q-TERMS, SINGULARITIES AND THE EXTENDED BLOCH GROUP

STAVROS GAROUFALIDIS

Dedicated to S. Bloch on the occasion of his sixtieth birthday.

ABSTRACT. Our paper originated from a generalization of the Volume Conjecture to multisums of q-hypergeometric terms. This generalization was sketched by Kontsevich in a problem list in Aarhus University in 2006; [Ko]. We introduce the notion of a q-hypergeometric term (in short, q-term). The latter is a product of ratios of q-factorials in linear forms in several variables.

In the first part of the paper, we show how to construct elements of the Bloch group (and its extended version) given a q-term. Their image under the Bloch-Wigner map or the Rogers dilogarithm is a finite set of periods of weight 2, in the sense of Kontsevich-Zagier.

In the second part of the paper we introduce the notion of a special q-term, its corresponding sequence of polynomials, and its generating series. Examples of special q-terms come naturally from Quantum Topology, and in particular from planar projections of knots.

The two parts are tied together by a conjecture that relates the singularities of the generating series of a special q-term with the periods of the corresponding elements of the extended Bloch group. In some cases (such as the 4_1 knot), the conjecture is known.

CONTENTS

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1. Introduction

1.1. **A brief summary of our results.** Our paper originated from a generalization of the Volume Conjecture to multisums of q-hypergeometric terms. This generalization was sketched by Kontsevich in a problem list in Aarhus University in 2006; [Ko]. Our paper expands Kontsevich's problem, and reveals a close and precise relation between special q-hypergeometric terms (defined below), elements of the extended Bloch group and its conjectural relation to singularities of generating series. Oddly enough, setting $q = 1$ also provides a relation between special hypergeometric terms and elements of the *extended additive* Bloch group. This is explained in a separate publication; see [Ga1] and [Ga2].

Our paper consists of two, rather disjoint parts, that are tied together by a conjecture.

In the first part, we introduce the notion of a q-term, and assign to it elements of the Bloch group and its extended version. The image of these elements of the extended Bloch group under the Rogers dilogarithm is a finite set of periods in the sense of Kontsevich-Zagier.

In the second part of the paper we introduce the notion of special q-term, its corresponding sequence of polynomials, and its generating series. Examples of special q-terms come naturally from Quantum Topology, and in particular from planar projections of knots.

Finally, we formulate a conjecture that relates the singularities of the generating series of a special q-term with the periods of the corresponding elements of the extended Bloch group. In some cases (such as the 4_1 knot, the conjecture is known, and implies the Volume Conjecture to all orders, with exponentially small terms included.

1.2. **What is a q-term?** The next definition, taken from [WZ], plays a key role in our paper.

Definition 1.1. A *q-term* $\mathfrak{t}$ in the $r+1$ variables $k = (k_0, k_1, \dots, k_r) \in \mathbb{N}^{r+1}$ is a list that consists of

- An integral symmetric quadratic form $Q(k)$ in k,
- An integral linear form L in k and a vector $\epsilon = (\epsilon_0, \dots, \epsilon_r)$ with $\epsilon_i = \pm 1$ for all i,
- An integral linear form A_j in k for $j = 1, \dots, J$

A q-term $\mathfrak{t}$ gives rise to an expression of the form

$$\mathfrak{t}_k(q) = q^{Q(k)} \epsilon^{L(k)} \prod_{j=1}^{J} (q)^{\epsilon_j}_{A_j(k)} \in \mathbb{Q}(q) \tag{1}$$

valid for $k \in \mathbb{N}^{r+1}$ such that $A_j(k) \geq 0$ for all $j = 1, \dots, J$. Here, the standard *q-factorial* of a natural number n and the *q-binomial* for $0 \leq m \leq n$ defined by (see for example [St]):

$$(q)_n = \prod_{j=1}^{n} (1 - q^j), \qquad \binom{n}{m}_q = \frac{(q)_n}{(q)_m (q)_{n-m}} \tag{2}$$

q-terms can be easily constructed and may be combinatorially encoded by matrices of the coefficients of the forms A_j, L, Q much in the spirit of Neumann-Zagier and Nahm; see [NZ, Ko, Na].

1.3. **The Bloch group.** Let us recall the symbolic-definition of the Bloch group from [B1]; see also [DS, Ne2, Su1, Su2]. Below F denotes a field of characteristic zero.

Definition 1.2. (a) The *pre-Bloch group* $\mathcal{P}(F)$ is the quotient of the freeabelian group generated by symbols $[z]$, $z \in F \setminus \{0,1\}$, subject to the *5-term relation*:

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0. \tag{3}$$

(b) The *Bloch group* $\mathcal{B}(F)$ is the kernel of the homomorphism

$$\nu : \mathcal{P}(F) \longrightarrow F^* \wedge F^*, \qquad [z] \mapsto z \wedge (1-z) \tag{4}$$

to the second exterior power of the abelian group F^* defined by mapping a generator $[z]$ to $z \wedge (1-z)$. The second exterior power $G \wedge G$ of an abelian group G is defined by:

$$G \wedge G = G \otimes_{\mathbb{Z}} G / (a \otimes b + b \otimes a) \tag{5}$$

Recall the *Bloch-Wigner function* (see for example, [NZ, Eqn.39])

$$\mathcal{D}_2 : \mathbb{C} \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2(z) := i\Im(\mathrm{Li}_2(z) + \log(1-z)\log|z|), \tag{6}$$

which is continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus \{0,1\}$. Since $\mathcal{D}_2$ satisfies the 5-term relation, we can define a map on the pre-Bloch group of the complex numbers:

$$R : \mathcal{P}(\mathbb{C}) \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2([z]) \tag{7}$$

and its restriction to the Bloch group (denoted by the same notation):

$$R : \mathcal{B}(\mathbb{C}) \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2([z]) \tag{8}$$

1.4. **From q-terms to the Bloch group.** In order to state our results, let us introduce some useful notation. Given a linear form A in $r+1$ variables $k = (k_0, k_1, \dots, k_r)$ let us define

$$v_i(A) = a_i, \qquad \text{where} \qquad A(k) = \sum_{i=0}^r a_i k_i. \tag{9}$$

If $z = (z_0, \dots, z_r) \in \mathbb{C}^*$ and A is an integral linear form A in $k = (k_0, k_1, \dots, k_r)$, let us abbreviate

$$z^A = \prod_{i=0}^r z_i^{v_i(A)}. \tag{10}$$

If Q is a symmetric quadratic form in $k = (k_0, k_1, \dots, k_r)$, we can write it in the form:

$$Q(k) = \frac{1}{2} \sum_{i,j=0}^r Q_{ij} k_i k_j + \sum_{i=0}^r \mathrm{QL}_i k_i.$$

Q gives rise to the linear forms Q_i (for $i = 0, \dots, r$) in k defined by

$$Q_i(k) = \sum_{j=0}^r Q_{ij} k_j.$$

The next definition associates to a q-term $\mathfrak{t}$ the complex points of an affine scheme defined over $\mathbb{Q}$.

Definition 1.3. Given a q-term $\mathfrak{t}$, let $X_\mathfrak{t}$ denote the set of points $z=(z_0,\dots,z_r)\in(\mathbb{C}^{**})^{r+1}$ that satisfy the following system of *Variational Equations*:

$$z^{Q_i}\epsilon^{v_i(L)}\prod_{j=1}^{J}(1-z^{A_j})^{v_i(\epsilon_j A_j)}=1 \tag{11}$$

for $i=0,\dots,r$. Here, $\mathbb{C}^{**}=\mathbb{C}\setminus\{0,1\}$.

Generically, $X_\mathfrak{t}$ is a zero-dimensional set. In that case, $X_\mathfrak{t}\subset(\overline{\mathbb{Q}}^*)^{r+1}$, where $\overline{\mathbb{Q}}$ is the set of algebraic numbers. The next definition assigns elements of the pre-Bloch group to a q-term $\mathfrak{t}$.

