NON-PERIPHERAL IDEAL DECOMPOSITIONS OF ALTERNATING KNOTS

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ABSTRACT. An ideal triangulation $\mathcal{T}$ of a hyperbolic 3-manifold M with one cusp is non-peripheral if no edge of $\mathcal{T}$ is homotopic to a curve in the boundary torus of M. For such a triangulation, the gluing and completeness equations can be solved to recover the hyperbolic structure of M. A planar projection of a knot gives four ideal cell decompositions of its complement (minus 2 balls), two of which are ideal triangulations that use 4 (resp., 5) ideal tetrahedra per crossing. Our main result is that these ideal triangulations are non-peripheral for all planar, reduced, alternating projections of hyperbolic knots. Our proof uses the small cancellation properties of the Dehn presentation of alternating knot groups, and an explicit solution to their word and conjugacy problems. In particular, we describe a planar complex that encodes all geodesic words that represent elements of the peripheral subgroup of an alternating knot group. This gives a polynomial time algorithm for checking if an element in an alternating knot group is peripheral. Our motivation for this work comes from the Volume Conjecture for knots.

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1. INTRODUCTION

1.1. **Motivation: the Volume Conjecture.** The motivation of our paper comes from the Kashaev's *Volume Conjecture* for knots in 3-space, which states that for a hyperbolic knot K in S^3 we have:

$$\lim_{n \to \infty} \frac{1}{n} \log |\langle K \rangle_N| = \frac{\text{Vol}(K)}{2\pi}$$

where $\langle K \rangle_N$ is the *Kashaev invariants* of K; see [Kas97, MM01]. This gives a precise connection between quantum topology and hyperbolic geometry. The Volume Conjecture has been verified for only a handful hyperbolic knots: initially for the simplest hyperbolic 4_1 knot and now, due to the work of Ohtsuki [Oht17], and Ohtsuki and Yokota [OY16], for all hyperbolic knots with at most 6 crossings.

The Volume Conjecture requires a common input for computing both the Kashaev invariant and the hyperbolic volume. Such an input turns out to be a planar projection of a knot K which allows one to express the Kashaev invariant as a multi-dimensional state sum whose summand is a ratio of quantum factorials (4 or 5, depending on the model used).

On the other hand, a planar projection gives four ideal cell decompositions of its complement (minus 2 balls), two of which are ideal triangulations that use 4 (resp., 5) ideal tetrahedra per crossing. These ideal triangulations are well-known from the early days of hyperbolic geometry, and were used by Weeks [Wee05] (in his computer program `SnapPy` [CDW]), by the third author [Thu99], Yokota [Yok02, Yok11], Sakuma-Yokota [SY] and others.

An approach to the Volume Conjecture initiated by the third author in [Thu99], and also by Yokota, Kashaev, Hikami, the first author and others (see [Gar08, KY, Hik01, Yok02]), is to convert multi-dimensional state-sum formulas for the Kashaev invariant to multi-dimensional state-integral formulas over suitable cycles, and then to apply a steepest descent method to study the asymptotic behaviour of the Kashaev invariant. The summand (and hence, the integrand) depends on the planar projection and the steepest descend method is applied to a leading term of the integrand, the so-called potential function. The critical points of the potential function have a geometric meaning, namely they are solutions to the *gluing equations*. The latter are a special system of polynomial equations (studied by W. Thurston and Neumann-Zagier in [Thu77, NZ85]) that are associated to the ideal triangulations of the knot complement discussed above. A suitable solution to the gluing equations recovers the hyperbolic structure, and the value of the potential function is the volume of the knot.

The problem is that every planar projection leads to ideal triangulations, hence to gluing equations, and even if we know that the knot is hyperbolic, it is by no means obvious that those gluing equations have a suitable solution (or in fact, any solution) that recovers the complete hyperbolic structure. It turns out that if a knot is hyperbolic, the lack of a suitable solution occurs only when edges of the ideal triangulation are homotopic to peripheral curves in the boundary tori.

1.2. **Non-peripheral ideal triangulations of alternating knots.** *Ideal triangulations* of hyperbolic 3-manifolds with cusps were introduced by W. Thurston in his study of Geometrization of 3-manifolds; see [Thu77]. For thorough discussions, see [BP92, CDW, NZ85, Wee05]. An ideal triangulation $\mathcal{T}$ of a hyperbolic 3-manifold M with one cusp is *non-peripheral* if no edge of $\mathcal{T}$ is homotopic to a curve in the boundary torus of M. For such a triangulation, the gluing and completeness equations of [NZ85] can be solved to recover the hyperbolic structure of M. For a proof, see [Til12, Lem.2.2] and also the discussion in [DG12, Sec.3].

