomial invariants $c = (c_1, c_2, c_3, c_{41}, c_{42})$, $C = (C_1, C_2, C_3, c_4, C_5)$ as defined in Eq. (7), the alternating link L and the 6 stable coefficients of the Jones polynomial of L.

G_0^8	$8,8,0,0,0$	$1,0,0,0,0$	8_1^2	$1-q$
G_1^8	$5,8,4,1,0$	$4,4,3,0,-6$	8_{18}	$1-4q+2q^2+9q^3-5q^4-8q^5$
G_2^8	$6,8,0,5,0$	$3,0,-5,-7,-4$	8_{14}^2	$1-3q+3q^2+4q^3-8q^4-2q^5$
G_3^8	$6,8,2,0,0$	$3,2,2,4,6$	8_{15}^3	$1-3q+q^2+3q^3-3q^4+3q^5$
G_4^8	$6,8,1,2,0$	$3,1,-1,0,2$	8_{16}	$1-3q+2q^2+3q^3-6q^4+q^5$
G_5^8	$6,8,0,6,0$	$3,0,-6,-10,-7$	8_1^4	$1-3q+3q^2+5q^3-8q^4-5q^5$
G_6^8	$7,8,0,1,0$	$2,0,-1,-1,-3$	8_1^3	$1-2q+q^2+q^3-q^4+2q^5$
G_7^8	$7,8,0,0,0$	$2,0,0,2,1$	8_5	$1-2q+q^2-2q^4+3q^5$

Fig. 7. Fig. 6 continued.

G_1^9	$5,9,7,0,2$	$5,7,11,17,31$	9_{40}	$1-5q+3q^2+14q^3-6q^4-15q^5$
G_2^9	$6,9,2,5,0$	$4,2,-3,-9,-13$	9_{12}^3	$1-4q+4q^2+7q^3-13q^4-7q^5$
G_3^9	$6,9,3,1,0$	$4,3,2,3,6$	9_{42}^2	$1-4q+3q^2+6q^3-9q^4$
G_4^9	$6,9,2,4,0$	$4,2,-2,-5,-5$	9_{34}	$1-4q+4q^2+6q^3-13q^4-3q^5$
G_5^9	$6,9,2,3,0$	$4,2,-1,-1,3$	9_{40}	$1-4q+4q^2+5q^3-13q^4+q^5$
G_6^9	$7,9,0,3,0$	$3,0,-3,-3,-4$	9_{40}^2	$1-3q+3q^2+2q^3-6q^4+4q^5$
G_7^9	$7,9,1,0,0$	$3,0,-3,-3,-4$	9_{41}	$1-3q+2q^2+q^3-4q^4+7q^5$
G_8^9	$7,9,2,0,0$	$3,2,2,2,0$	9_{31}^2	$1-3q+q^2+3q^3-q^4+3q^5$
G_9^9	$7,9,0,3,0$	$3,0,-3,-2,0$	9_{36}^2	$1-3q+3q^2+2q^3-7q^4+3q^5$
G_{10}^9	$7,9,1,1,0$	$3,1,0,1,0$	9_{35}^2	$1-3q+2q^2+2q^3-4q^4+4q^5$
G_{11}^9	$7,9,0,2,0$	$3,0,-2,1,3$	9_{29}	$1-3q+3q^2+q^3-7q^4+6q^5$
G_{12}^9	$7,9,0,3,0$	$3,0,-3,-1,3$	9_3^3	$1-3q+3q^2+2q^3-8q^4+3q^5$
G_{13}^9	$7,9,0,2,0$	$3,0,-2,2,6$	9_9^3	$1-3q+3q^2+q^3-8q^4+6q^5$
G_{14}^9	$8,9,0,0,0$	$2,0,0,1,0$	9_{19}^2	$1-2q+q^2-q^4+2q^5$
G_{15}^9	$8,9,0,1,0$	$2,0,0,1,0$	9_{13}^2	$1-2q+q^2+q^3-q^4$
G_{16}^9	$8,9,0,0,0$	$2,0,0,0,-3$	9_{35}	$1-2q+q^2+3q^5$

Fig. 8. Fig. 6 continued.