Definition 1.4. Given a q-term $\mathfrak{t}$, consider the map:

$$\beta_\mathfrak{t}: X_\mathfrak{t}\longrightarrow\mathcal{P}(\mathbb{C}) \tag{12}$$

given by:

$$z\mapsto\beta_\mathfrak{t}(z):=\sum_{j=1}^{J}\epsilon_j[z^{A_j}]. \tag{13}$$

The next theorem, communicated to us by Kontsevich, assigns elements of the Bloch group to a q-term $\mathfrak{t}$. In [Za2], Zagier attributes the result to Nahm [Na].

Theorem 1. (a) The map $\beta_\mathfrak{t}$ descends to a map

$$\beta_\mathfrak{t}: X_\mathfrak{t}\longrightarrow\mathcal{B}(\mathbb{C}) \tag{14}$$

which we denote by the same name.
(b) The image of

$$R\circ\beta_\mathfrak{t}: X_\mathfrak{t}\longrightarrow i\mathbb{R}$$

is a finite subset of $i\mathbb{R}\cap\mathcal{P}$, where $\mathcal{P}$ is the set of periods in the sense of Kontsevich-Zagier; [KZ].

1.5. **An example.** We illustrate Theorem 1 with the following example which is nontrivial and of interest to Quantum Topology. Consider the special q-term

$$\mathfrak{t}_n(q)=q^{a\frac{n(n+1)}{2}}\epsilon^n(q)_n^b \tag{15}$$

where $a\in\mathbb{Z}$, $\epsilon=\pm1$ and $b\in\mathbb{N}$. In other words,

$$r=0,\qquad k=(n),\qquad Q(n)=a\frac{n(n+1)}{2},\qquad J=b,\qquad A_j(n)=n, j=1,\dots,b.$$

Then,

$$X_\mathfrak{t}=\{z\in\mathbb{C}^*|z^a(1-z)^b\epsilon=1\} \tag{16}$$

The map (12) is given by:

$$X_\mathfrak{t}\longrightarrow\mathcal{B}(\mathbb{C}),\qquad z\mapsto\beta_\mathfrak{t}(z):=a[z]. \tag{17}$$

Equation (16) and the corresponding element of the Bloch group was also studied by Lewin, using his method of *ladders*. In that sense, the Variational Equations (11) is a generalization of the method of ladders.

Theorem 1 is a practical way of constructing elements of the Bloch group $\mathcal{B}(\mathbb{C})$ that are typically defined over number fields. Even if $X_\mathfrak{t}$ is not 0-dimensional, its image under $R\circ\beta_\mathfrak{t}$ always is finite. This finiteness is positive evidence for Bloch's *Rigidity Conjecture*, which states that

$$\mathcal{B}(\overline{\mathbb{Q}})\otimes\mathbb{Q}\cong\mathcal{B}(\mathbb{C})\otimes\mathbb{Q}.$$

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2. Proof of Theorem 1

This section is devoted to a proof of Theorem 1, using elementary symbol-like manipulations. See also [Za2, p.43]. Zagier attributes the following proof to Nahm.

Proof. (of Theorem 1) Let us fix $z = (z_0, \dots, z_r)$ that satisfies the Variational Equations (11). We will show that $\nu(\beta_{\mathfrak{t}}(z)) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*$, where ν is given by (4). We compute as follows:

$$\beta_{\mathfrak{t}}(z) \quad = \quad \sum_{j=1}^{J} \epsilon_j (z^{A_j} \wedge (1 - z^{A_j})).$$

On the other hand, we have:

$$z^{A_j} = \prod_{i=0}^{r} z_i^{v_i(A_j)}.$$

Thus,

$$\begin{aligned} \epsilon_j (z^{A_j} \wedge (1 - z^{A_j})) &= \epsilon_j \sum_{i=0}^{r} z_i^{v_i(A_j)} \wedge (1 - z^{A_j}) \\ &= \sum_{i=0}^{r} \epsilon_j v_i(A_j) (z_i \wedge (1 - z^{A_j})) \\ &= \sum_{i=0}^{r} z_i \wedge \left((1 - z^{A_j})^{v_i(\epsilon_j A_j)} \right). \end{aligned}$$

Since z satisfies the Variational Equations (11), after we interchange the j and i summation, we obtain that:

$$\begin{aligned} \beta_{\mathfrak{t}}(z) &= \sum_{i=0}^{r} \sum_{j=1}^{J} z_i \wedge \left((1 - z^{A_j})^{v_i(\epsilon_j A_j)} \right) \\ &= \sum_{i=0}^{r} z_i \wedge \prod_{j=1}^{J} (1 - z^{A_j})^{v_i(\epsilon_j A_j)} \\ &= \sum_{i=0}^{r} z_i \wedge \left(z^{-Q_i} \epsilon^{-v_i(L)} \right) \\ &= -\sum_{i=0}^{r} z_i \wedge z^{\frac{\partial Q}{\partial z_i}} - \sum_{i=0}^{r} z_i^{v_i(L)} \wedge \epsilon \\ &= -\sum_{i=0}^{r} z_i \wedge z^{Q_i} - z^L \wedge \epsilon. \end{aligned}$$

On the other hand, we have:

$$\sum_{i=0}^{r} z_i \wedge z^{Q_i} = \sum_{i=0}^{r} z_i \wedge \prod_{j=0}^{r} z_j^{\frac{\partial Q}{\partial z_i \partial z_j}} = \sum_{i,j} \frac{\partial Q}{\partial z_i \partial z_j} (z_i \wedge z_j)$$

Since Q is a symmetric bilinear form and $\wedge$ is skew-symmetric, it follows that

$$\sum_{i=0}^{r} z_i \wedge z^{Q_i} = 0.$$

In addition, we claim that for $\epsilon = \pm 1$, we have:

$$z \wedge \epsilon = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*. \tag{18}$$

Indeed, if $\epsilon = 1$, then

$$z \wedge 1 = z \wedge (1.1) = z \wedge 1 + z \wedge 1.$$

If $\epsilon = -1$, then

$$z^2 \wedge (-1) = 2(z \wedge (-1)) = z \wedge ((-1)^2) = z \wedge 1 = 0.$$

Since $\mathbb{C}^* = (\mathbb{C}^*)^2$, the result follows. Thus,

$$\nu(\beta_{\mathfrak{t}}(z)) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*.$$

This proves that $\beta_{\mathfrak{t}}$ takes values in the Bloch group, and concludes part (a) of Theorem 2.

Part (b) follows from a stronger result; see part (c) of Theorem 2 below. The idea is that any analytic function is constant on a connected component of its critical set. In our case, there exists a potential function whose points are the complex points of an affine variety defined over $\mathbb{Q}$ by the Variational Equations. The complex points of an affine variety have finitely many connected components. Finiteness of the image of $R \circ \beta_{\mathfrak{t}}$ follows. $\square$

3. From q-terms to the extended Bloch group

In the remainder of the first part of the paper, we will extend our results from the Bloch group to the extended Bloch group. As a guide, given a q-term $\mathfrak{t}$, we will assign

(a) A potential function $V_{\mathfrak{t}}$; see Definition 3.10.
(b) A set of Logarithmic Variational Equations (see (11)), which typically have a finite set of solutions, given by algebraic numbers. The variational equations may be identified with the critical points $\widehat{X_{\mathfrak{t}}}$ of the potential $V_{\mathfrak{t}}$; see Proposition 3.12.
(c) A map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{B}(\mathbb{C})}$$

where $\widehat{\mathcal{B}(\mathbb{C})}$ is the extended Bloch group; see Definition 3.1 and Theorem 2. When $\hat{\beta}_{\mathfrak{t}}$ is composed with a map $\hat{R}$ (given in Equation (29)), it essentially coincides with the evaluation of the potential on its critical values; see Theorem 2.
(d) The image of the composition $e^{\hat{R}/(2\pi i)} \circ \hat{\beta}_{\mathfrak{t}}$ is a finite set of complex numbers, given by exponentials of periods; see Theorem 2.