A planar projection Δ of a knot gives rise to four ideal cell decompositions of its complement (namely, $\mathcal{T}_{2B}(\Delta)$, $\mathcal{T}_O^\circ(\Delta)$, $\mathcal{T}_{4T}^\circ(\Delta)$ and $\mathcal{T}_{5T}^\circ(\Delta)$), the last two of which are ideal triangulations that use 4 (resp., 5) ideal tetrahedra per crossing. We will briefly recall these decompositions here, although their precise definition is not needed for the statement and proof of Theorem 1.3 below.

• $\mathcal{T}_{2B}(\Delta)$ is a decomposition of the knot complement into one ball above and one ball below the planar projection. These two balls have a cell-decomposition that matches the planar projection of the knot, and were originally studied by W. Thurston, and more recently by Lackenby [Lac04].

- $\mathcal{T}_O^\circ(\Delta)$ is a decomposition of the knot complement minus two balls into ideal octahedra, one at each crossing of Δ. This was described by Weeks [Wee05], and also by the third author [Thu99], and by Yokota [Yok02, Yok11].
- Each ideal octahedron can be subdivided into 4 ideal tetrahedra, or into 5 ideal tetrahedra. Thus, a subdivision of $\mathcal{T}_O^\circ(\Delta)$ gives rise to two ideal triangulations of the knot complement minus two balls, denoted by $\mathcal{T}_{4T}^\circ(\Delta)$ and $\mathcal{T}_{5T}^\circ(\Delta)$.

Theorem 1.1. *If Δ is a prime, reduced, alternating projection of a non-torus knot K, then the four ideal cell decompositions $\mathcal{T}_{2B}(\Delta)$, $\mathcal{T}_O^\circ(\Delta)$, $\mathcal{T}_{4T}^\circ(\Delta)$ and $\mathcal{T}_{5T}^\circ(\Delta)$ are non-peripheral. Consequently, the gluing equations have a solution that recovers the complete hyperbolic structure.*

1.3. **Alternating knots and small cancellation theory.** The above theorem follows from proving that all edges of the above ideal triangulations are homotopically non-peripheral. Luckily, we can describe those edges directly in terms of the planar projection of the knot as follows.

Definition 1.2. Let $\Delta \subset \mathbb{R}^2$ be a knot diagram with n crossings. Consider the projection plane $\mathbb{R}^2$ as the xy-plane of $\mathbb{R}^3$, and consider the knot $K \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ obtained from Δ by "pulling" the overcrossing arcs above the plane and undercrossing arcs under the plane in the standard way. Fix a basepoint for $\pi_1(S^3 \setminus K)$ in the unbounded region near one strand of K. We distinguish four kinds of loops in $\pi_1(S^3 \setminus K)$.

(1) A *Wirtinger arc* follows the double of Δ through k crossings with $1 < k < 2n$ and then returns to the basepoint through either the upper or lower half-space.

(2) A *Wirtinger loop* starts at the basepoint, travels in either the upper (resp. lower) half-space to pass through a region R of Δ, passes through a region adjacent to R, and then returns through the upper (resp. lower) half-space to the basepoint. We forbid the short loop around the strand near the basepoint, which is manifestly a meridian.

(3) A *Dehn arc* starts at the basepoint, travels in the upper (resp. lower) half-space through a region of Δ and then returns to the basepoint through the lower (resp. upper) half-space without passing through the projection plane.

(4) A *short arc* follows the double of Δ from the basepoint until some crossing, where it jumps to the other strand in the crossing and then follows the double back to the basepoint.

There four types of arc are illustrated in Figure 1.

These arcs are denoted by the letters A, B, C and D in [SY].

Theorem 1.3. *If Δ is a prime, reduced, alternating projection of a non-torus knot K, then all Wirtinger arcs, Wirtinger loops, Dehn arcs and short arcs are non-peripheral.*

Theorem 1.1 immediately follows from Theorem 1.3, since all of the arcs that appear in any of the decompositions in Theorem 1.1 are of one of the four types in Theorem 1.3.