Proof. (a) If w is a vertex in the interior incident to v and v' then it contributes $+1$ to $\deg(v)$, $+1$ to $\deg(v')$ and -1 to itself. Hence totally such w's contribute c'_{40}. If w is a vertex in the interior incident to $v,\ v',\ v''$ then it contributes $+1$ to each $\deg(v),\ \deg(v'),\ \deg(v'')$ and -1 to itself. So totally such w's contribute $2c_{42}$. If w is a boundary vertex then its contribution to each of $\deg(v),\ \deg(v'),\ \deg(v'')$ is $+2$ and the total contribution of the 3 boundary vertices is $+6$ which cancels the -6 in d_3. Thus we

G_0^{10}	$6, 10, 5, 2, 1$	$5, 5, 5, 6, 11$	10_{121}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 7q^5$
G_1^{10}	$6, 10, 5, 0, 0$	$5, 5, 5, 6, 10$	10_{123}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 6q^5$
G_2^{10}	$6, 10, 4, 4, 0$	$5, 4, 0, -8, -20$	10^4_{17}	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
G_3^{10}	$6, 10, 4, 4, 0$	$5, 4, 0, -8, -20$	10^2_{155}	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
G_4^{10}	$6, 10, 4, 3, 0$	$5, 4, 1, -3, -6$	10^2_{137}	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
G_5^{10}	$7, 10, 0, 10, 0$	$4, 0, -10, -20, -15$	10^5_1	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
G_6^{10}	$7, 10, 0, 8, 0$	$4, 0, -8, -13, -7$	10^3_{25}	$1 - 4q + 6q^2 + 4q^3 - 18q^4 + 3q^5$
G_7^{10}	$7, 10, 0, 7, 0$	$4, 0, -7, -10, -5$	10_{120}	$1 - 4q + 6q^2 + 3q^3 - 17q^4 + 7q^5$
G_8^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 1, 3$	10^2_{33}	$1 - 4q + 4q^2 + 4q^3 - 11q^4 + 5q^5$
G_9^{10}	$7, 10, 3, 0, 0$	$4, 3, 3, 4, 3$	10_{112}	$1 - 4q + 3q^2 + 5q^3 - 6q^4 + 4q^5$
G_{10}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10_{116}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{11}^{10}	$7, 10, 1, 3, 0$	$4, 1, -2, 1, 7$	10^2_{151}	$1 - 4q + 5q^2 + 2q^3 - 14q^4 + 11q^5$
G_{12}^{10}	$7, 10, 1, 4, 0$	$4, 1, -3, -3, 0$	10_{119}	$1 - 4q + 5q^2 + 3q^3 - 14q^4 + 7q^5$
G_{13}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10_{114}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{14}^{10}	$7, 10, 1, 3, 0$	$4, 1, -2, 0, 3$	10^2_{156}	$1 - 4q + 5q^2 + 2q^3 - 13q^4 + 11q^5$
G_{15}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 4, 7$	10^2_{147}	$1 - 4q + 4q^2 + 3q^3 - 10q^4 + 9q^5$
G_{16}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 3, 3$	10_{122}	$1 - 4q + 4q^2 + 3q^3 - 9q^4 + 9q^5$
G_{17}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10^3_{74}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{18}^{10}	$7, 10, 1, 4, 0$	$4, 1, -3, -2, 4$	10^2_{28}	$1 - 4q + 5q^2 + 3q^3 - 15q^4 + 7q^5$