In the last part of the paper, which was a motivation all along, we mention that quantum invariants of 3-dimensional knotted objects (or general statistical-mechanical sums) give rise to special q-terms, a fact that was initially observed in [GL1].

3.1. The extended Bloch group.

Our aim is to associate elements of the extended Bloch group to every q-term. The extended Bloch group was introduced by Neumann in his investigation of the Cheeger-Chern-Simons classes and hyperbolic 3-manifolds; see [Ne1]. The definition below is called the very-extended Bloch group by Neumann, see also [DZ].

There is a close relation between the Bloch group $\mathcal{B}(F)$ and $K_3^{\mathrm{ind}}(F)$, (indecomposable K-theory of F); see [B1, B2] and [Su1, Su2], as well as Section 3.3 below. Unfortunately, the relation among $\mathcal{B}(F)$ and $K_3^{\mathrm{ind}}(F)$ is known modulo torsion, and the torsion does not match. For $F = \mathbb{C}$, this is exactly what the extended Bloch group captures. The idea is that:

$$\boxed{\text{torsion is encoded by the choice of the branch of the logarithms}}.$$

More precisely, consider the doubly punctured plane

$$\mathbb{C}^{**} := \mathbb{C} \setminus \{0,1\} \tag{19}$$

and let $\hat{\mathbb{C}}$ denote the universal abelian cover of $\mathbb{C}^{**}$. We can represent the Riemann surface of $\mathbb{C}^{**}$ as follows. Let $\mathbb{C}_{\text{cut}}$ denote $\mathbb{C}^{**}$ cut open along each of the intervals $(-\infty, 0)$ and $(1,\infty)$ so that each real number r outside $[0,1]$ occurs twice in $\mathbb{C}_{\text{cut}}$. Let us denote the two occurrences of r by $r+0i$ and $r-0i$ respectively. It is now easy to see that $\hat{\mathbb{C}}$ is isomorphic to the surface obtained from $\mathbb{C}_{\text{cut}} \times 2\mathbb{Z} \times 2\mathbb{Z}$ by the following identifications:

$$\begin{aligned}(x+0i;2p,2q) &\sim (x-0i;2p+2,2q) \quad \text{for} \quad x \in (-\infty,0)\\ (x+0i;2p,2q) &\sim (x-0i;2p,2q+2) \quad \text{for} \quad x \in (1,\infty)\end{aligned}$$

This means that points in $\hat{\mathbb{C}}$ are of the form (z,p,q) with $z \in \mathbb{C}^{**}$ and p,q even integers. Moreover, $\hat{\mathbb{C}}$ is the Riemann surface of the function

$$\mathbb{C}^{**} \longrightarrow \mathbb{C}^2, \qquad z \mapsto (\mathrm{Log}(z), \mathrm{Log}(1-z)).$$

where Log denotes the *principal branch* of the logarithm. Consider the set

$$\mathrm{FT} := \{(x, y, \frac{y}{x}, \frac{1-x^{-1}}{1-y^{-1}}, \frac{1-x}{1-y})\} \subset (\mathbb{C}^{**})^5$$

of 5-tuples involved in the 5-term relation. Also, let

$$\mathrm{FT}_0 := \{(x_0,\dots,x_4) \in \mathrm{FT} | 0 < x_1 < x_0 < 1\}$$

and define $\widehat{\mathrm{FT}} \subset \hat{\mathbb{C}}^5$ to be the component of the preimage of FT that contains all points $((x_0;0,0),\dots,(x_4;0,0))$ with $(x_0,\dots,x_4) \in \mathrm{FT}_0$. See also [DZ, Rem.2.1].

We can now define the extended Bloch group, following [GZ, Def.1.5].

Definition 3.1. (a) The *extended pre-Bloch group* $\widehat{\mathcal{P}(\mathbb{C})}$ is the abelian group generated by the symbols $[z;p,q]$ with $(z;p,q) \in \hat{\mathbb{C}}$, subject to the relation:

$$\sum_{i=0}^{4} (-1)^i [x_i;p_i,q_i] = 0 \quad \text{for} \quad ((x_0;p_0,q_0),\dots,(x_4;p_4,q_4)) \in \widehat{\mathrm{FT}}. \tag{20}$$

(b) The *extended Bloch group* $\widehat{\mathcal{B}(\mathbb{C})}$ is the kernel of the homomorphism

$$\hat{\nu} : \widehat{\mathcal{P}(\mathbb{C})} \longrightarrow \mathbb{C} \wedge \mathbb{C}, \qquad [z;p,q] \mapsto (\mathrm{Log}(z) + p\pi i) \wedge (-\mathrm{Log}(1-z) + q\pi i) \tag{21}$$

(c) We will call the 2-term complex (21) the *extended Bloch-Suslin* complex.

For a comparison between the extended Bloch-Suslin complex and the Suslin complex, see Section 6.3.

Remark 3.2. There are four versions of the extended Bloch group, depending on whether $(p,q) \in \mathbb{Z}^2$ versus $(p,q) \in (2\mathbb{Z})^2$, and whether we include the transfer relation of [Ne1, Eqn.3]. For a comparison of Neumann's version of the extended Bloch group with the above definition, see [GZ, Rem.4.3].

3.2. **The Rogers dilogarithm.** The extended Bloch-Suslin complex (21) is very closely related to the functional properties of a normalized form of the dilogarithm. In fact, it would be hard to motivate the extended Bloch-Suslin complex without knowing the Rogers dilogarithm and vice-versa.

Following [Ne1, DZ], the *Rogers dilogarithm* is the following function defined on the open interval $(0,1)$:

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\mathrm{Log}(z)\mathrm{Log}(1-z) - \frac{\pi^2}{6} \tag{22}$$

where

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{Log}(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is the classical *dilogarithm function*. In [GZ, Defn.1.5], Goette and Zickert extended $L(z)$ to a multivalued analytic function on $\hat{\mathbb{C}}$ as follows (see [Ne1, Prop.2.5]):

$$\hat{R} : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{23}$$

$$\begin{aligned} \hat{R}(z;p,q) &:= \mathrm{Li}_2(z) + \frac{1}{2}(\mathrm{Log}(z) + \pi i p)(\mathrm{Log}(1-z) + \pi i q) - \frac{\pi^2}{6} &\quad (24)\\ &= L(z) + \frac{\pi i}{2}(q\mathrm{Log}(z) + p\mathrm{Log}(1-z)) - \frac{\pi^2 pq}{2}. &\quad (25)\end{aligned}$$

where as usual, for a subgroup K of $(\mathbb{C},+)$ and an integer $n \in \mathbb{Z}$, we denote $K(n) = (2\pi i)^n K \subset \mathbb{C}$.

Let us comment a bit on the properties of the Rogers dilogarithm.

Remark 3.3. A computation involving the monodromy of the Li_2 function shows that the dilogarithm Li_2 has an analytic extension:

$$\mathrm{Li}_2 : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{26}$$

See [Oe, Prop.1]. Compare also with [Ne1, Prop.2.5]. In [GZ, Lem.2.2] it is shown that $\hat{R}$ takes values in $\mathbb{C}/\mathbb{Z}(2)$ and that the Rogers dilogarithm is defined on the extended pre-Bloch group; we will denote the extension by $\hat{R}$. In other words, we have:

$$\hat{R} : \widehat{\mathcal{P}(\mathbb{C})} \longrightarrow \mathbb{C}/\mathbb{Z}(2). \tag{27}$$

In [DZ] and [GZ] it was shown that the Rogers dilogarithm coincides with the $\mathbb{C}/\mathbb{Z}(2)$-valued *Cheeger-Chern-Simons class* (denoted $\hat{C}_2$ in [DZ]). For a discussion on this matter, see [DZ, Thm.4.1].