The proof of Theorem 1.3 uses the small cancellation property of the Dehn presentation of hyperbolic alternating knots. Curiously, our proof uses an explicit solution to the conjugacy problem of the Dehn presentation of a prime reduced alternating planar projection Δ. See Remark 2.17 below. More generally, we emphasise that the approach we take in this paper is an algorithmic one. Moreover our algorithms run in polynomial-time (in the length of the word).

Wirtinger arc   Wirtinger loop

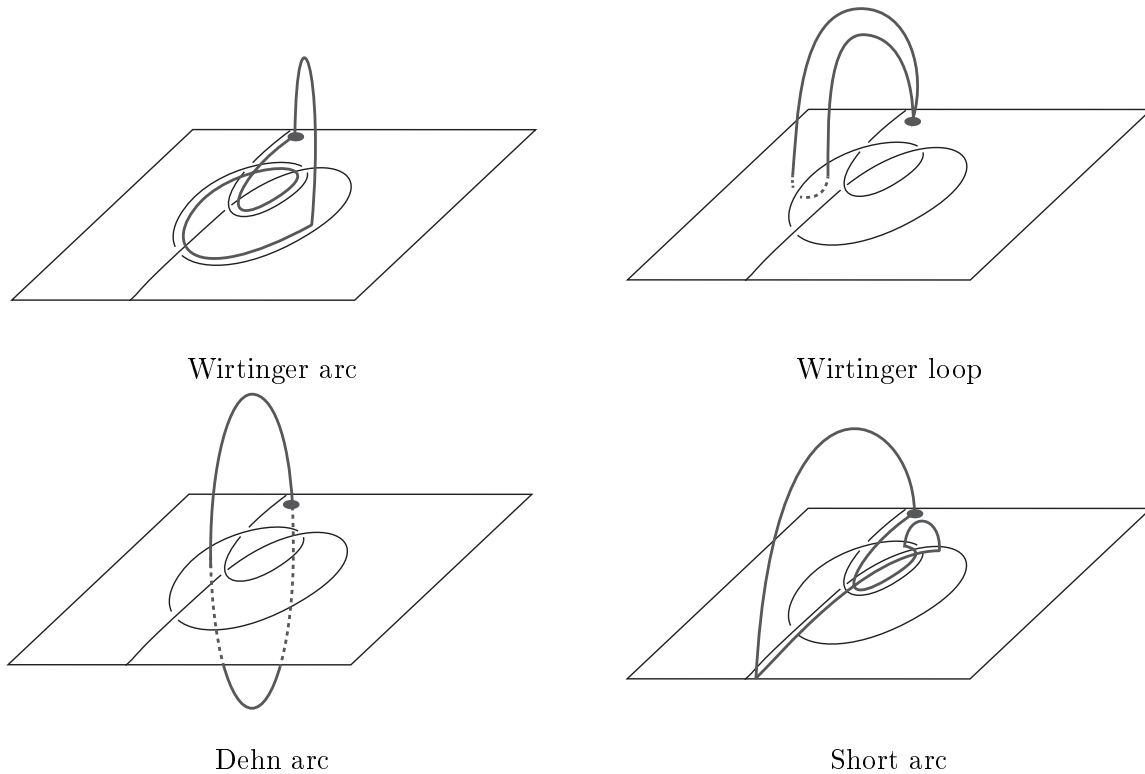Dehn arc   Short arc

FIGURE 1. Four types of loop in a knot complement.

2. SMALL CANCELLATION THEORY

2.1. **The (augmented) Dehn presentation of a knot group.** We begin with a discussion of the augmented Dehn presentation of a knot diagram. As it turns out, the augmented Dehn presentation (defined below) is a small cancellation group and this structure provides a quick and implementable solution to its word problem. Background on small cancellation groups and combinatorial group theory can be found in [LS77].

Throughout this paper we implicitly *symmetrize* all group presentations. This means that when we write a set of relators R, we actually mean the set of all relators which can be obtained from R by inversion and cyclic permutation.

Let Δ be a n crossing planar diagram of a link L. Of the $n+2$ regions of the diagram Δ, exactly $n+1$ of these regions are bounded. Assign a unique label $1, 2, \ldots, n+1$ to each of these bounded region and the label 0 to the unbounded region. We identify each region with its label.

