Knots and tropical curves

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ABSTRACT. A sequence of rational functions in a variable q is q-holonomic if it satisfies a linear recursion with coefficients polynomials in q and q^n. In the paper, we assign a tropical curve to every q-holonomic sequence, which is closely related to the degree of the sequence with respect to q. In particular, we assign a tropical curve to every knot which is determined by the Jones polynomial of the knot and its parallels. The topical curve explains the relation between the AJ Conjecture and the Slope Conjecture (which relate the Jones polynomial of a knot and its parallels to the $SL(2, \mathbb{C})$ character variety and to slopes of incompressible surfaces). Our discussion predicts that the tropical curve is dual to a Newton subdivision of the A-polynomial of the knot. We compute explicitly the tropical curve for the 4_1, 5_2 and 6_1 knots and verify the above prediction.

1. Introduction

1.1. What is a q-holonomic sequence? A sequence of rational functions $f_n(q) \in \mathbb{Q}(q)$ in a variable q is *q-holonomic* if it satisfies a linear recursion with coefficients polynomials in q and q^n. In other words, we have

$$(1) \qquad \sum_{i=0}^{d} a_i(q^n, q) f_{n+i}(q) = 0$$

where the coefficients $a_i(M, q) \in \mathbb{Z}[M, q]$ are polynomials for $i = 0, \ldots, d$ where $a_d(M, q) \neq 0$. The term was coined by Zeilberger in [**Z**] and further studied in [**WZ**]. q-holonomic sequences appear in abundance in Enumerative Combinatorics; [**PWZ, St**]. The fundamental theorem of Wilf-Zeilberger states that a multi-dimensional finite sum of a (proper) q-hyper-geometric term is always q-holonomic; see [**WZ, Z, PWZ**]. Given this result, one can easily construct q-holonomic sequences. Combining this fundamental theorem with the fact that many *state-sum* invariants in Quantum Topology are multi-dimensional sums of the above shape,

it follows that Quantum Topology provides us with a plethora of q-holonomic sequences of *natural origin*; [**GL**]. For example, the sequence of *Jones polynomials* of a knot and its parallels which we will study below (technically, the colored Jones function) is q-holonomic.

The goal of our paper is to assign a tropical curve to a q-holonomic sequence. To motivate the connection between q-holonomic sequences and tropical curves, we will write Equation (1) in operator form using the operators M, L which act on a sequence $f_n(q) \in \mathbb{Q}(q)$ by

$$(Mf)_n(q) = q^n f_n(q), \qquad (Lf)_n(q) = f_{n+1}(q).$$

It is easy to see that $LM = qML$ generate the q-*Weyl algebra*

$$(2) \qquad\qquad W = \mathbb{Z}[q^{\pm 1}]\langle M, L \rangle / (LM - qML)$$

Equation (1) becomes

$$(3) \qquad\qquad\qquad\qquad Pf = 0$$

where

$$(4) \qquad\qquad\qquad P = \sum_{i=0}^{d} a_i(M, q) L^i \in W.$$

In other words, Equation (4) says that P annihilates f. Although a q-holonomic sequence f is annihilated by many operators $P \in W$, it was observed in [**Ga2**] that it is possible to canonically choose an operator P_f with coefficients $a_i(M, q) \in \mathbb{Z}[M, q]$. Likewise, there is a unique non-homogeneous linear recursion relation of the form $P_f f = b_f$ where $b_f \in \mathbb{Z}[M, q]$. For a detailed definition, see Section 2 below.

DEFINITION 1.1. We call P_f and (P_f^{nh}, b_f) the *homogeneous* and the *non-homogeneous* annihilator of the q-holonomic sequence f.

1.2. What is a tropical curve? In this section we will recall the definition of a tropical curve. For a survey on tropical curves, see [**RGST, SS**]. With those conventions, a *tropical polynomial* $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of the form:

$$(5) \qquad\qquad P(x, y) = \min\{a_1 x + b_1 y + c_1, \dots, a_r x + b_r y + c_r\}$$

where a_i, b_i, c_i are rational numbers for $i = 1, \dots, r$. P is convex and piecewise linear. The *tropical curve* $\mathcal{T}(P)$ of the tropical polynomial P is the set of points $(x, y) \in \mathbb{R}^2$ such that P is not linear at (x, y). Equivalently, $\mathcal{T}(P)$ is the set of points where the minimum is attained at two or more linear functions. A *rational graph* Γ is a finite union of rays and segments whose endpoints and directions are rational numbers, and each ray has a positive integer multiplicity. A *balanced rational graph* is defined in [**RGST**, Eqn.10]: at every vertex the sum of the slope vectors with multiplicities adds to zero. Every tropical curve is a balanced rational graph and vice-versa; see [**RGST**, Thm.3.6]. Tropical curves are very computable objects. For example, the vertices of a rational curve are the points (x, y) where the minimum in (5) is attained at least three times. The coordinates of such points can be solved by solving a system of linear equations. An explicit algorithm to compute the vertices and the slopes of a tropical curve is given in [**RGST**, Sec.3], and a computer implementation in `Singular` is available from [**Ma**]. This allows us to compute the tropical curves of the 4_1, 5_2 and 6_1 knots in Sections 5.1-5.4 below. In the case of the 6_1 knot, the non-homogeneous tropical curve is defined by an explicit polynomial with $x = 346$ terms.

Tropical curves arise from 2-variable polynomials $P_t(x, y)$ whose coefficients depend on an additional parameter t as follows. Consider

$$(6) \qquad P_t(x, y) = \sum_{i=1}^{r} \gamma_i(t) x^{a_i} y^{b_i}$$

where $\gamma_i(t)$ are algebraic functions of t with order at $t = 0$ equal to c_i. Then, the corresponding tropical polynomial is given by (5). $P_t(x, y)$ gives rise to two Newton polytopes:

- The 3-dimensional Newton polytope N_P, i.e., the convex hull of the exponents of (x, y, t) in $P_t(x, y)$.
- The 2-dimensional Newton polygon $N_{P,0}$, i.e., the convex hull of the exponents of (x, y) in $P_t(x, y)$.

In fact, $N_{P,0}$ is the image of N_P under the projection map $(x, y, t) \longrightarrow (x, y)$. The *lower faces* of N_P give rise to a Newton subdivision of $N_{P,0}$ which is combinatorially dual to the tropical curve $\mathcal{T}(P)$; see [**RGST**].

The polynomials $P_t(x, y)$ appear frequently in numerical problems of *Path Homotopy Continuation* where one is interested to connect $P_0(x, y)$ to $P_1(x, y)$. They also appear in *Quantization problems* in Physics, where t (or $\log t$) plays the role of Planck's constant. We will explain below that they also appear in Quantum Topology, and they are a natural companion of the AJ and the Slope Conjecture.

1.3. The tropical curve of a q-holonomic sequence. In this section we associate a tropical surve to a q-holonomic sequence. The main observation is that an element of the q-Weyl algebra is a polynomial in 3 variables M, L, q. Two of those q-commute (i.e., satisfy $LM = qML$) but we can always sort the powers of L to the right and the powers of M to the left. In other words, there is an *additive* map

$$(7) \qquad \mathbb{Z}[q^{\pm 1}]\langle M, L \rangle / (LM - qML) \longrightarrow \mathbb{Z}[M, L, q^{\pm 1}]$$

Let us change variables $(x, y, 1/t) = (L, M, q)$ and ignore the coefficients of the monomials of $x^i y^j t^k$, and record only their exponents. They give rise to a tropical curve. Explicitly, let

$$(8) \qquad P = \sum_{(i,j,k) \in \mathcal{A}} a_{i,j,k}\, q^k M^j L^i \in W$$

denote an element of the q-Weyl elgebra, where $\mathcal{A}$ is a finite set and $a_{i,j,k} \in \mathbb{Z} \setminus \{0\}$ for all $(i, j, k) \in \mathcal{A}$.