G_{19}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 5, 11$	10^4_{12}	$1 - 4q + 4q^2 + 3q^3 - 11q^4 + 9q^5$
G_{20}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 3, 4$	10^2_{106}	$1 - 3q + 3q^2 - 6q^4 + 8q^5$
G_{21}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 2, 1$	10^2_{20}	$1 - 3q + 3q^2 - 5q^4 + 8q^5$
G_{22}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 1, -1$	10^2_{141}	$1 - 3q + 3q^2 - 4q^4 + 7q^5$
G_{23}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 2, 2$	10_{93}	$1 - 3q + 3q^2 - 5q^4 + 7q^5$
G_{24}^{10}	$8, 10, 1, 1, 0$	$3, 1, 0, 0, -1$	10_{85}	$1 - 3q + 2q^2 + 2q^3 - 3q^4 + 2q^5$
G_{25}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -1, -1$	10_{100}	$1 - 3q + 3q^2 + q^3 - 5q^4 + 4q^5$
G_{26}^{10}	$8, 10, 1, 0, 0$	$3, 1, 1, 3, 3$	10^3_{33}	$1 - 3q + 2q^2 + q^3 - 3q^4 + 5q^5$
G_{27}^{10}	$8, 10, 2, 0, 0$	$3, 2, 2, 2, 2$	10^3_{40}	$1 - 3q + q^2 + 3q^3 - q^4 + q^5$
G_{28}^{10}	$8, 10, 0, 3, 0$	$3, 0, -3, -4, -5$	10^2_{59}	$1 - 3q + 3q^2 + 2q^3 - 5q^4 + 2q^5$
G_{29}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 3, 5$	10^3_{37}	$1 - 3q + 3q^2 - 6q^4 + 7q^5$
G_{30}^{10}	$8, 10, 0, 0, 0$	$3, 0, 0, 4, 2$	10^2_{37}	$1 - 3q + 3q^2 - q^3 - 4q^4 + 10q^5$
G_{31}^{10}	$8, 10, 1, 0, 0$	$3, 1, 1, 2, 0$	10_{108}	$1 - 3q + 2q^2 + q^3 - 2q^4 + 5q^5$
G_{32}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -5$	10^2_{40}	$1 - 3q + 3q^2 + q^3 - 4q^4 + 5q^5$
G_{33}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -6$	10^4_3	$1 - 3q + 3q^2 + q^3 - 4q^4 + 6q^5$
G_{34}^{10}	$8, 10, 0, 3, 0$	$3, 0, -3, -4, -6$	10^4_{22}	$1 - 3q + 3q^2 + 2q^3 - 5q^4 + 3q^5$
G_{35}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -6$	10^4_3	$1 - 3q + 3q^2 + q^3 - 4q^4 + 6q^5$
G_{36}^{10}	$9, 10, 0, 1, 0$	$2, 0, -1, -1, -1$	10^3_{69}	$1 - 2q + q^2 + q^3 - q^4$
G_{37}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 1, 1$	10_{46}	$1 - 2q + q^2 - q^4 + q^5$
G_{38}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 0, -2$	10^3_{65}	$1 - 2q + q^2 + 2q^5$
G_{39}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 0, -1$	10_{61}	$1 - 2q + q^2 + q^5$
G_{40}^{10}	$10, 10, 0, 0, 0$	$1, 0, 0, 0, 0$	10^2_1	$1 - q$

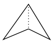
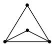
(a).

Fig. 9. The irreducible graphs G with 6 vertices, the vector $C = (C_1, \ldots, C_5)$, the alternating link L and the 6 stable coefficients of the Jones polynomial of L.