Remark 3.4. It is natural to ask why to modify the dilogarithm as in Rogers extension. The motivation is that $\hat{L}$ (but *not* the dilogarithm function $\mathrm{Li}_2(z)$ nor the Bloch-Wigner dilogarithm) satisfies the extended 5-term relation $\widehat{\mathrm{FT}}$; see [DZ, Sec.2] and [GZ, Lem.2.2].

Remark 3.5. For the purposes of comparing the Rogers dilogarithm with other special functions (such as the entropy function discussed in [Ga1]) It is easy to see that the derivative of the Rogers dilogarithm is given by:

$$L'(z) = -\frac{1}{2}\left(\frac{\log(1-z)}{z} + \frac{\log(z)}{1-z}\right). \tag{28}$$

Thus, $\hat{R}$ descends to a map:

$$\hat{R} : \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{29}$$

which can be exponentiated to a map:

$$e^{\frac{1}{2\pi i}\hat{R}} : \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathbb{C}^*. \tag{30}$$

The maps R and $\hat{R}$ of the Bloch group and its extended version are part of the following commutative diagram:

$$\begin{tikzcd} \widehat{\mathcal{B}(\mathbb{C})} \arrow[r, "\hat{R}"] \arrow[d] & \mathbb{C}/\mathbb{Z}(2) \arrow[r, "e^{\frac{\bullet}{2\pi i}}"] \arrow[d, "i\Im"] & \mathbb{C}^* \arrow[d, "|\bullet|"] \\ \mathcal{B}(\mathbb{C}) \arrow[r, "R"] & i\mathbb{R} \arrow[r, "e^{\frac{\bullet}{2\pi i}}"] & \mathbb{R}^+ \end{tikzcd}$$

For a proof of the commutativity of the left square, see [DZ, Prop.4.6]. Let us end this section with a remark.

Remark 3.6. Even though the regulators R and $\hat{R}$ are defined on the (extended) pre-Bloch groups, the following diagram is *not* commutative:

$$\begin{tikzcd} \widehat{\mathcal{B}(\mathbb{C})} \arrow[r, "\hat{R}"] \arrow[d] & \mathbb{C}/(2\pi^2\mathbb{Z}) \arrow[d, "i\Im"] \\ \mathcal{B}(\mathbb{C}) \arrow[r, "R"] & i\mathbb{R} \end{tikzcd}$$

3.3. **Two cousins of the extended Bloch group: $K_3^{\text{ind}}(\mathbb{C})$ and $\mathrm{CH}^2(\mathbb{C},3)$.** The Bloch group $\mathcal{B}(F)$ of a field has two cousins: $K_3^{\text{ind}}(F)$ and the higher Chow groups $\mathrm{CH}^2(F,3)$, also defined by Bloch; see [B2]. A spectral sequence argument (attributed to Bloch and Bloch-Lichtenbaum) and some low-degree computations imply that for every infinite field F we have an isomorphism:

$$K_3^{\text{ind}}(F) \cong \mathrm{CH}^2(F,3). \tag{31}$$

For a detailed discussion, see [E-V, Prop.5.5.20]. On the other hand, in [Su1, Thm.5.2] Suslin proves the existence of a short exact sequence:

$$0 \longrightarrow \widetilde{\mathrm{Tor}(\mu_F,\mu_F)} \longrightarrow K_3^{\text{ind}}(F) \longrightarrow \mathcal{B}(F) \longrightarrow 0. \tag{32}$$

where $\widetilde{\mathrm{Tor}(\mu_F,\mu_F)}$ is the unique nontrivial extension of $\mathrm{Tor}(\mu_F,\mu_F)$, where μ_F is the roots of unity of F. For $F=\mathbb{C}$, the above short exact sequence becomes:

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow K_3^{\text{ind}}(\mathbb{C}) \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0. \tag{33}$$

Moreover, Suslin proves in [Su1] that $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space.

On the other hand, in [GZ, Thm.3.12] Goette and Zickert prove that the extended Bloch group fits in a short exact sequence:

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\hat{\psi}} \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0, \tag{34}$$

for an explicit map $\hat{\psi}$. Equations (31), (33), (34), together with the fact that $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space, imply the following.

Proposition 3.7. *There exist abstract isomorphisms:*

$$\widehat{\mathcal{B}(\mathbb{C})} \cong K_3^{\text{ind}}(\mathbb{C}) \cong \mathrm{CH}^2(F,3). \tag{35}$$

In addition there are maps (sometimes also known by the name of *cycle maps* or *Abel-Jacobi maps*):

$$\begin{aligned} R' : K_3^{\text{ind}}(\mathbb{C}) &\longrightarrow H^1_D(\mathrm{Spec}(\mathbb{C}),\mathbb{Z}(2)) = \mathbb{C}/\mathbb{Z}(2) \\ R'' : \mathrm{CH}^2(F,3) &\longrightarrow H^1_D(\mathrm{Spec}(\mathbb{C}),\mathbb{Z}(2)) = \mathbb{C}/\mathbb{Z}(2) \end{aligned}$$

where H_D denotes Deligne cohomology. For an explicit formula for R'', see [KLM-S, Sec.5.7].

Let us end this section with a problem and a question. Recall that the extended Bloch group can be defined for any subfield F of the complex numbers, as discussed in [Ne1].

Problem 3.8. Define explicit isomorphisms in (35) that commute with the maps R, R' and R''.

We will come back to this problem in a forthcoming publication; [GZ]. For a careful relation between $\widehat{\mathcal{B}(F)}$ and $K_3^{\mathrm{ind}}(F)$ in the case of a number field F, see [Zi].

3.4. **q-terms and potential functions.** In this section, we will assign elements of $\widehat{\mathcal{B}(\mathbb{C})}$ to a q-term $\mathfrak{t}$; see Theorem 2 below.

Definition 3.9. Consider the following multivalued function on $\mathbb{C}^{**}$:

$$\Phi : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(1), \qquad \Phi(x) = \frac{1}{2\pi i}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(x)\right). \tag{36}$$

Remark 3.3 shows that indeed Φ takes values in $\mathbb{C}/\mathbb{Z}(1)$. The function $x \mapsto \Phi(e^{2\pi i x})$ appears in work of Kashaev on the Volume Conjecture; see [Ks]. The relevance of the special function Φ is that it describes the growth rate of the q-factorials at complex roots of unity; see Lemma 6.1 in Section 6.1. The next definition associates a potential function to a q-term.

Definition 3.10. Given a q-term $\mathfrak{t}$ as in (1), let us define its potential function $V_{\mathfrak{t}}$ by:

$$V_{\mathfrak{t}}(z) = \frac{1}{2\pi i} Q(\mathrm{Log}(z)) + \frac{1}{2\pi i}\mathrm{Log}\epsilon \cdot \mathrm{Log}(z^L) + \sum_{j=1}^{J} \epsilon_j \Phi(z^{A_j}) \tag{37}$$

where $z = (z_0, z_1, \dots, z_r)$ and $\mathrm{Log}(z) = (\mathrm{Log}(z_0), \dots, \mathrm{Log}(z_r))$. Let $\widehat{X_{\mathfrak{t}}}$ and denote the *critical points* of the potential $V_{\mathfrak{t}}$.