DEFINITION 1.2. There is a map

$$(9) \qquad W \longrightarrow \{\text{Tropical Curves in } \mathbb{R}^2\}, \qquad P \mapsto \Gamma_P$$

which assigns to P in (9) the tropical polynomial $P_t(x, y)$ given by:

$$P_t(x, y) = \min_{(i,j,k) \in \mathcal{A}} \{ix + jy - k\}$$

Γ_P is the tropical curve of $P_t(x, y)$.

Combining Definitions 1.1 and 1.2 allows us to assign a tropical curve to a q-holonomic sequence f.

DEFINITION 1.3. (a) If f is a q-holonomic sequence, let Γ_f and Γ_f^{nh} denote the tropical curves of $P_f(y, x, 1/t)$ and $P_f^{nh}(y, x, 1/t)$ respectively, where $P_f(M, L, q)$ and $P_f^{nh}(M, L, q)$ are given in Definition 1.1.

The tropical curve Γ_f of a q-holonomic sequence f is closely related to the degree (with respect to q) of the sequence of rational functions $f_n(q)$. If $\delta_n = \deg_q(f_n(q))$ denotes this degree, then it was shown in [**Ga4**] that for large enough n, δ_n is a quadratic quasi-polynomial with slope recorded by the rays of the tropical curve Γ_f.

1.4. 3 polytopes of a q-holonomic sequence. In this section we assign 3 polytopes to a q-holonomic sequence.

DEFINITION 1.4. (a) If $P \in W$ is given by Equation (8), it defines 3 polytopes:
- N_P is the convex hull of the exponents of the polynomial $P(M, L, q)$ with respect to the variables (M, L, q).
- $N_{P,0}$ is the projection of N_P under the projection map $(M, L, q) \longrightarrow (L, M)$.
- $N_{P,1}$ is the convex hull of the exponents of the polynomial $P(L, M, 1)$.

(b) If f is a q-holonomic sequence, its annihilator P_f gives rise to the polytopes N_{P_f}, $N_{P_f,0}$ and $N_{P_f,1}$.

Note that N_P is a 3-dimensional convex lattice polytope, and $N_{P,0}, N_{P,1}$ are 2-dimensional convex lattice polygons. Since every exponent of $P(M, L, 1)$ comes from some exponents of $P(M, L, q)$, it follows that

$$(10) \qquad\qquad\qquad N_{P,1} \subset N_{P,0}$$

REMARK 1.1. *It follows by* [**RGST**] *that the tropical curve Γ_P is dual to a Newton subdivision of $N_{P,0}$.*

We will say that $P(M, L, q)$ is *good* if $N_{P,1} = N_{P,0}$. It is easy to see that goodness is a generic property.

1.5. The slopes of a q-holonomic sequence. In this section we discuss the slopes of a q-holonomic sequence and their relation with its tropical curve. The proof of the following theorem uses differential Galois theory and the Lech-Mahler-Skolem theorem from number theory.

THEOREM 1.5. [**Ga4**] *The degree with respect to q of a q-holonomic sequence $f_n(q) \in \mathbb{Q}(q)$ is given (for large values of n) by a quadratic quasi-polynomial.*

Recall that a *quadratic quasi-polynomial* is a function of the form:

$$(11) \qquad p : \mathbb{N} \longrightarrow \mathbb{N}, \qquad p(n) = \gamma_2(n)\binom{n}{2} + \gamma_1(n)n + \gamma_0(n)$$

where $\gamma_j(n)$ are rational-valued periodic functions of n. Quasi-polynomials appear in lattice point counting problems, and also in Enumerative Combinatorics; see [**BP, BR, Eh, St**] and references therein.

The set of *slopes* $s(p)$ of a quadratic quasi-polynomial is the finite set of values of the periodic function $\gamma_2(n)$. These are essentially the quadratic growth rates of the quasi-polynomial. More precisely, recall that $x \in \mathbb{R}$ is a *cluster point* of a sequence (x_n) of real numbers if for every $\epsilon > 0$, there are infinitely many indices

$n \in \mathbb{N}$ such that $|x - x_n| < \epsilon$. Let $\{x_n\}'$ denote the set of *cluster points* of a sequence (x_n). It is easy to show that for every quadratic quasi-polynomial p we have:

$$(12) \qquad s(p) = \{\frac{2}{n^2} p(n) \mid n \in \mathbb{N}\}' \subset \mathbb{Q}$$

Given a q-holonomic sequence $f_n(q) \in \mathbb{Q}(q)$, let $s(f)$ denote the slopes of the quadratic quasi-polynomial $\deg_q f_n(q)$. Let $s(N)$ denote the set of slopes of the edges of a convex polygon N in the plane. The next proposition relates the slopes of a q-holonomic sequence with its tropical curve. See also [**Ga4**, Prop.4.4].

PROPOSITION 1.2. *If f is q-holonomic, then $s(f) \subset -s(N_{P_f,0})$.*

PROOF. Let $\delta(n) = \deg_q f_n(q)$ denote the degree of $f_n(q)$ with respect to q, and let P denote the annihilator of f. We expand P in terms of monomials as in Equation (8). For every monomial $q^k M^j L^i$ and every n we have

$$\deg_q((q^k M^j L^i) f_n(q)) = k + jn + \delta(n + i).$$

Since P annihilates f, for every n the following maximum is attained *at least twice* (from now on, twice will mean at least twice as is common in Tropical Geometry):

$$(13) \qquad \max_{(i,j,k)} \{jn + k + \delta(n + i)\}$$

Subtracting $\delta(n)$, it follows that the maximum is obtained twice:

$$(14) \qquad \max_{(i,j,k)} \{jn + k + \delta(n + i) - \delta(n)\}$$

Now $\delta(n)$ is a quadratic quasi-polynomial given by

$$\delta(n) = \gamma_2(n) \binom{n}{2} + \gamma_1(n)n + \gamma_0(n)$$

Theorem 1.5 implies that for large enough n, in a fixed arithmetic progression, we have $\gamma_i(n) = \widehat{\gamma}_i$ for $i = 1, 2$, thus

$$\delta(n + i) - \delta(n) = \widehat{\gamma}_2 \, i \, n + \widehat{\gamma}_2 \binom{i}{2} + \widehat{\gamma}_1 \, i$$

Substituting into (14), it follows that for large enough n in an arithmetic progression, the max is obtained twice:

$$(15) \qquad \max_{(i,j,k)} \{jn + k + \widehat{\gamma}_2 \, i \, n + \widehat{\gamma}_2 \binom{i}{2} + \widehat{\gamma}_1 \, i\}$$

It follows that there exists $(i', j') \neq (i, j)$ such that

$$(16) \qquad \widehat{\gamma}_2 = -\frac{j - j'}{i - i'}.$$

This proves Proposition 1.2.   □

2. The q-Weyl alegbra and its localization

In this section we will discuss some algebraic properties of the q-Weyl algebra and its localization, which will justify Definition 1.1.

Recall the q-Weyl algebra from (2). We will say that an element P of W is *reduced* if it has the form (4) where $a_i(M, q) \in \mathbb{Z}[M, q]$ for all i, and the greatest common divisor of $a_i(M, q) \in \mathbb{Z}[M, q]$ is 1.

Consider the *localized q-Weyl algebra* W_{loc} given by

$$(17) \qquad W_{\text{loc}} = \mathbb{Q}(M, q)\langle L \rangle / (Lf(M, q) - f(Mq, q)L)$$

It was observed in [**Ga2**] that W is not a principal left-ideal domain, but becomes so after localization; see [**Cou**]. If f is a sequence of rational functions, consider the left ideal M_f

$$M_f = \{P \in W_{\text{loc}} \,|\, Pf = 0\}$$

M_f is a principal ideal, which is nonzero if f is q-holonomic. Let P' denote the monic generator of M_f. Left multiply it by a polynomial in M, q so as to obtain a reduced annihilator P_f of f.