have

$$d_3 = c'_{40} + 2c_{42}.$$

(b) We have

$c_{40} - c'_{40} = [\![$  $]\!] - [\![$  $]\!]$

$= [\![$  $]\!]$

$= c_{41}. \quad \square$

G	C	L	$\Phi_L(q) + O(q^6)$
Gv_1^6	$3, 0, -6, -10, -7$	8_1^4	$1 - 3q + 3q^2 + 5q^3 - 8q^4 - 5q^5$
Gv_2^6	$2, 0, -1, 1, 2$	7_4^2	$1 - 2q + q^2 + q^3 - 3q^4 + q^5$
Gv_3^6	$4, 2, -3, -9, -13$	9_{12}^3	$1 - 4q + 4q^2 + 7q^3 - 13q^4 - 7q^5$
Gv_4^6	$1, 0, 0, 0, -1$	6_1^2	$1 - q + q^5$
Gv_5^6	$3, 2, 2, 4, 6$	8_5^3	$1 - 3q + q^2 + 3q^3 - 3q^4 + 3q^5$
Gv_6^6	$4, 3, 2, 3, 6$	9_{42}^2	$1 - 4q + 3q^2 + 6q^3 - 9q^4$
Gv_7^6	$5, 5, 5, 6, 11$	10_{121}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 7q^5$
Gv_8^6	$5, 5, 5, 6, 10$	10_{123}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 6q^5$
Gv_9^6	$5, 4, 0, -8, -20$	10_{17}^4	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
Gv_{10}^6	$3, 1, -1, 0, 2$	8_{16}	$1 - 3q + 2q^2 + 3q^3 - 6q^4 + q^5$
Gv_{11}^6	$4, 2, -2, -5, -5$	9_{34}	$1 - 4q + 4q^2 + 6q^3 - 13q^4 - 3q^5$
Gv_{12}^6	$5, 4, 0, -8, -20$	10_{155}^2	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
Gv_{13}^6	$5, 4, 0, -8, -20$	9_{40}	$1 - 4q + 4q^2 + 5q^3 - 13q^4 + q^5$
Gv_{14}^6	$5, 4, 0, -8, -20$	10_{137}^2	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
Gv_{15}^6	$6, 7, 8, 8, 9$	11_{314}	$1 - 6q + 8q^2 + 14q^3 - 29q^4 - 17q^5$
Gv_{16}^6	$6, 6, 3, -7, -28$	$L11a520$	$1 - 6q + 9q^2 + 13q^3 - 35q^4 - 17q^5$
Gv_{17}^6	$7, 10, 16, 25, 46$	$L12a1183$	$1 - 7q + 11q^2 + 19q^3 - 43q^4 - 33q^5$
Gv_{18}^6	$7, 8, 5, -13, -65$	$L12a2008$	$1 - 7q + 13q^2 + 16q^3 - 57q^4 - 28q^5$
Gv_{19}^6	$3, 0, -5, -7, -4$	8_{14}^2	$1 - 3q + 3q^2 + 4q^3 - 8q^4 - 2q^5$

(b).

Fig. 9. (*continued*)

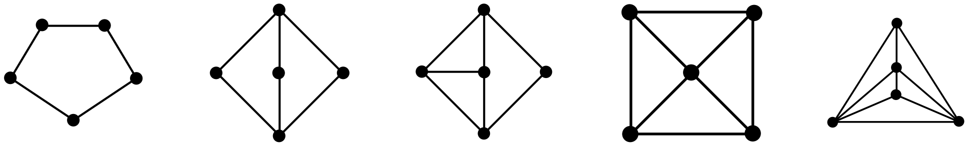


Fig. 10. The irreducible planar graphs Gv_i^5 for $i = 1, \ldots, 5$ (from the left to the right) with 5 vertices.

Therefore Eq. (37) combined with Lemmas 2.3 and 4.1 gives that

$$
\begin{aligned}
a_3 &= c_{41} - 2c_{42} - c_3c_1 + c_1 + c_2 + 2\left(\frac{c_1(c_1-1)}{2} - c_2\right) + \gamma \\
&\quad + \delta + c_2 + 2c_2(c_1 - 1) - 2\delta + 2c_2 \\
&= 2c_1c_2 + c_1^2 - c_3c_1 + c_{41} - 2c_{42} + \gamma - \delta \\
&= \frac{{c_1}^3}{6} + \frac{{c_1}^2}{2} + c_1c_2 - c_1c_3 + \frac{c_1}{3} + c_2 - c_3 + c_{41} - 2c_{42}.
\end{aligned}
$$