We obtain a group presentation from the labelled diagram Δ as follows. Take one generator X_i for each region $i = 0, 1, 2, \ldots, n+1$ of Δ. Take one relator R_i for each of the n crossings of Δ which is read from the diagram thus

$$
\begin{array}{c|c} a & b \\ \hline d & c \end{array} \qquad \rightsquigarrow \qquad X_a X_b^{-1} X_c X_d^{-1}
$$

If we choose a base point above the projection plane, and we choose a point p_i in the interior of each region i. Then the generator X_i can be described geometrically by a loop in the knot complement which passes from the base point, downwards through the region p_i then back up to the base point

through the point p_0 which lies in the unbounded region. Dehn showed that

$$\mathcal{D}_\Delta \stackrel{def}{=} \langle X_0, X_1, \ldots, X_{n+1} \mid R_1, R_2, \ldots R_n, X_0 \rangle$$

is a presentation for the knot group $\pi_1(S^3 \setminus L)$. We call this the *Dehn presentation* of $\pi_1(S^3 \setminus L)$ read from the diagram Δ. In what follows, we use a minor modification of the Dehn presentation which has better small cancellation properties.

The *augmented Dehn presentation*, $\mathcal{A}_\Delta$, of Δ is the group presentation

$$\mathcal{A}_\Delta \stackrel{def}{=} \langle X_0, X_1, \ldots, X_{n+1} \mid R_1, R_2, \ldots R_n \rangle .$$

The augmented Dehn presentation arises as a Dehn presentation of a link. Given a labelled link diagram Δ, construct a new labelled link diagram $\Delta \cup \mathcal{O}$ by adding a zero-crossing component $\mathcal{O}$, which bounds Δ. This is called the *augmented link diagram*. The augmented Dehn presentation of Δ is a presentation for the for the *augmented link group* $\pi_1(S^3 \setminus (K \cup \mathcal{O}))$, i.e.,

$$(1) \qquad\qquad \mathcal{A}_\Delta \cong \mathcal{D}_{\Delta \cup \mathcal{O}} \cong \pi_1(S^3 \setminus K) * \mathbb{Z} .$$

We will solve the word problem in $\mathcal{D}_\Delta$ by solving it in $\mathcal{A}_\Delta$. For completeness, let us say a few words about why it is sufficient to solve the word problem in $\mathcal{A}_\Delta$. This is a consequence of some standard facts about group presentations that can be found in, for example, [LS77]. Let $\mathcal{P}_G = \langle g_1, \ldots, g_k \mid r_1, \ldots r_j \rangle$ and $\mathcal{P}_H = \langle h_1, \ldots, h_l \mid s_1, \ldots s_m \rangle$ be presentations for groups G and H respectively. Then the *standard presentation*, which we denote by $\mathcal{P}_G * \mathcal{P}_H$, for the free product $G * H$ is

$$\mathcal{P}_G * \mathcal{P}_H = \langle g_1, \ldots, g_k, h_1, \ldots, h_l \mid r_1, \ldots r_j, s_1, \ldots s_m \rangle .$$

A standard consequence of the *normal form for free products* (again see [LS77]) is that with $\mathcal{P}_G$, $\mathcal{P}_H$ and $\mathcal{P}_G * \mathcal{P}_H$ as above, if w is a word in the generators $g_1, \ldots, g_k$ and their inverses, then $w =_G 1$ if and only if $w =_{G*H} 1$. Thus, by (1), the word problem in $\mathcal{D}_\Delta \cong \pi_1(S^3 \setminus K)$ can be solved by the word problem in $\mathcal{A}_\Delta \cong \pi_1(S^3 \setminus K) * \mathbb{Z}$.

An an explicit isomorphism of the augmented Dehn presentation with a standard presentation for the free product $\pi_1(S^3 \setminus K) * \mathbb{Z}$ is given by

$$(2) \qquad\qquad \phi : \mathcal{A}_\Delta \to \mathcal{D}_\Delta * \langle Y \mid \ \rangle$$

where

$$\phi : X_i \mapsto \begin{cases} Y & \text{if } i = 0 \\ X_i Y^{-1} & \text{otherwise} \end{cases} .$$

Geometrically, ϕ corresponds to isotoping the component $\mathcal{O}$ of the augmented link in S^3 away from the subdiagram Δ so that it bounds a disc in the projection plane.