Now, we discuss non-homogeneous recursion relations of the form

$$\sum_{i=0}^{d} a_i(q^n, q) f_{n+i}(q) = b(q^n, q)$$

where $a_i(M, q), b(M, q) \in \mathbb{Q}(M, q)$ for all i. In operator form, we can write the above recursion as

$$Pf = b.$$

Consider the left ideal

$$(18) \qquad M_f^{nh} = \{P \in W_{\text{loc}} \,|\, \exists b \in \mathbb{Q}(M, q) \,:\, Pf = b\}$$

It is easy to see that M_f^{nh} is a left ideal. If f is q-holonomic, $M_f^{nh} \neq 0$. Let P'' denote the monic generator of M_f^{nh}. There exists $b'' \in \mathbb{Q}(M, q)$ such that

$$P'' f = b''$$

There are two cases: $b'' \neq 0$ or $b'' = 0$. If $b'' \neq 0$, then dividing by b'' we obtain that $1/b \cdot P'' f = 1$. We left multiply both sides by a polynomial in M, q so as to obtain $P_f^{nh} f = b_f$ where P^{nh} is reduced. If $b'' = 0$ then multiply by a polynomial in M, q so as to obtain $P_f^{nh} f = 0$ and define $b_f = 0$ in tha case. This concludes Definition 1.1.

The next lemma relates the homogeneous and the non-homogeneous annihilator of a q-holonomic sequence. It is well-known that one can convert an non-homogeneous recursion relation $Pf = b$ where $b \neq 0$ into a homogeneous recursion relation of order one more. Indeed, $Pf = b$ where $b \neq 0$ is equivalent to

$$(L - 1)b^{-1}Pf = 0$$

This implies the following conversion between (P_f^{nh}, b_f) and P_f. Fix a q-holonomic sequence $f_n(q) \in \mathbb{Q}(q)$.

LEMMA 2.1. (a) If $b_f = 0$ then $P_f^{nh} = P_f$. If $b_f \neq 0$, then P_f^{nh} is obtained by clearing denominators of $(L - 1)b_f^{-1}P_f$ by putting the powers of L on the right and the elements of $\mathbb{Q}(M, q)$ on the left.

(b) If P_f is not left divisible by $L - 1$ in W, then $P_f = P^{nh}$ and $b_f = 0$. If P_f is left

divisible by $L-1$ in W, then $P_f = (L-1)Q_f$ and if d is the common denominator of Q_f, then $(dQ_f, d) = (P_f^{nh}, b_f)$.

DEFINITION 2.2. We say that a q-holonomic sequence f is *homogeneous* if $b_f = 0$–else f is non-homogeneous.

In other words, a q-holonomic sequence f is *homogeneous* if and only if P_f is left-divisible by $L-1$ in W.

3. Quantum Topology

3.1. The tropical curve of a knot. Quantum Topology is a source of q-holonomic sequences attached to knotted 3-dimensional objects. Let $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ denote the *colored Jones polynomial* of a knot K in 3-space, colored by the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ and normalized to be 1 at the unknot; [**Jo, Tu**]. The sequence $J_{K,n}(q)$ for $n = 0, 1, \dots$ essentially encodes the Jones polynomial of a knot and all of its parallels; see [**Tu**]. In [**GL**, Thm.1] it was shown that the sequence $J_{K,n}(q)$ of colored Jones polynomials of a knot K is q-holonomic.

DEFINITION 3.1. (a) If K is a knot, we denote by $A_K(M, L, q)$ and $(A_K^{nh}(M, L, q), B_K(M, q))$ the homogeneous and the non-homogeneous annihilator of the q-holonomic sequence $J_{K,n}(q)$. These are the non-commutative and the non-homogeneous non-commutative A-polynomials of the knot.
(b) If K is a knot, let Γ_K and Γ_K^{nh} denote the *tropical curves* of A_K and A_K^{nh} respectively.

The non-homogeneous non-commutative A-polynomial of a knot appeared first in [**GS**].

3.2. The AJ Conjecture. The AJ Conjecture (resp. the Slope Conjecture) relates the Jones polynomial of a knot and its parallels to the $\mathrm{SL}(2, \mathbb{C})$ character variety (resp. to slopes of incompressible surfaces) of the knot complement. We will relate the two conjectures using elementary ideas from Tropical Geometry.

The A-polynomial of a knot is a polynomial in two commuting variables M and L that essentially encodes the image of the $\mathrm{SL}(2, \mathbb{C})$ character variety of K, projected in $\mathbb{C}^* \times \mathbb{C}^*$ by the eigenvalues of a meridian and longitude of K. It was defined in [**CCGLS**].

CONJECTURE 3.1. [**Ga2**] *The AJ Conjecture states that*

$$A_K(M, L, 1) = B_K(M) A_K(M^{1/2}, L) \tag{19}$$

where $A_K(M, L)$ is the A-polynomial of K and $B_K(M) \in \mathbb{Z}[M]$ is a polynomial that depends on M and of course K.

The AJ Conjecture is known for infinitely many 2-bridge knots; see [**Le**].

It is natural to ask whether the q-holonomic sequence $J_{K,n}(q)$ is of non-homogeneous type or not. Based on geometric information (the so-called *loop expansion* of the colored Jones polynomial, see [**Ga1**]), as well as experimental evidence for all knots whose non-commutative A-polynomial is known (these are the torus knots in [**Hi**] and the twist knots in [**GS**]) we propose the following conjecture.

CONJECTURE 3.2. *For every knot K, $J_{K,n}(q)$ is non-homogeneous.*

The above conjecture implies that $B_K(M, q) \in \mathbb{Z}[M, q] \setminus \{0\}$ is an invariant which is independent and *invisible* from the classical A-polynomial of the knot. There is a close connection between the $B_K(M, q)$ invariant of a knot and the torsion polynomial of the knot introduced in [**DbG**]. We will discuss this in a future publication.

3.3. The Slope Conjecture. The Slope Conjecture of [**Ga3**] relates the degree of the colored Jones polynomial of a knot and its parallels to slopes of incompressible surfaces in the knot complement. To recall the conjecture, let $\delta_K(n) = \deg_q J_{K,n}(q)$ (resp. $\delta_K^*(n) = \deg_q^* J_{K,n}(q)$) denote the maximum (resp. minimum) *degree* of the polynomial $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ (or more generally, of a rational function) with respect to q.

For a knot K, define the *Jones slopes* js_K by:

$$(20) \qquad js_K = \{\frac{2}{n^2}\delta_K(n) \mid n \in \mathbb{N}\}'$$

(b) Let $bs_K \subset \mathbb{Q} \cup \{1/0\}$ denote the set of boundary slopes of incompressible surfaces of K; [**Ha, HO**].

CONJECTURE 3.3. [**Ga3**] *The* Slope Conjecture *states that for every knot K we have*

$$(21) \qquad 2js_K \subset bs_K.$$

Note that the Slope Conjecture applied to the mirror of K implies that $2js_K^* \subset bs_K$. The Slope Conjecture is known for alternating knots and torus knots (see [**Ga3**]), for adequate knots (which include all alternating knots; see [**FKP**]), for $(-2, 3, n)$ pretzel knots (see [**Ga3**]), and for 2-fusion knots; see [**DnG**]. A general method for verifying the Slope Conjecture is discussed in [**Ga5, DnG**].

3.4. The AJ Conjecture and the Slope Conjecture. In this section we will see how the AJ Conjecture relates to the Slope Conjecture, expanding a comment of [**Ga3**, Sec.2]. We will specialize Definition 1.4 to knot theory when $P = A_K$ is the non-commutative A-polynomial of a knot K, and we will denote by N_K, $N_{K,0}$ and $N_{K,1}$ the three polytopes associated to A_K. Proposition 1.2 implies that

$$(22) \qquad js_K \subset -N_{K,0}$$

Let bs_K^A denote the slopes of the A-polynomial of K. The AJ Conjecture implies that up to possibly excluding the slope $1/0$ from $2N_{K,1}$, we have:

$$(23) \qquad 2N_{K,1} = bs_K^A.$$

For a careful proof, see Proposition 3.4 and Remark 3.5 below. Culler and Shalen show that edges of the Newton polygon of the A-polynomial of K give rise to ideal points of the $SL(2, \mathbb{C})$ character variety of K; see [**CS, CGLS, CCGLS**]. For every ideal point, Culler and Shalen construct an incompressible surface whose slope is a boundary slope of K; see [**CS, CCGLS**]. bs_K^A is the set of the so-called *strongly detected boundary slopes* of K, and satisfies the inclusion:

$$(24) \qquad bs_K^A \subset bs_K.$$

If $A_K(M, L, q)$ is good, then

$$(25) \qquad N_{K,0} = N_{K,0}$$

If K^* denotes the mirror of K, then $J_{K^*,n}(q) = K_{K,n}(q^{-1})$ which implies that $-N_{K,0} = N_{K^*,0}$. Combining Equations (22)-(25), it follows that

$$2\mathrm{js}_K \subset \mathrm{bs}_{K^*}$$

which is the Slope Conjecture, up to a harmless mirror image. This derivation also explains two independent factors of 2, one in Equation (20) and another one in Equation (19).