Therefore the coefficient $\phi_{G,3}$ of q^3 in $\Phi_G(q)$ is given by

$$
\begin{aligned}
\phi_{G,3} &= a_3 + b_3 + a_1b_2 + a_2b_1 - a_2 - b_2 - a_1b_1 \\
&= c_{41} - 2c_{42} + \frac{c_2}{6} + c_3c_2 - \frac{c_2^3}{6} - \frac{c_1}{6} - c_3c_1 + \frac{c_2^2c_1}{2} - \frac{c_2c_1^2}{2} + \frac{c_1^3}{6}.
\end{aligned}
$$

This completes the proof of Theorem 1.3. □

Acknowledgments

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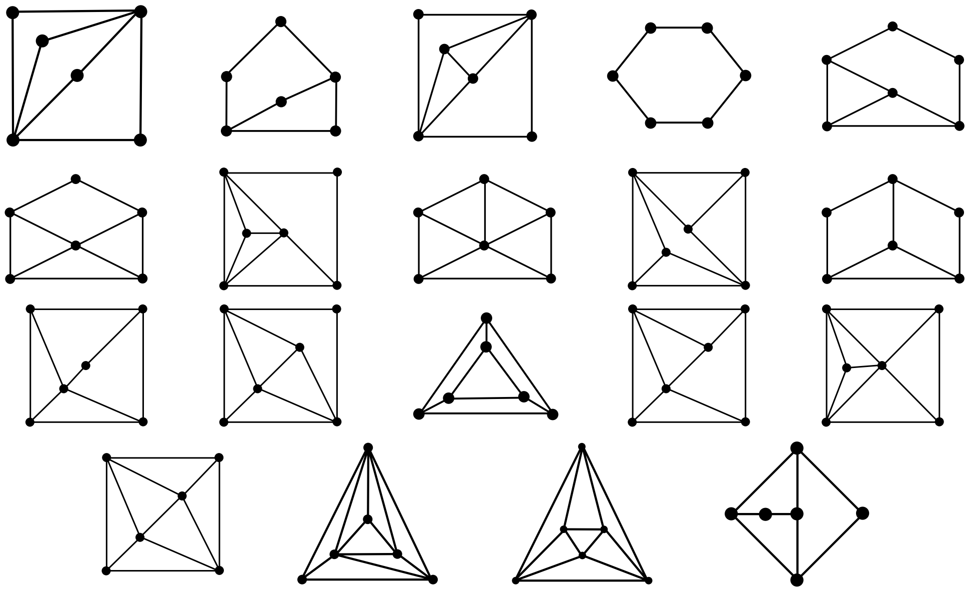

Fig. 11. The irreducible planar graphs Gv_i^6 for $i = 1, \ldots, 19$ (from the left to the right) with 6 vertices.

Appendix A. Computations

Figs. 6 and 9 illustrate Theorem 1.3 and confirm Conjecture 1.5 for all alternating links with at most 10 crossings and all irreducible planar graphs with at most 7 vertices. These tables were compiled as follows.

- We use `Sage` to list all irreducible planar graphs with at most 10 edges (using the notation of [6, Appendix A]).
- We use a `Mathematica` program to compute the corresponding vectors c and C and the series $\Phi_G(q) + O(q^4)$ of Theorem 1.3.
- To identify the corresponding alternating links L, we use a `Mathematica` program that converts the adjacency matrix of a planar graph G to the Dowker–Thistlethwaite code of the corresponding alternating link L, and then use `SnapPy` (see [11]) to identify the link with one of the Rolfsen's table [15] (if L has at most 10 crossings) or Thistlethwaite's table (if L has more than 10 crossings).
- We compute the stable coefficients $\Phi_L(q) + O(q^6)$ using `KnotAtlas` (see [2]) which computes the colored Jones polynomials of a link.