Since $V_{\mathfrak{t}}(z)$ is a multivalued function, let us explain its domain. Given a q-term $\mathfrak{t}$ as in (1), let denote

$$H_{\mathfrak{t}} = \{z \subset (\mathbb{C}^*)^{r+1} \,|z^L \prod_{j=1}^{J}(1 - z^{A_j}) \prod_{i=0}^{r}(1 - z_i) = 0\}. \tag{38}$$

Let $\mathcal{D}_{\mathfrak{t}}$ denote the universal abelian cover of $(\mathbb{C}^*)^{r+1} \setminus H_{\mathfrak{t}}$. Observe that for every $i = 0, \dots, r$, $j = 1, \dots, J$ there are well-defined analytic maps:

$$\pi_i, \pi_{A_j}, \pi_L : (\mathbb{C}^*)^{r+1} \setminus H_{\mathfrak{t}} \to \mathbb{C}^{**} \tag{39}$$

given by $\pi_i(z) = z_i$, $\pi_{A_j}(z) = z^{A_j}$ and $\pi_L(z) = z^L$; using the notation of (10). Lifting them to the universal abelian cover, gives rise to analytic maps:

$$\hat{\pi}_i, \hat{\pi}_{A_j}, \hat{\pi}_L : \mathcal{D}_{\mathfrak{t}} \to \hat{\mathbb{C}}. \tag{40}$$

We denote the image of $z \in \mathcal{D}_{\mathfrak{t}}$ under these maps by z_i, z^{A_j} and z^L respectively. With these conventions we have:

Lemma 3.11. *Equation* (37) *defines an analytic function:*

$$V_{\mathfrak{t}} : \mathcal{D}_{\mathfrak{t}} \longrightarrow \mathbb{C}/\mathbb{Z}(1). \tag{41}$$

The next proposition describes the critical points and the critical values of the potential function.

Proposition 3.12. (a) The critical points z of $V_{\mathfrak{t}}$ are the solutions to the following system of *Logarithmic Variational Equations*:

$$\sum_{j=1}^{J} \epsilon_j v_i(A_j)\mathrm{Log}(1 - z^{A_j}) + \frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) + \mathrm{Log}\epsilon \cdot v_i(L) = 0 \tag{42}$$

for $i = 0, \dots, r$.
(b) There is a map:

$$\widehat{X_{\mathfrak{t}}} \longrightarrow X_{\mathfrak{t}}, \qquad z \mapsto (\pi(z_1), \dots, \pi(z_r)) \tag{43}$$

where $\pi : \hat{\mathbb{C}} \to \mathbb{C}^{**}$ is the projection map.

For $A = A_j$, $(j = 1, \dots, J)$ or $A = L$, let $p_{z,A} \in 2\mathbb{Z}$ (or simply, p_A if z is clear) be defined so that we have:

$$\mathrm{Log}(z^A) = \sum_{i=0}^{r} v_i(A)\mathrm{Log}(z_i) - p_{z,A}\pi i. \tag{44}$$

Given $w = (x; p_0, q_0) \in \hat{\mathbb{C}}$ and even integers $p, q \in 2\mathbb{Z}$ let us denote

$$T_1^p T_0^q(w) := (x; p_0 + p, q_0 + q) \in \hat{\mathbb{C}} \tag{45}$$

In other words, T_1 and T_0 are generators for the deck transformations of the $\mathbb{Z}^2$-cover:

$$\pi : \hat{\mathbb{C}} \longrightarrow \mathbb{C}^{**}. \tag{46}$$

We need one more piece of notation: for $z \in \mathcal{D}_{\mathfrak{t}}$, consider the corresponding elements z^L and z_i of $\hat{\mathbb{C}}$ for $i = 0, \dots, r$. We denote by $z^{-L/2} \in \hat{\mathbb{C}}$ the unique element of $\hat{\mathbb{C}}$ that solves the equation:

$$-\mathrm{Log}(z^L) - \mathrm{Log}(z^{-L/2}) + \frac{1}{2}\sum_{i=0}^{r} v_i(L)\mathrm{Log}(z_i) = 0. \tag{47}$$

Definition 3.13. With the above conventions, consider the map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{P}(\mathbb{C})} \tag{48}$$

given by:

$$w \mapsto \hat{\beta}_{\mathfrak{t}}(w) := [z^{-L/2}; 0, 2\frac{\mathrm{Log}\epsilon}{\pi i}] - [z^{-L/2}; 0, 0] + \sum_{j=1}^{J} \epsilon_j [T_1^{p_{z,A_j}}(z^{A_j})] \tag{49}$$

Our next definition assigns a numerical invariant to a special term $\mathfrak{t}$.

Definition 3.14. For a q-term $\mathfrak{t}$, let

$$\mathrm{CV}_{\mathfrak{t}} = \{e^{-V_{\mathfrak{t}}(z)} \,|z \text{ satisfies } (42)\}. \tag{50}$$

Our next main theorem links the exponential of the critical values of $V_{\mathfrak{t}}$ with the values of $\hat{\beta}_{\mathfrak{t}}$. Let $\mathcal{P}$ denote the set of *periods* in the sense of Kontsevich-Zagier, [KZ]. $\mathcal{P}$ is a countable subset of the complex numbers.

Theorem 2. (a) The map $\hat{\beta}_{\mathfrak{t}}$ descends to a map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{B}(\mathbb{C})} \tag{51}$$

which we denote by the same name.
(b) We have a commutative diagram:

$$\begin{tikzcd} \widehat{X_{\mathfrak{t}}} \arrow[rr, "\hat{\beta}_{\mathfrak{t}}"] \arrow[dr, "e^{-V_{\mathfrak{t}}}"'] & & \widehat{\mathcal{B}(\mathbb{C})} \arrow[dl, "e^{\frac{1}{2\pi i}\hat{R}}"] \\ & \mathbb{C}^* & \end{tikzcd}$$

(c) $\mathrm{CV}_{\mathfrak{t}}$ is a finite subset of $e^{\mathcal{P}}$. Thus, we get a map:

$$\mathrm{CV} : q\text{-terms} \longrightarrow \text{Finite subsets of } e^{\mathcal{P}}. \tag{52}$$

Example 3.15. Let us continue with the Example 1.5. When $a = -1$, $b = 2$ and $\epsilon = -1$, the Variational Equations (16) have solutions

$$X_{\mathfrak{t}} = \{z^{\pm 1} \,|\, z = e^{2\pi i/6}\}.$$

In that case, the corresponding elements of the extended Bloch group are

$$2[z^{\pm 1}; 0, 0] + [z^{-1/2}; 0, 2] - [z^{-1/2}; 0, 0],$$

and their values under the map $\hat{R}$ are given by:

$$0 \pm i\, 2.0298832128193074\ldots$$

whose imaginary part equals to the *Volume* $\mathrm{Vol}(4_1)$ of the 4_1 knot; see [Th]. Moreover,

$$\begin{aligned} \mathrm{CV}_{\mathfrak{t}} &= \{e^{\pm \frac{1}{2\pi}\mathrm{Vol}(4_1)}\} \\ &= \{0.7239261119\ldots, 1.3813564445\ldots\}. \end{aligned}$$

4. Proof of Theorem 2

In this section we will give the proofs of Proposition 3.12 and Theorem 2.