PROPOSITION 3.4. *If the non-commutative A-polynomial of K is good, and if the AJ Conjecture holds, then Γ_K is dual to the Newton subdivision of the A-polynomial of K (multiplied by a polynomial in M).*

PROOF. Let P denote the non-commutative A-polynomial of a knot K. Γ_K is dual to $N_{P,0}$. If P is good, then $N_{P,0} = N_{P,1}$. With the notation of Conjecture 3.1, the AJ Conjecture implies that

$$P(M, L, 1) = A_K(M^{1/2}, L)B_K(M)$$

where $B_K(M)$ is a polynomial of M, and A_K is the A-polynomial of A. The Newton polygon of of the product of two polynomials is the Minkowski sum of their Newton polygons. Moreover, the Newton polygon of $B_K(M)$ is a vertical line segment in the (L, M)-plane. It follows that the Newton polygon of $A_K(M^{1/2}, L)B_K(M)$ is the Newton polygon of the A-polynomial of K and its translation by a vertical segment. On the other hand, the Newton polygon of $P(M, L, 1)$ is $N_{P,1}$. The result follows. □

REMARK 3.5. *Note that the Newton polygon of $A_K(M^{1/2}, L)B_K(M)$ is the Newton polygon of $A_K(M^{1/2}, L)$ and its shift by a vertical line segment. It follows that the slopes of the Newton polygon of $A_K(M^{1/2}, L)B_K(M)$ are the slopes of $A_K(M^{1/2}, L)$ plus the slope of a vertical segment (i.e., $1/0$). For concrete examples, see Section 5 where the Newton polygons of the non-homogeneous A-polynomials of $4_1, 5_2, 6_1, 8_1$ is shown and it differs from the Newton polygon of the A-polynomial by a shift by a vertical segment.*

The only knots with explicitly known non-commutative A-polynomials (homogeneous and non-homogeneous) are the handful of twist knots K_p of [**GS**] for $p = -8, \ldots, 11$. An explicit check shows that these non-commutative A-polynomials (both the homogeneous and the non-homogeneous) are good. For details, see Section 5.

4. Quantization and Tropicalization

Quantization is the process of producing the non-commutative A-polynomial of a knot from the usual A-polynomial. In other words, Quantization starts with $P_1(x, y)$ and produces $P_t(x, y)$ as in Equation (6). On the other hand, Tropical Geometry expands $P_t(x, y)$ at $t = 0$ (or equivalently at $q = \infty$) and produces a tropical curve. Schematically, we have a diagram:

$$\begin{pmatrix} A\text{-polynomial} \\ q = 1 \end{pmatrix} \xleftarrow{\text{Classical limit}} \begin{pmatrix} \text{non-commutative} \\ A\text{-polynomial} \\ q \end{pmatrix} \xrightarrow{\text{Tropicalization}} \begin{pmatrix} \text{Tropical curve} \\ q = \infty \end{pmatrix}$$

Quantization is a map reverse to the Classical Limit map in the above diagram. Both sides of the above diagram (i.e., the limits at $q = 1$ and $q = \infty$) are classical

dual invariants of the knot. Indeed, the tropical curve ought to be dual to a Newton subdivision of the A-polynomial of K. This duality is highly nontrivial, even for the simple case of the 4_1 knot, computed in Section 5.1 below.

This conjectured duality may be related to the duality between Chern-Simons theory (i.e., colored U(N) polynomials of a knot) and Enumerative Geometry (i.e., BPS states) of the corresponding Calabi-Yau 3-fold. For a discussion of the latter duality, see [**ADKMV, DGKV, LMV, DV**] and references therein.

Physics principles concerning Quantization of complex Lagrangians in Chern-Simons theory suggest that the A-polynomial of a knot should determine the non-commutative A-polynomial. In particular, it should determine the polynomial invariant $B_K(M, q)$ of Definition 1.1, and it should determine the tropical curves Γ_K and Γ_K^{hn}.

Aside from duality conjectures, let us concentrate on a concrete question. It is well-known that the A-polynomial of a knot is a *triangulated curve* in the sense of *algebraic K-theory*. In other words, if X is the curve of zeros $A_K(M, L) = 0$ of the A-polynomial then there exist nonzero rational functions $z_1, \ldots, z_r \in C(X)^*$ in X such that

$$(26) \qquad M \wedge L = 2 \sum_{i=1}^{r} z_i \wedge (1 - z_i) \in \wedge_{\mathbb{Z}}^2(C(X)^*)$$

where $C(X)$ is the field of rational functions of X and $M, L \in C(X)^*$ are the eigenvalues of the meridian and the longitude. For a proof of (26) (which uses the symplectic nature of the so-called Neumann-Zagier matrices), see [**Ch**, Lem.10.1]. For an excellent discussion of triangulated curves X and for a plethora of examples and computations, see [**BRVD**]. Geometrically, a triangulation of X comes from an ideal triangulation of the knot complement with r ideal tetrahedra with shape parameters $z_1, \ldots, z_r$ which satisfy some gluing equations. The symplectic nature of these gluing equations, introduced and studied by Neumann and Zagier in [**NZ**], implies (26). The triangulation of X has important arithmetic consequences regarding the *volume* of the knot complement and its Dehn fillings, and it is closely related to the *Bloch group* of the complex numbers. It is important to realize that X has infinitely many triangulations, and in general it is not possible to choose a canonical one. In addition, triangulations tend to work well with hyperbolic knots. On the contrary, the non-commutative A-polynomial and its corresponding tropical curve exist for every knot in 3-space, hyperbolic or not. Let us end with some questions, which aside from its theoretical interest, may play a role in the Quantization of the A-polynomial.

QUESTION 4.1. *Is the tropical curve Γ_K of a hyperbolic knot K related to a triangulation of its A-polynomial curve?*

To formulate our next question, recall that the tropical curve Γ_K is dual to a Newton subdivision of the 2-dimensional Newton polytope of the polynomial $A_K(M, L, q)$ with respect to the variables L and M. Assuming that $A_K(M, L, q)$ is good, and assuming the AJ Conjecture, it follows that Γ_K is dual to the Newton polygon of the A-polynomial of K. Γ_K is a balanced rational graph that consists or edges and rays, and the above assumptions imply that the slopes of the rays

are negative inverses of the slopes of the A-polynomial of K. Consequently, Culler-Shalen theory (see [**CS**]) implies that the slopes of the rays of Γ_K are negative inverses of boundary slopes of K, appropriately normalized by a factor of 2.