The equality $\Phi_G(q) = \Phi_L(q)$ of Theorem 1.3 is observed up to $O(q^4)$ and Conjecture 1.5 is verified for all such graphs (see Figs. 5, 7 and 8).

Remark A.1. If G is a connected planar graph with v vertices and e edges, the following inequalities bound e in terms of v and vice-versa

$$v \leq e \quad \text{and} \quad e \leq 3v - 6.$$

Appendix B. Tables of irreducible planar graphs

See Figs. 10–16.

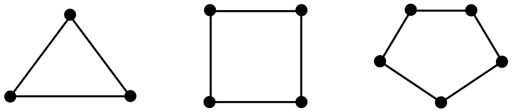

Fig. 12. The irreducible planar graphs G_0^3, G_0^4 and G_0^5 with 3, 4 and 5 edges.

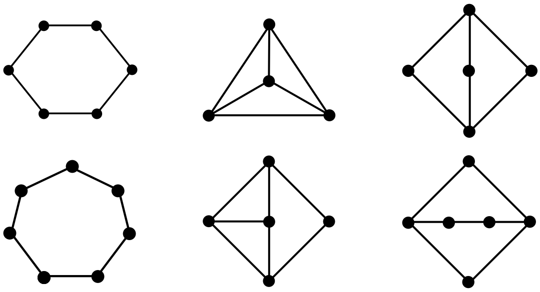

Fig. 13. The irreducible planar graphs with 6 and 7 edges: G_0^6, G_1^6, G_2^6 on the top and G_0^7, G_1^7, G_2^7 on the bottom.

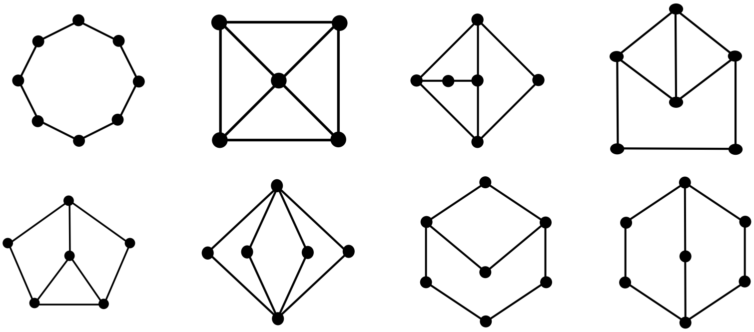

Fig. 14. The irreducible planar graphs with 8 edges: $G_0^8, \ldots, G_3^8$ on the top (from left to right) and $G_4^8, \ldots, G_7^8$ on the bottom.

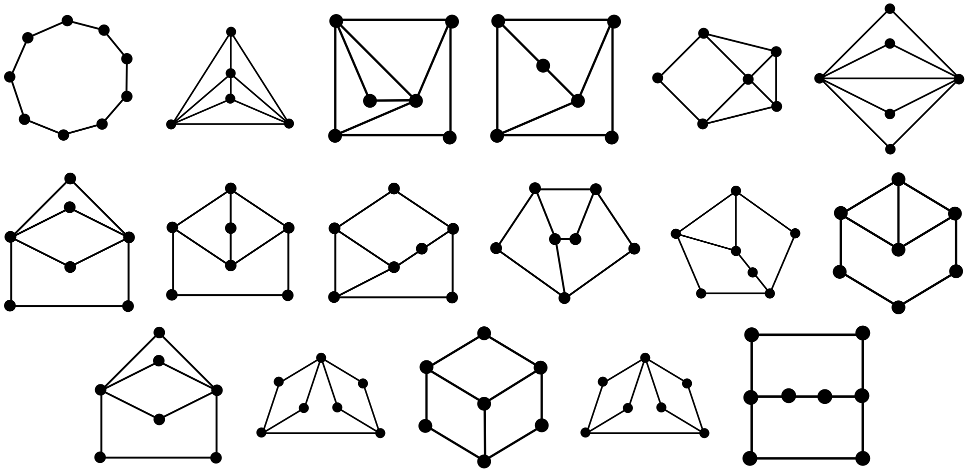

Fig. 15. The irreducible planar graphs with 9 edges: $G_0^9, \ldots, G_5^9$ on the top, $G_6^9, \ldots, G_{11}^9$ on the middle and $G_{12}^9, \ldots, G_{16}^9$ on the bottom.