4.1. **Proof of Proposition 3.12.** It suffices to show that for every $i = 0, \ldots, r$, $2\pi\sqrt{-1} z_i \frac{\partial V_{\mathfrak{t}}}{\partial z_i}$ is given by the left hand side of Equation (42). This follows easily from the definition of $V_{\mathfrak{t}}$ and the elementary computation:

$$\Phi'(x) = \frac{1}{2\pi i} \frac{\mathrm{Log}(1-x)}{x}. \tag{53}$$

4.2. **Proof of Theorem 2.** The proof is similar to the proof of Theorem 1, once we keep track of the branches of the logarithms. Fix a q-term $\mathfrak{t}$ and consider the map given by (49). Without loss of generality, we assume that

$$Q(k) = \frac{1}{2}\sum_{i,j=0}^{r} k_i k_j$$

Let us fix $z = (z_0, \ldots, z_r)$ that satisfies the Variational Equations (11). We will show that $\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z)) = 0 \in \mathbb{C} \wedge \mathbb{C}$, where $\hat{\nu}$ is given by (21). We have

$$\hat{\beta}_{\mathfrak{t}}(z) = \hat{\beta}_{1,\mathfrak{t}}(z) + \hat{\beta}_{2,\mathfrak{t}}(z)$$

where

$$\begin{aligned} \hat{\beta}_{1,\mathfrak{t}}(z) &= \sum_{j=1}^{J} \epsilon_j [T_1^{p_{z,A_j}}(z^{A_j})] \\ \hat{\beta}_{2,\mathfrak{t}}(w) &= [z^{-L/2}; 0, 2\frac{\mathrm{Log}\epsilon}{\pi i}] - [z^{-L/2}; 0, 0]. \end{aligned}$$

It follows that

$$\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z)) = \hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) + \hat{\nu}(\hat{\beta}_{2,\mathfrak{t}}(z))$$

where

$$\begin{aligned} \hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) &= \sum_{j=1}^{J} \epsilon_j (\mathrm{Log}(z^{A_j}) + p_{z,A_j}\pi i) \wedge (-\mathrm{Log}(1 - z^{A_j})) \\ \hat{\nu}(\hat{\beta}_{2,\mathfrak{t}}(z)) &= \mathrm{Log}(z^{-L/2}) \wedge 2\mathrm{Log}\epsilon \\ &= 2\mathrm{Log}(z^{-L/2}) \wedge \mathrm{Log}\epsilon. \end{aligned}$$

On the other hand, using Equation (44), we have:

$$\begin{aligned}(\mathrm{Log}(z^A)+p_{z,A}\pi i)\wedge\mathrm{Log}(1-z^A) &= \sum_{i=0}^{r} v_i(A)(\mathrm{Log}(z_i)\wedge\mathrm{Log}(1-z^A))\\ &= \sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(v_i(A)\mathrm{Log}(1-z^A)).\end{aligned}$$

Since z satisfies the Logarithmic Variational Equations (42), after we interchange the j and i summation, we obtain that:

$$\begin{aligned}\hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) &= \sum_{i=0}^{r}\sum_{j=1}^{J}\mathrm{Log}(z_i)\wedge(-\epsilon_j v_i(A_j)\mathrm{Log}(1-z^{A_j}))\\ &= \sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(\sum_{j=1}^{J}-\epsilon_j v_i(A_j)\mathrm{Log}(1-z^{A_j}))\\ &= \sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z))+\mathrm{Log}\epsilon\cdot v_i(L)).\end{aligned}$$

Since Q is an integral symmetric bilinear form and $\wedge$ is skew-symmetric, it follows that

$$\sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) = 0.$$

Moreover,

$$\begin{aligned}\sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(\mathrm{Log}\epsilon\cdot v_i(L)) &= \sum_{i=0}^{r} v_i(L)\mathrm{Log}(z_i)\wedge\mathrm{Log}\epsilon\\ &= (\mathrm{Log}(z^L)+p_{z,L}\pi i)\wedge\mathrm{Log}\epsilon\\ &= \mathrm{Log}(z^L)\wedge\mathrm{Log}\epsilon.\end{aligned}$$

Thus,

$$\hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z))=\mathrm{Log}(z^L)\wedge\mathrm{Log}\epsilon$$

which implies that

$$\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z))=(2\mathrm{Log}(z^{-L/2})+\mathrm{Log}(z^L))\wedge\mathrm{Log}\epsilon.$$

On the other hand, reducing Equation (47) modulo $\pi i\mathbb{Z}$, and using (44) we have:

$$\begin{aligned}0 &= -\mathrm{Log}(z^L)-\mathrm{Log}(z^{-L/2})+\frac{1}{2}\sum_{i=0}^{r} v_i(L)\mathrm{Log}(z_i)\\ &= -\mathrm{Log}(z^L)-\mathrm{Log}(z^{-L/2})+\frac{1}{2}(\mathrm{Log}(z^L)+\pi i p_{z,L})\\ &= -\frac{1}{2}(\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2}))+\frac{\pi i p_{z,L}}{2}\\ &= -\frac{1}{2}(\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2})).\end{aligned}$$

In other words, we have:

$$\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2})\in\frac{\pi i}{2}\mathbb{Z}. \tag{54}$$

Since $\mathrm{Log}\epsilon \in \pi i \mathbb{Z}$, it follows that $\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z)) = 0$ and concludes part (a) of Theorem 2.

For part (b), observe first that for every $w \in \hat{\mathbb{C}}$ and every even integers p and q we have:

$$\hat{R}([T_1^p T_0^q(w)]) = \hat{R}([w]) + \frac{\pi i}{2}(q\mathrm{Log}(z) + p\mathrm{Log}(1-z)).$$

Moreover, by the definition of Φ and by Equation (44), for any integral linear form $A = A_j$ for $j = 1, \dots, J$ or $A = L$, we have:

$$-\Phi(z^A) \quad = \quad \frac{1}{2\pi i}\left(\hat{R}([T_1^{p_{z,A}}(z^A)]) - \frac{1}{2}\mathrm{Log}(z^A)\mathrm{Log}(1-z^A) - \frac{\pi i}{2}p_{z,A}\mathrm{Log}(1-z^A)\right).$$

Thus, using the definition of the potential function from Equation (37), we obtain that:

$$-V_{\mathfrak{t}}(z) = \frac{1}{2\pi i}R(\hat{\beta}_{\mathfrak{t}}(z)) + T_1 + T_2$$

where

$$\begin{aligned}
T_1 &= \frac{1}{2\pi i}\left(-\frac{1}{2}\sum_j \epsilon_j \mathrm{Log}(z^{A_j})\mathrm{Log}(1-z^{A_j}) - \frac{\pi i}{2}p_{z,A_j}\mathrm{Log}(1-z^{A_j})\right) \\
T_2 &= -\frac{1}{2\pi i}Q(\mathrm{Log}(z)) - \frac{1}{2\pi i}\mathrm{Log}\epsilon \cdot \mathrm{Log}(z^L) - \frac{1}{2\pi i}(\hat{R}([z^{-L/2}; 0, \frac{2\mathrm{Log}\epsilon}{\pi i}] - [z^{-L/2}; 0, 0]) \\
&= -\frac{1}{2\pi i}Q(\mathrm{Log}(z)) - \frac{1}{2\pi i}\mathrm{Log}\epsilon \cdot \mathrm{Log}(z^L) - \frac{\mathrm{Log}\epsilon}{2\pi i}\mathrm{Log}(z^{-L/2}).
\end{aligned}$$

Using (44), and interchanging i and j summation, and using the Logarithmic Variational Equations (42), it follows that:

$$\begin{aligned}
\sum_{j=0}^{J}\epsilon_j \mathrm{Log}(z^{A_j})\mathrm{Log}(1-z^{A_j}) &= \sum_{j=0}^{J}(\sum_{i=0}^{r}\epsilon_j v_i(A_j)\mathrm{Log}(z_i) - p_{z,A_j}\pi i)\mathrm{Log}(1-z^{A_j}) \\
&= \sum_{i=0}^{r}\mathrm{Log}(z_i)\sum_{j=0}^{J}\epsilon_j v_i(A_j)\mathrm{Log}(1-z^{A_j}) - \sum_{j=0}^{J}\epsilon_j p_{z,A_j}\pi i \mathrm{Log}(1-z^{A_j}) \\
&= \sum_{i=0}^{r}\mathrm{Log}(z_i)(-\frac{\partial Q}{\partial z_i} - \mathrm{Log}\epsilon \cdot v_i(L)) - \sum_{j=0}^{J}\epsilon_j p_{z,A_j}\pi i \mathrm{Log}(1-z^{A_j}).
\end{aligned}$$

Thus,

$$T_1 \quad = \quad \frac{1}{4\pi i}\sum_{i=0}^{r}\mathrm{Log}(z_i)\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) + \frac{\mathrm{Log}\epsilon}{4\pi i}\sum_{i=0}^{r}v_i(L)\mathrm{Log}(z_i).$$

Since Q is a symmetric bilinear form, it follows that

$$\frac{1}{2}\sum_{i=0}^{r}\mathrm{Log}(z_i)\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) - Q(\mathrm{Log}(z)) = 0.$$

This, together with Equation (47) implies that $T_1 + T_2 = 0$. Exponentiating, it follows that

$$e^{-V_{\mathfrak{t}}(z)} = e^{\frac{1}{2\pi i}\hat{R}(\hat{\beta}_{\mathfrak{t}}(z))} \in \mathbb{C}^*$$

which concludes part (b) of Theorem 2.