QUESTION 4.2. *What is the geometric meaning of the vertices of Γ_K (those are points in $\mathbb{Q}^2$) and of the slopes of the edges of Γ_K?*

5. Computations of tropical curves of knots

5.1. The homogeneous tropical curve of the 4_1 knot. The non-commutative A-polynomial $A_{4_1}(M, L, q)$ of 4_1 was computed in [**GL**, Sec.6.2] and also [**Ga2**, Sec.3.2] using the WZ method of [**WZ, Z**] implemented by [**PR**] in Mathematica. The non-commutative A-polynomial is given by

$$
\begin{aligned}
A_{4_1}(y, x, 1/t) \;=\; & \frac{x^3\,(t^2 - y)\,(t^3 - y)\,y^2(t + y)\,(t - y^2)\,(t^3 - y^2)}{t^{14}} \\
& + \frac{(t^2 - y)\,(-1 + y)y^2\,(t^2 + y)\,(t^3 - y^2)\,(t^5 - y^2)}{t^{15}} \\
& - \frac{1}{t^{18}}x^2\,(t^2 - y)^2\,(t^2 + y)\,(t - y^2)\,(t^3 - y^2) \\
& \quad \cdot (t^8 - 2t^6 y + t^7 y - t^3 y^2 + t^4 y^2 - t^5 y^2 + t y^3 - 2t^2 y^3 + y^4) \\
& + \frac{1}{t^{17}}x(t - y)\,(t^2 - y)\,(t + y)\,(t^3 - y^2)\,(t^5 - y^2) \\
& \quad \cdot (t^4 + y^4 - t^3 y(2 + y) - t y^2(1 + 2y) + t^2 y\,(1 + y + y^2))
\end{aligned}
$$

Notice that

$$
A_{4_1}(x, y, 1) = (-1 + x)(-1 + y)^4(1 + y)^3(-x + xy + y^2 + 2xy^2 + x^2 y^2 + xy^3 - xy^4)
$$

confirms the AJ Conjecture, since the last factor is the geometric component of the A-polynomial of 4_1, the first term is the abelian component of the A-polynomial, and the remaining second and third terms depend only on $y = M$. Expanding out the terms, we obtain that:

$A_{4_1}(y, x, 1/t) = \frac{1}{t^{18}} \cdot x^2 y^{11} + \frac{-1}{t^{14}} \cdot x^3 y^9 + \frac{1-3 \cdot t}{t^{17}} \cdot x^2 y^{10} + \frac{-1}{t^{17}} \cdot xy^{11} + \frac{-1+t+t^2}{t^{17}} \cdot x^3 y^8 + \frac{-1-3 \cdot t^2 + 2 \cdot t^3 - t^4}{t^{17}} \cdot x^2 y^9 + \frac{2}{t^{16}} \cdot xy^{10} + \frac{1+2 \cdot t^2 + t^3 - t^4}{t^{13}} \cdot x^3 y^7 + \frac{-1+3 \cdot t - t^2 + 3 \cdot t^3 + 2 \cdot t^5}{t^{16}} \cdot x^2 y^8 + \frac{1+t^3+t^4}{t^{16}} \cdot xy^9 + \frac{1-t-t^3 - 2 \cdot t^4}{t^{12}} \cdot x^3 y^6 + \frac{3-2 \cdot t + 3 \cdot t^2 + t^4}{t^{14}} \cdot x^2 y^7 + \frac{-1-t-t^2 - t^3 - 2 \cdot t^4}{t^{15}} \cdot xy^8 + \frac{1}{t^{15}} \cdot y^9 + \frac{-2-t-t^3+t^4}{t^{10}} \cdot x^3 y^5 + \frac{1-3 \cdot t - 2 \cdot t^3 - t^4 - 2 \cdot t^6}{t^{13}} \cdot x^2 y^6 + \frac{-1-t^2 - t^3 - t^4 - t^6}{t^{14}} \cdot xy^7 + \frac{-1}{t^{15}} \cdot y^8 + \frac{-1+t+2 \cdot t^2 + t^4}{t^9} \cdot x^3 y^4 + \frac{-2-t^2 - 2 \cdot t^3 - 3 \cdot t^5 + t^6}{t^{11}} \cdot x^2 y^5 + \frac{1+t^2+t^3 + 2 \cdot t^4 + t^5 + t^6}{t^{13}} \cdot xy^6 + \frac{-1-t-t^2}{t^{12}} \cdot y^7 + \frac{1+t-t^2}{t^7} \cdot x^3 y^3 + \frac{1+3 \cdot t^2 - 2 \cdot t^3 + 3 \cdot t^4}{t^8} \cdot x^2 y^4 + \frac{1+t+2 \cdot t^2 + t^3 + t^4 + t^6}{t^{11}} \cdot xy^5 + \frac{1+t+t^2}{t^{12}} \cdot y^6 + \frac{-1}{t^{12}} \cdot x^3 y^2 + \frac{2+3 \cdot t^2 - t^3 + 3 \cdot t^4 - t^5}{t^7} \cdot x^2 y^3 + \frac{-1-t^2 - t^3 - t^4 - t^6}{t^{10}} \cdot xy^4 + \frac{1+t+t^2}{t^8} \cdot y^5 + \frac{-1+2 \cdot t - 3 \cdot t^2 - t^4}{t^5} \cdot x^2 y^2 + \frac{-2-t-t^2 - t^3 - t^4}{t^7} \cdot xy^3 + \frac{-1-t-t^2}{t^8} \cdot y^4 + \frac{-3+t}{t^2} \cdot x^2 y + \frac{1+t+t^4}{t^6} \cdot xy^2 + \frac{-1}{t^3} \cdot y^3 + x^2 + \frac{2}{t^2} \cdot xy + \frac{1}{t^3} \cdot y^2 + \frac{-1}{t} \cdot x$

Inspection of the above formula shows that $A_{4_1}(y, x, 1/t)$ is good. Using the drawing polymake program of [**Ma**] implemented in Singular one can compute the vertices of the tropical curve:
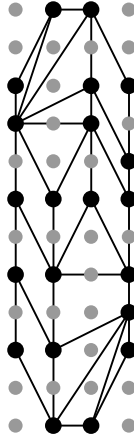
$$(3, -1/2),\ (-1, -1/3),\ (-3/4, -1/2),\ (-2, 0),\ (2, -1),\ (-1/2, -1),\ (1, -3/2),$$

$$(0, -3/2),\ (-1/2, -5/4),\ (1/2, -7/4),\ (-1, -3/2),\ (1/2, -2),\ (2, -3),\ (3/4, -5/2),$$

$$(1, -8/3),\ (-2, -2),\ (-3, -5/2),$$

The tropical curve (with the convention that unmarked edges or rays have multiplicity 1) is:
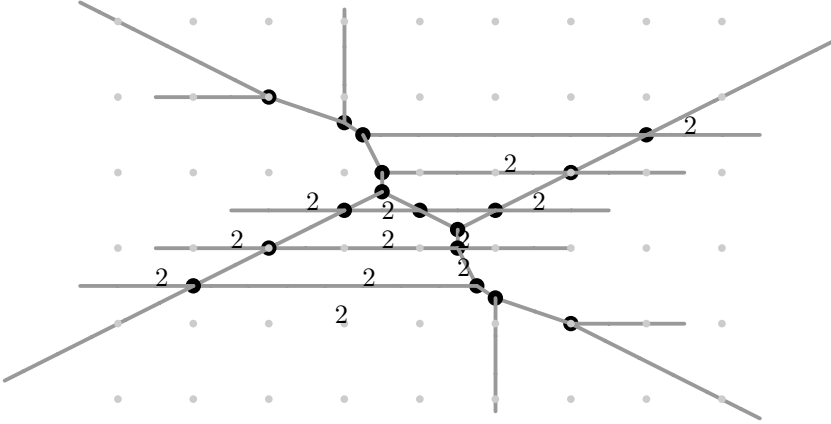
The Newton subdivision of the Newton polygon is:

The reader may observe that the above Newton polygon is the Minkowski sum of the Newton polygon of the A-polynomial of 4_1 with a vertical segment.