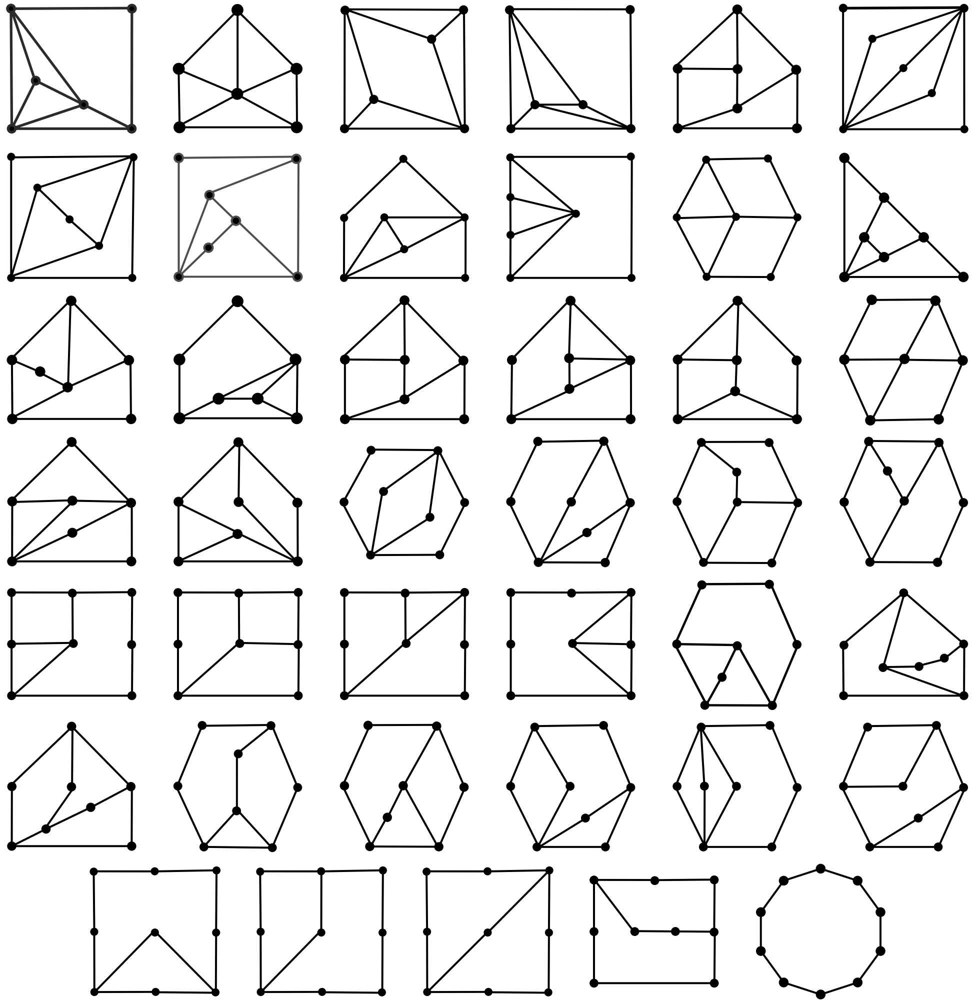

Fig. 16. The irreducible planar graphs with 10 edges: $G_0^{10}, \ldots, G_5^{10}$ on the top, $G_6^{10}, \ldots, G_{35}^{10}$ on the middle and $G_{36}^{10}, \ldots, G_{40}^{40}$ on the bottom.

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