For part (c), since an analytic function is constant on a connected component of its sets of critical points, and since the set of critical points $\widehat{X_{\mathfrak{t}}}$ is the set of complex points of an affine variety defined over $\mathbb{Q}$, it follows that $\widehat{X_{\mathfrak{t}}}$ has finitely many connected components. Thus, $\mathrm{CV}_{\mathfrak{t}}$ is a finite subset of $\mathbb{C}$. Moreover, since the value of the map (29) at a point $[z;p,q]$ with $z \in \overline{\mathbb{Q}}$ is a period, in the sense of Kontsevich-Zagier, and since every connected component of $\widehat{X_{\mathfrak{t}}}$ has a point w so that $z = e^{2\pi i w} \in (\overline{\mathbb{Q}})^{r+1}$ (due to the Variational Equations (11)), it follows that $\mathrm{CV}_{\mathfrak{t}}$ is a subset of $e^{\mathcal{P}}$. This concludes the proof of Theorem 2. □

5. Special q-terms, generating series and singularities

The second part of the paper assigns to germs of analytic functions $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ to a special q-term, and formulates a conjecture regarding their analytic continuation in the complex plane minus a set of singularities related to the image of the composition $\hat{R} \circ \hat{\beta}_{\mathfrak{t}}$.

5.1. What is a special q-term?

In this section we introduce the notion of a special q-term. Of course, every special q-term is a q-term. Examples of special q-terms that come naturally from Quantum Topology are discussed in Section 5.3 below.

Definition 5.1. A *special q-hypergeometric term* $\mathfrak{t}$ (in short, *special q-term*) in the $r+1$ variables $k = (k_0, k_1, \dots, k_r) \in \mathbb{N}^r$ is a list that consists of

- An integral symmetric quadratic form $Q(k)$ in k,
- An integral linear form L in k and a vector $\epsilon = (\epsilon_0, \dots, \epsilon_r)$ with $\epsilon_i = \pm 1$ for all i,
- Integral linear forms B_j, C_j, D_j, E_j in k for $j = 1, \dots, J$

The list $\mathfrak{t}$ satisfies the following condition. Its *Newton polytope* $P_{\mathfrak{t}}$ defined by

$$P_{\mathfrak{t}} = \{w \in \mathbb{R}^r \mid B_j(1,w) \geq C_j(1,w) \geq 0, \quad D_j(1,w) \geq E_j(1,w) \geq 0, \; j = 1, \dots, J\} \tag{55}$$

is a nonempty rational compact polytope of $\mathbb{R}^r$.

This accurate but rather obscure definition is motivated from the fact that a special q-term $\mathfrak{t}$ gives rise to an expression of the form

$$\mathfrak{t}_k(q) = q^{Q(k)} \epsilon^{L(k)} \prod_{j=1}^{J} \binom{B_j(k)}{C_j(k)}_q \frac{(q)_{D_j(k)}}{(q)_{E_j(k)}} \in \mathbb{Z}[q^{\pm 1}] \tag{56}$$

valid for $k \in \mathbb{N}^{r+1}$ such that $B_j(k) \geq C_j(k) \geq 0$ and $D_j(k) \geq E_j(k) \geq 0$ for all $j = 1, \dots, J$.

5.2. From special q-terms to generating series.

We now discuss how a special q-term $\mathfrak{t}$ gives rise to a sequence of Laurent polynomials $(a_{\mathfrak{t},n}(q))$ and to a generating series $G_{\mathfrak{t}}(z)$. Given a special q-term $\mathfrak{t}$ in $r+1$ variables $k = (k_0, \dots, k_r)$, it will be convenient to single out the first variable k_0, and denote it by n as follows:

$$n = k_0, \qquad k' = (k_1, \dots, k_r) \tag{57}$$

With the above convention, we have $k = (n, k')$.

Definition 5.2. A special q-term $\mathfrak{t}$ gives rise to a sequence of Laurent polynomials $(a_{\mathfrak{t},n}(q))$ as follows:

$$a_{\mathfrak{t},n}(q) = \sum_{k' \in nP_{\mathfrak{t}} \cap \mathbb{N}^r} \mathfrak{t}_{n,k'}(q) \in \mathbb{Z}[q^{\pm 1}] \tag{58}$$

where the summation is over the finite set $nP_{\mathfrak{t}}\mathbb{N}^r$ of lattice points of the translated Newton polytope of $\mathfrak{t}$.

Definition 5.3. A special q-term $\mathfrak{t}$ gives rise to a power series $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ defined by:

$$L^{\mathrm{np}}_{\mathfrak{t}}(z) = \sum_{n=0}^{\infty} a_{\mathfrak{t},n}(e^{\frac{2\pi i}{n}}) z^n. \tag{59}$$

The next lemma proves that $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ is an analytic function at $z=0$.

Lemma 5.4. *For every special q-term $\mathfrak{t}$, $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ is analytic at $z=0$.*

Proof. Fix a special q-term $\mathfrak{t}$ as Definition (56). Let $||f(q)||_1$ denote the sum of the absolute values of the coefficients of a Laurent polynomial $f(q) \in \mathbb{Q}[q^{\pm 1}]$. It suffices to show that there exists $C > 0$ so that

$$||a_{\mathfrak{t},n}(q)||_1 \leq C^n \tag{60}$$

for all n. In that case, since $|e^{2\pi i/n}| = 1$, it follows that

$$|a_{\mathfrak{t},n}(e^{\frac{2\pi i}{n}})| \leq C^n$$

for all n, thus $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ is analytic for $|z| < 1/C$. Now, recall that for natural numbers a, b with $a \geq b \geq 0$ we have:

$$\binom{a}{b}_q \in \mathbb{N}[q^{\pm 1}], \qquad ||\binom{a}{b}_q||_1 = \binom{a}{b} \leq 2^a.$$

(see for example, [St]). In addition for natural numbers c, d with $c \geq d \geq 0$ we have:

$$\frac{(q)_c}{(q)_d} = \prod_{j=d+1}^{c} (1 - q^j), \qquad ||\frac{(q)_c}{(q)_d}||_1 \leq 2^{c-d} \leq 2^c.$$

Equation (58) implies that

$$||a_{\mathfrak{t},n}(q)||_1 \leq \sum_{k' \in nP_{\mathfrak{t}} \cap \mathbb{N}^r} ||\mathfrak{t}_{n,k'}(q)||_1$$

Since the number of terms in the above sum is bounded by a polynomial function of n, and the summand is bounded by an exponential function of n, Equation (60) follows. □

5.3. **Examples of special q-terms from Quantum Topology.** Quantum Topology gives a plethora of special q-terms $\mathfrak{t}$ to knotted 3-dimensional objects whose corresponding sequence $\alpha_{\mathfrak{t},n}(q)$ depends on the knotted object itself. A concrete example is given in [GL1, Sec.3]. To state it, let $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ denote the *colored Jones polynomial* of a *knot* K, colored by the n dimensional irreducible representation of $\mathfrak{sl}_2$ and normalized to equal to 1 at the unknot; see [Tu]. In [GL1, Lem.3.2] the following is shown.