5.2. The non-homogeneous tropical curve of the 4_1 knot. The non-homogeneous A-polynomial of the 4_1 knot was computed in Theorem 1 of [**GS**] (with the notation $A_{-1}(E, Q, q)$ where $E = L$ and $Q = M$). It has 22 terms and it is given by:

$$
\begin{aligned}
A^{nh}_{4_1}(M, L, q) &= L^2 M^2 q^2 \left(-1 + M^2 q\right)\left(-1 + M q^2\right) + (-1 + M)M^2 q^2 \left(-1 + M^2 q^3\right) \\
&\quad -L(-1 + Mq)^2(1 + Mq)\left(1 - Mq - M^2 q - M^2 q^3 - M^3 q^3 + M^4 q^4\right) \\
B_{4_1}(M, L) &= Mq(1 + Mq)\left(-1 + M^2 q\right)\left(-1 + M^2 q^3\right)
\end{aligned}
$$

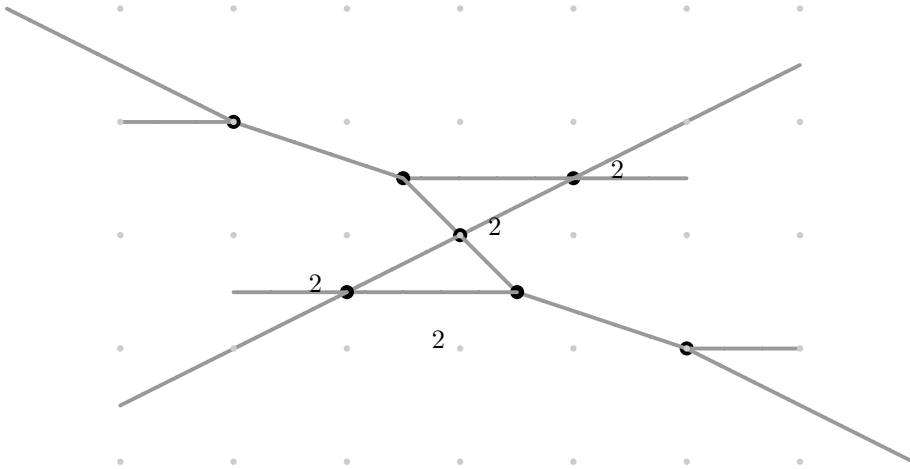It follows that:

$A^{nh}_{4_1}(y, x, 1/t) = \frac{-1}{t^7} \cdot xy^7 + \frac{1}{t^5} \cdot x^2 y^5 + \frac{2}{t^6} \cdot xy^6 + \frac{-1}{t^3} \cdot x^2 y^4 + \frac{1+t^2}{t^6} \cdot xy^5 + \frac{-1}{t^4} \cdot x^2 y^3 + \frac{-1-t-t^2}{t^5} \cdot xy^4 + \frac{1}{t^5} \cdot y^5 + \frac{1}{t^2} \cdot x^2 y^2 + \frac{-1-t-t^2}{t^4} \cdot xy^3 + \frac{-1}{t^5} \cdot y^4 + \frac{1+t^2}{t^3} \cdot xy^2 + \frac{-1}{t^2} \cdot y^3 + \frac{2}{t} \cdot xy + \frac{1}{t^2} \cdot y^2 - x$
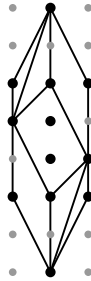
It is easy to see that the above polynomial is good. The vertices of the corresponding tropical curve are:

$$(1, -1/2), \ (-1/2, -1/2), \ (-2, 0), \ (0, -1), \ (2, -2), \ (1/2, -3/2), \ (-1, -3/2)$$

The tropical curve is:

The Newton subdivision of the Newton polygon is:

This example exhibits that the non-homogeneous tropical curve is much simpler than the homogeneous one.

5.3. The non-homogeneous tropical curve of the 5_2 knot. The non-homogeneous non-commutative A-polynomial $A_{5_2}^{nh}(M, L, q)$ has 98 terms, and it is given by [**GS**] (with the notation $A_2^{nh}(E, Q, q)$ where $E = L$, $Q = M$):

$A_{5_2}^{nh}(y, x, 1/t) = \frac{-1}{t^{19}} \cdot xy^{12} + \frac{1}{t^{17}} \cdot x^2 y^{10} + \frac{3}{t^{18}} \cdot xy^{11} + \frac{1}{t^{18}} \cdot y^{12} + \frac{-2}{t^{15}} \cdot x^2 y^9 + \frac{1-t+t^3+t^4}{t^{18}} \cdot xy^{10} + \frac{-1}{t^{18}} \cdot y^{11} + \frac{-1-t+t^2+t^3-t^4}{t^{16}} \cdot x^2 y^8 + \frac{-2-2\cdot t+t^2-2\cdot t^3-3\cdot t^4}{t^{17}} \cdot xy^9 + \frac{-1-t}{t^{14}} \cdot y^{10} + \frac{2+2\cdot t-t^2+t^3+2\cdot t^4}{t^{14}} \cdot x^2 y^7 + \frac{1-2\cdot t-t^2+t^3-t^5}{t^{15}} \cdot xy^8 + \frac{1+t}{t^{14}} \cdot y^9 + \frac{1}{t^6} \cdot x^3 y^5 + \frac{1-t-t^2+t^3+t^4-2\cdot t^5}{t^{14}} \cdot x^2 y^6 + \frac{2-t+t^2+4\cdot t^3+2\cdot t^4-t^5+2\cdot t^6}{t^{15}} \cdot xy^7 + \frac{1}{t^9} \cdot y^8 + \frac{-1}{t^3} \cdot x^3 y^4 + \frac{-2+t-t^2-4\cdot t^3-2\cdot t^4+t^5-2\cdot t^6}{t^{12}} \cdot x^2 y^5 + \frac{-1+t+t^2-t^3-t^4+2\cdot t^5}{t^{14}} \cdot xy^6 + \frac{-1}{t^9} \cdot y^7 + \frac{-1-t}{t^5} \cdot x^3 y^3 + \frac{-1+2\cdot t+t^2-t^3+t^5}{t^9} \cdot x^2 y^4 + \frac{-2-2\cdot t+t^2-t^3-2\cdot t^4}{t^{11}} \cdot xy^5 + \frac{1+t}{t^2} \cdot x^3 y^2 + \frac{2+2\cdot t-t^2+2\cdot t^3+3\cdot t^4}{t^8} \cdot x^2 y^3 + \frac{1+t-t^2-t^3+t^4}{t^{10}} \cdot xy^4 + \frac{1}{t^3} \cdot x^3 y + \frac{-1+t-t^3-t^4}{t^6} \cdot x^2 y^2 + \frac{2}{t^6} \cdot xy^3 - x^3 + \frac{-3}{t^3} \cdot x^2 y + \frac{-1}{t^5} \cdot xy^2 + \frac{1}{t} \cdot x^2$
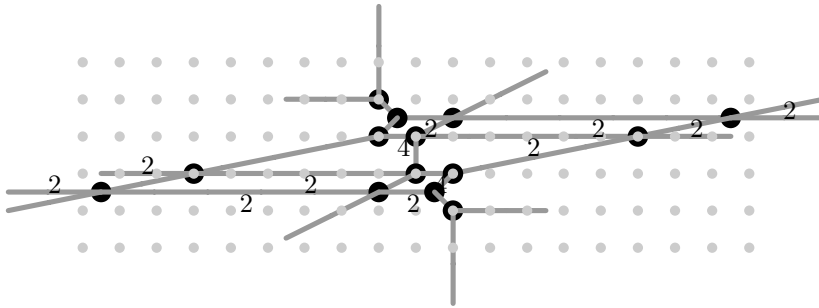
The vertices of the tropical curve are:

$(1, -1/2), \ (-1, 0), \ (-1/2, -1/2), \ (17/2, -1/2), \ (-1, -1), \ (0, -1), \ (-6, -2),$

$$(6, -1), \ (-17/2, -5/2), \ (0, -2), \ (1, -2), \ (-1, -5/2), \ (1/2, -5/2), \ (1, -3)$$

The Newton subdivision of the tropical curve is:
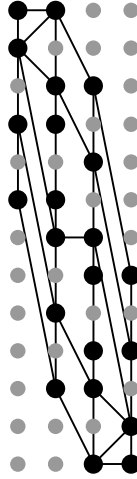
The tropical curve is:

5.4. The non-homogeneous tropical curve of the 6_1 knot.

The non-homogeneous non-commutative A-polynomial $A_{6_1}^{nh}(M, L, q)$ has 346 terms, and it is given by [**GS**] (with the notation $A_{-2}^{nh}(E, Q, q)$ where $E = L$, $Q = M$):

$$A_{6_1}^{nh}(y, x, 1/t) = \frac{1}{t^{31}} \cdot x^2 y^{15} + \frac{-1-t}{t^{28}} \cdot x^3 y^{13} + \frac{-1-3\cdot t}{t^{30}} \cdot x^2 y^{14} + \frac{-1}{t^{30}} \cdot xy^{15} + \frac{1}{t^{22}} \cdot$$

$$x^4 y^{11} + \frac{1+3\cdot t+t^2}{t^{26}} \cdot x^3 y^{12} + \frac{-1-t+2\cdot t^2+2\cdot t^3-t^4-t^5-t^6}{t^{30}} \cdot x^2 y^{13} + \frac{2}{t^{29}} \cdot xy^{14} + \frac{-1}{t^{18}} \cdot x^4 y^{10} +$$

$$\frac{1+2\cdot t+2\cdot t^2+t^3-t^4-2\cdot t^5+2\cdot t^6+t^7}{t^{27}} \cdot x^3 y^{11} + \frac{1+4\cdot t+3\cdot t^2-t^3+2\cdot t^5+4\cdot t^6+3\cdot t^7}{t^{29}} \cdot x^2 y^{12} +$$

$$\frac{1-t-t^2+t^4+t^5+t^6}{t^{29}} \cdot xy^{13} + \frac{-1-t-t^2}{t^{21}} \cdot x^4 y^9 + \frac{-1-4\cdot t-4\cdot t^2-3\cdot t^3-2\cdot t^6-3\cdot t^7-t^8}{t^{25}} \cdot x^3 y^{10} +$$

$$\frac{1-2\cdot t-3\cdot t^2+3\cdot t^4+3\cdot t^5+t^6-4\cdot t^7-t^8+t^9+t^{10}}{t^{28}} \cdot x^2 y^{11} + \frac{-2+2\cdot t^2-2\cdot t^4-2\cdot t^5-2\cdot t^6}{t^{28}} \cdot xy^{12} + \frac{1+t+t^2}{t^{17}} \cdot$$

$$x^4 y^8 + \frac{-1-2\cdot t-2\cdot t^2+3\cdot t^4-2\cdot t^6-4\cdot t^7-t^8+3\cdot t^9+t^{10}-t^{11}}{t^{25}} \cdot x^3 y^9 +$$

$$\frac{-1-3\cdot t+t^2-3\cdot t^4-6\cdot t^5-6\cdot t^6-4\cdot t^7+t^8-t^9-2\cdot t^{10}-3\cdot t^{11}}{t^{27}} \cdot x^2 y^{10} + \frac{1+2\cdot t-t^3+t^5+2\cdot t^6+t^7-t^8-t^9-t^{10}}{t^{27}} \cdot$$

$$xy^{11} + \frac{1+t+t^2}{t^{19}} \cdot x^4 y^7 + \frac{1+3\cdot t+3\cdot t^2+2\cdot t^3+2\cdot t^5+5\cdot t^6+5\cdot t^7+3\cdot t^8-t^{10}+t^{11}+t^{12}}{t^{23}} \cdot x^3 y^8 +$$

$$\frac{1+t-2\cdot t^2-2\cdot t^3-2\cdot t^4+2\cdot t^5-t^7-3\cdot t^8-3\cdot t^9+2\cdot t^{11}-t^{12}}{t^{27}} \cdot x^2 y^9 +$$

$$\frac{1-3\cdot t-t^2+3\cdot t^3+3\cdot t^4+t^5-2\cdot t^6-2\cdot t^7+2\cdot t^8+2\cdot t^9+2\cdot t^{10}}{t^{26}}\cdot xy^{10}+\frac{1}{t^{26}}\cdot y^{11}+\frac{-1-t-t^2}{t^{15}}\cdot x^4y^6+$$

$$\frac{1+t-t^2-2\cdot t^3+t^4+3\cdot t^5+4\cdot t^6-2\cdot t^7-4\cdot t^8-2\cdot t^9+t^{10}+2\cdot t^{11}-2\cdot t^{13}}{t^{22}}\cdot x^3y^7+$$

$$\frac{1+2\cdot t+2\cdot t^2+3\cdot t^3+t^4+3\cdot t^5+3\cdot t^6+3\cdot t^7+3\cdot t^8+t^9+t^{11}}{t^{22}}\cdot x^2y^8+$$

$$\frac{-2-2\cdot t^3-4\cdot t^4-4\cdot t^5-2\cdot t^6+t^7-t^9-2\cdot t^{10}-t^{11}+t^{13}}{t^{25}}\cdot xy^9+\frac{-1}{t^{26}}\cdot y^{10}+\frac{-1}{t^{16}}\cdot x^4y^5+$$

$$\frac{-2-2\cdot t^3-4\cdot t^4-4\cdot t^5-2\cdot t^6+t^7-t^9-2\cdot t^{10}-t^{11}+t^{13}}{t^{19}}\cdot x^3y^6+$$

$$\frac{1+2\cdot t+2\cdot t^2+3\cdot t^3+t^4+3\cdot t^5+3\cdot t^6+3\cdot t^7+3\cdot t^8+t^9+t^{11}}{t^{20}}\cdot x^2y^7+$$

$$\frac{1+t-t^2-2\cdot t^3+t^4+3\cdot t^5+4\cdot t^6-2\cdot t^7-4\cdot t^8-2\cdot t^9+t^{10}+2\cdot t^{11}-2\cdot t^{13}}{t^{24}}\cdot xy^8+\frac{-1-t-t^2}{t^{21}}\cdot y^9+\frac{1}{t^{12}}\cdot$$

$$x^4y^4+\frac{1-3\cdot t-t^2+3\cdot t^3+3\cdot t^4+t^5-2\cdot t^6-2\cdot t^7+2\cdot t^8+2\cdot t^9+2\cdot t^{10}}{t^{16}}\cdot x^3y^5+$$

$$\frac{1+t-2\cdot t^2-2\cdot t^3-2\cdot t^4+2\cdot t^5-t^7-3\cdot t^8-3\cdot t^9+2\cdot t^{11}-t^{12}}{t^{19}}\cdot x^2y^6+$$

$$\frac{1+3\cdot t+3\cdot t^2+2\cdot t^3+2\cdot t^5+5\cdot t^6+5\cdot t^7+3\cdot t^8-t^{10}+t^{11}+t^{12}}{t^{21}}\cdot xy^7+\frac{1+t+t^2}{t^{21}}\cdot y^8+$$

$$\frac{1+2\cdot t-t^3+t^5+2\cdot t^6+t^7-t^8-t^9-t^{10}}{t^{13}}\cdot x^3y^4+\frac{-1-3\cdot t+t^2-3\cdot t^4-6\cdot t^5-6\cdot t^6-4\cdot t^7+t^8-t^9-2\cdot t^{10}-3\cdot t^{11}}{t^{17}}\cdot$$

$$x^2y^5+\frac{-1-2\cdot t-2\cdot t^2+3\cdot t^4-2\cdot t^6-4\cdot t^7-t^8+3\cdot t^9+t^{10}-t^{11}}{t^{19}}\cdot xy^6+\frac{1+t+t^2}{t^{15}}\cdot y^7+$$

$$\frac{-2+2\cdot t^2-2\cdot t^4-2\cdot t^5-2\cdot t^6}{t^{10}}\cdot x^3y^3+\frac{1-2\cdot t-3\cdot t^2+3\cdot t^4+3\cdot t^5+t^6-4\cdot t^7-t^8+t^9+t^{10}}{t^{14}}\cdot x^2y^4+$$

$$\frac{-1-4\cdot t-4\cdot t^2-3\cdot t^3-2\cdot t^6-3\cdot t^7-t^8}{t^{15}}\cdot xy^5+\frac{-1-t-t^2}{t^{15}}\cdot y^6+\frac{1-t-t^2+t^4+t^5+t^6}{t^7}\cdot x^3y^2+$$