Lemma 5.5. *Let β denote a braid whose closure is a knot K. Then, there exists a special q-term $\mathfrak{t}_\beta$ such that*

$$J_{K,n}(q) = a_{\mathfrak{t}_\beta,n}(q) \tag{61}$$

for all $n \in \mathbb{N}$. It follows that $a_{\mathfrak{t}_\beta,n}(e^{\frac{2\pi i}{n}})$ is the n-th Kashaev invariant *of K;* [Ks].

Please observe that the special q-term $\mathfrak{t}_\beta$ depends on the braid β, and that a fixed knot K can always be obtained as the closure of infinitely many braids β. Nonetheless, for all such braids β, the sequence of polynomials $(a_{\mathfrak{t}_\beta,n}(q))$ depend only on the knot K.

Remark 5.6. A computer implementation of Lemma 5.5 is available from [B-N]. It uses as input a braid word in the standard generators of the braid group, and outputs the expression (56) of the corresponding special q-term.

Quantum Topology constructs many more examples of special q-terms, that depend on a pair $(K, \mathfrak{g})$ of a knot K and a simple Lie algebra $\mathfrak{g}$, or to a pair $(M, \mathfrak{g})$ of a closed 3-manifold M with the integer homology of S^3 and a simple Lie algebra $\mathfrak{g}$.

5.4. **An ansatz for the singularities of $L^{\mathrm{np}}_{\mathfrak{t}}(z)$.** In this section we connect the two different parts of the paper. Namely, a special q-term $\mathfrak{t}$ gives rise to

(a) a finite set $\mathrm{CV}_{\mathfrak{t}} \subset \mathbb{C}^*$ from Definition 3.14,
(b) a generating series $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ from Definition 5.3.

Conjecture 1. *For every special q-term $\mathfrak{t}$, the germ $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ has an analytic continuation as a multivalued function in $\mathbb{C} \setminus (\mathrm{CV}_{\mathfrak{t}} \cup \{0\})$. Moreover, the local monodromy of $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ is* quasi-unipotent, *i.e., its eigenvalues are complex roots of unity.*

Example 5.7. If the special q-term is given by (15), then Conjecture 1 is known; see [CG]. With the notation of (15), the case of $(a, b, \epsilon) = (-1, 2, -1)$ coincides with the Kashaev invariant of the 4_1 knot.

In a sequel to this paper [Ga3], we discuss in detail examples of Conjecture 1 for special q-terms that comes from Quantum Topology.

6. For completeness

6.1. **Motivation for the special function Φ and the potential function.** Recall the special function Φ given by Equation (36). The next lemma is our motivation for introducing Φ. Kashaev informs us that this computation was well-known to Faddeev, and was a starting point in the theory of q-dilogarithm function; see [FK].

Lemma 6.1. *For every $\alpha \in (0, 1)$ we have:*

$$\prod_{k=1}^{[\alpha N]} (1 - e^{\frac{2\pi i k}{N}}) = e^{N\Phi(e^{2\pi i \alpha}) + O\left(\frac{\log N}{N}\right)}. \tag{62}$$

Proof. The proof is similar to the proof of [GL2, Prop.8.2], and follows from applying the *Euler-MacLaurin summation formula*

$$\begin{aligned} \log(\prod_{k=1}^{[\alpha N]} (1 - e^{\frac{2\pi i k}{N}})) &= \sum_{k=1}^{[\alpha N]} \log(1 - e^{\frac{2\pi i k}{N}}) \\ &= \alpha N \int_0^1 \log(1 - e^{2\pi i \alpha x}) dx + O\left(\frac{\log N}{N}\right), \end{aligned}$$

together with the fact that:

$$\int_0^1 \log(1 - e^{2\pi i a x}) dx = \frac{1}{2\pi i \alpha}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{2\pi i \alpha})\right).$$

□

Fix a special q-term $\mathfrak{t}_k(q)$ where $k = (k_1, \dots, k_r)$ and a positive natural number $N \in \mathbb{N}$. Fix also $w = (w_0, \dots, w_r)$ where $w_i \in (0, 1)$ for $i = 0, \dots, r$. Let us abbreviate $([w_0 N], [w_1 N], \dots, [w_r N])$ by $[wN]$. Lemma 6.1 implies the following.

Lemma 6.2. *With the above assumptions, we have:*

$$\log \mathfrak{t}_{[wN]} = e^{N V_{\mathfrak{t}}(e^{2\pi i w}) + O\left(\frac{\log N}{N}\right)}. \tag{63}$$

This motivates our definition of the potential function.

6.2. **Laplace's method for a q-term.** There is an alternative way to derive the Variational Equations (11) from a q-term $\mathfrak{t}$.

Since $\mathfrak{t}_k$ is q-hypergeometric, and $k = (k_1, \dots, k_r)$, it follows that for every $i = 0, \dots, r$ we have

$$R_i(z_0, \dots, z_r, q) := \frac{\mathfrak{t}_{n, k_1, \dots, k_i+1, \dots, k_r}(q)}{\mathfrak{t}_{n, k_1, \dots, k_r}(q)} \in \mathbb{Q}(z_0, \dots, z_r, q)$$

where $z_i = q^{k_i}$ for $i = 1, \dots, r$ and $z_0 = q^n$. It is easy to see that the system of equations:

$$R_i(z_0, z_1, \dots, z_r) = 1, \qquad i = 0, \dots, r.$$

is identical to the system (11) of variational equations. In discrete math, the above system is known as *Laplace's method*.

6.3. **A comparison between the Bloch group and its extended version.** A comparison between the extended Bloch-Suslin complex and the Suslin complex is summarized in the following diagram with short exact rows and columns. The diagram is taken by combining [GZ, Sec.3] with [Ne1, Thm.7.7], and using the map $\hat{\chi}$ from [GZ, Eqn.3.11].

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "e^{2\pi i \bullet}"] \arrow[d, "\hat{\psi}"] & \mathbb{C}^* \arrow[r] \arrow[d, "\hat{\chi}"] & \mathbb{C}/(\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathcal{B}(\mathbb{C})} \arrow[r] \arrow[d] & \widehat{\mathcal{P}(\mathbb{C})} \arrow[r, "\hat{\nu}"] \arrow[d] & \mathbb{C} \wedge \mathbb{C} \arrow[r] \arrow[d, "-\exp\wedge\exp"] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{B}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r, "\nu"] \arrow[d] & \mathbb{C}^* \wedge \mathbb{C}^* \arrow[r] \arrow[d] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

where

$$
\begin{aligned}
G \wedge G &= G \otimes_{\mathbb{Z}} G/(a \otimes b + b \otimes a) \\
\hat{\psi}(z) &= \hat{\chi}(e^{2\pi i z}) \\
\tau(z) &= z \wedge 2\pi i \\
(-\exp \wedge \exp)(a \wedge b) &= -(e^{2\pi i a} \wedge e^{2\pi i b}).
\end{aligned}
$$

In addition, we have the following useful Corollary, from [GZ, 3.14].

Corollary 6.3. *For $z \in \mathbb{C}$ we have:*

$$\frac{1}{(2\pi i)^2} \hat{R}(\hat{\chi}(e^{2\pi i z})) = z$$

The restriction

$$\hat{R} : \mathrm{Ker}(\widehat{\mathcal{B}(\mathbb{C})} \to \mathcal{B}(\mathbb{C})) \longrightarrow \mathbb{C}/\mathbb{Z}(2)$$

is 1-1.

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA, `http://www.math.gatech.edu/~stavros`
E-mail address: `stavros@math.gatech.edu`