$$\frac{1+4\cdot t+3\cdot t^2-t^3+2\cdot t^5+4\cdot t^6+3\cdot t^7}{t^{11}}\cdot x^2y^3+\frac{1+2\cdot t+2\cdot t^2+t^3-t^4-2\cdot t^5+2\cdot t^6+t^7}{t^{13}}\cdot xy^4+\frac{-1}{t^8}\cdot y^5+\frac{2}{t^3}\cdot$$

$$x^3y+\frac{-1-t+2\cdot t^2+2\cdot t^3-t^4-t^5-t^6}{t^8}\cdot x^2y^2+\frac{1+3\cdot t+t^2}{t^8}\cdot xy^3+\frac{1}{t^8}\cdot y^4-x^3+\frac{-1-3\cdot t}{t^4}\cdot x^2y+$$

$$\frac{-1-t}{t^6}\cdot xy^2+\frac{1}{t}\cdot x^2$$
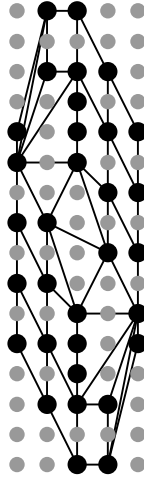
The vertices of the tropical curve are:

$(2,-1/2)$, $(-1,-1/2)$, $(5,-1/2)$, $(-3/2,-1/2)$, $(-4,0)$, $(1,-1)$, $(-1/2,-1)$,

$(-1,-2/3)$, $(4,-1)$, $(1/2,-3/2)$, $(3,-3/2)$, $(1/5,-8/5)$, $(-1/2,-5/4)$,

$(1/2,-11/4)$, $(-1/5,-12/5)$, $(-3,-5/2)$, $(4,-4)$, $(1/2,-3)$, $(1,-10/3)$,

$(3/2,-7/2)$, $(-1/2,-5/2)$, $(-4,-3)$, $(-1,-3)$,

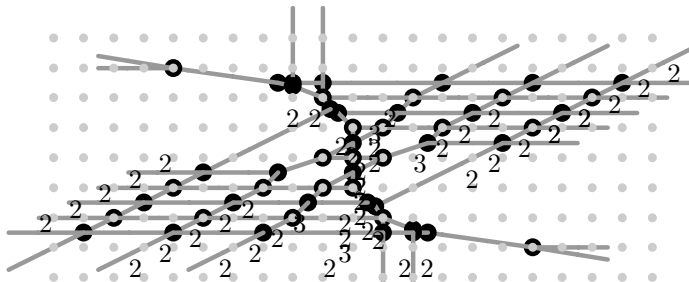$(-5,-7/2)$, $(1,-7/2)$, $(-2,-7/2)$

The tropical curve is:
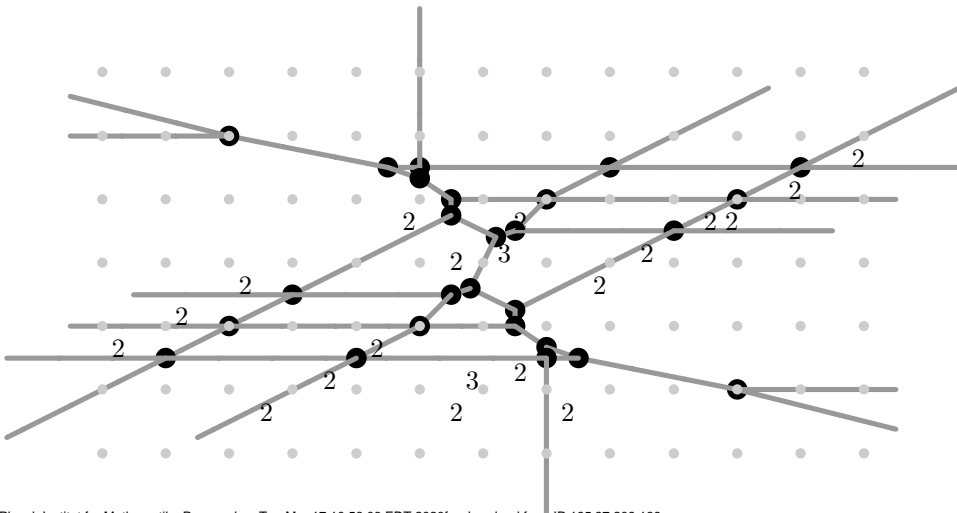
The Newton subdivision of the tropical curve is:

5.5. The non-homogeneous tropical curve of the 8_1 knot. The non-homogeneous non-commutative A-polynomial $A_{8_1}^{nh}(M, L, q)$ has 2112 terms, which we not present here. The vertices of the tropical curve are:
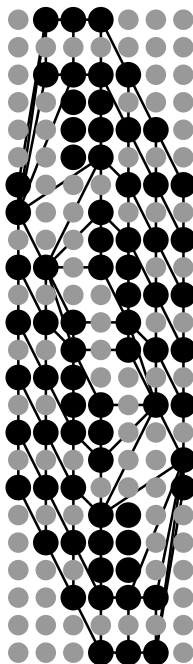
$$(3, -1/2), \ (-1, -1/2), \ (6, -1/2), \ (-2, -1/2), \ (9, -1/2), \ (2, -1), \ (-1, -1),$$
$$(-5/2, -1/2), \ (-6, 0), \ (5, -1), \ (-2, -3/5), \ (8, -1), \ (3/2, -3/2), \ (4, -3/2),$$
$$(-1/2, -3/2), \ (-3/4, -11/8), \ (7, -3/2), \ (1, -2), \ (3, -2), \ (0, -2), \ (6, -2),$$
$$(0, -5/2), \ (5/2, -5/2), \ (5, -5/2), \ (1, -3), \ (0, -3), \ (-5, -7/2), \ (0, -7/2),$$
$$(-1, -3), \ (-5/2, -7/2), \ (3/4, -37/8), \ (0, -4), \ (1/2, -9/2), \ (-6, -4), \ (6, -6),$$
$$(1, -5), \ (2, -27/5), \ (5/2, -11/2), \ (-3, -4), \ (-1, -4), \ (-7, -9/2), \ (-4, -9/2),$$
$$(-3/2, -9/2), \ (-8, -5), \ (-5, -5), \ (-2, -5), \ (-9, -11/2), \ (2, -11/2),$$
$$(-6, -11/2), \ (1, -11/2), \ (-3, -11/2)$$

The tropical curve is:

The Newton subdivision of the tropical curve is:

5.6. The number of terms of the non-homogeneous A-polynomial of twist knots. In [**GS**] we explicitly computed the non-homogeneous A-polynomial $(A_{K_p}^{nh}, B_{K_p})$ of the *twist knots* K_p for $p = -15, \ldots, 15$. K_p is the knot obtained by $1/p$ surgery on one component of the Whitehead link. This includes the following knots in the Rolfsen notation:

$$K_1 = 3_1 \qquad K_2 = 5_2 \qquad K_3 = 7_2 \qquad K_4 = 9_2$$
$$K_{-1} = 4_1 \qquad K_{-2} = 6_1 \qquad K_{-3} = 8_1 \qquad K_{-4} = 10_1$$

The computations reveal that for $p = 1, \ldots, 15$, $A_{K_p}^{nh}$ has (L, M, q) degree equal to

$$\left(2p - 1, 8p - 4, \frac{17}{2}p(p-1) + 2 \right)$$

The total number of terms of the 3-variable polynomial $A_{K_p}^{nh}$ is given by

$$139976, 80252, 41996, 19402, 7406, 2112, 346, 22$$

for $p = -8, \ldots, -1$, and by

$$4, 98, 908, 4100, 12236, 28978, 58668, 106800, 179814, 284998, 430652$$

for $p = 1, \ldots, 11$. Using the data from [**GS**], the author has computed the tropical curves (homogeneous or not) of all twist knots K_p with $p = -15, \ldots, 15$. Needless to say, the output of the computations it too large to be displayed in the paper.

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