Remark 2.1. Let $\iota : \mathcal{D}_\Delta \to \mathcal{D}_\Delta * \langle Y \mid \ \rangle$ denote the natural inclusion. Given a projection l of a loop $\ell \in \pi_1(S^3 \setminus K)$ in the diagram Δ, we can read off a representative $\phi^{-1}(\iota(w))$ as follows: follow the loop l from its basepoint in the direction of its orientation. When l "passes downwards" through a region i of Δ assign a generator X_i; and whenever l "passes upwards" through a region i of Δ assign a generator X_i^{-1}. The word thus obtained clearly represents the loop ℓ. Thus, $w \neq_{\pi_1(S^3 \setminus K)} 1$ if and only if $\phi^{-1}(\iota(w)) \neq_{\mathcal{A}_\Delta} 1$.

2.2. Square and grid presentations.

The augmented Dehn presentation of a prime, reduced, alternating knot diagram has *small cancellation* properties, as was first observed by Weinbaum in [Wei71].

Let $G = \langle X | R \rangle$ be a symmetrized group presentation. We call a non-empty word r a *piece* with respect to R if there exist distinct words $s, t \in R$ such that $s = ru$ and $t = rv$.

Definition 2.2. (a) A symmetrized presentation $\langle X | R \rangle$ is called a *square presentation* if it satisfies the following two small cancellation conditions:

Condition $C''(4)$. All relators have length four and no defining relator is a product of fewer than four pieces.

Condition $T(4)$. Let r_1, r_2 and r_3 be any three defining relators such that no two of the words are inverses to each other, then one of $r_1 r_2$, $r_2 r_3$ or $r_3 r_1$ is freely reduced without cancellation.

(b) A symmetrized presentation $\langle X | R \rangle$ is called a *grid presentation* if it is a square presentation and in addition X is colored by two colors (black or white) and every relator alternates in the two colors and in taking inverse.

Remark 2.3. There does not appear to be a standard terminology of the above definition. In [Wei71], Weinbaum calls square presentations $C''(4) - T(4)$ presentations. In [Joh97, Joh00], Johnsgard uses the term *parity* to denote the black/white coloring of a grid presentation. In [Wis06, Defn.3.1] and [Wis07, Defn.2.2], Wise uses the terms squared presentations and VH presentations for our square presentations and grid presentations.

We may depict a relator r of a grid presentation by a Euclidean square as follows:

$$ab^{-1}cd^{-1} \quad \longleftrightarrow \quad$$

It is easy to see that in a grid presentation the following holds:

- Relator squares have oriented edges, labelled from X. There are two sinks and two sources in each relator square.
- We call a two letter subword of a relator a *pair*. The $C''(4)$ condition says that a pair uniquely determines a relator up to cyclic permutation and inversion.
- $T(4)$ says that if ab and $b^{-1}c$ are pairs then ac is not.
- If a, b and c are letters such that ab and $b^{-1}c$ are both pairs (with $b \neq c$), then the word ac is called a *sister-set*. By the $T(4)$ condition, no pair is a sister-set.
- The edges of a relator square have an additional coloring: they are vertical or horizontal. Moreover, going around a relator square we alternate between black and white.
- We can invoke a convention that the black and white colorings correspond to horizontal and vertical line placement in our drawings of relator squares.
- A rotation or reflection of a relator square corresponds to the cyclic permutation or inversion of a relator.

We can now state Weinbaum's theorem.

Theorem 2.4. [Wei71] *The augmented Dehn presentation of a prime, reduced, alternating knot diagram is a grid presentation.*

In [LS77] Lyndon and Schupp show that square and grid presentations have have solvable word and conjugacy problems. Since the appearance of that work, polynomial time algorithms have been given for the word (see [Joh97, Sec.7])) and conjugacy problems ([Joh97]) of these groups. We use these more efficient algorithms here.

2.3. **The word problem for square presentations.** In this section we recall the solution to the word problem of square presentations. To any group presentation $G = \langle X | R \rangle$ we can associate a *standard 2-complex K* in the usual way: K consists of one 0-cell, one labelled 1-cell for each generator and one 2-cell for each relator, where the 2-cell D_r representing the relator $r \in R$ is attached to the 1-skeleton, $K^{(1)}$, by a continuous map which identifies the boundary ∂D_r with a loop representing r in the 1-skeleton. We impose a piece-wise Euclidean structure on the standard 2-complex and set all 1-cells to be of unit length.

A word w represents the identity in G if and only if there is a simply connected planar 2-complex Δ, and a map $\phi : (D, \partial D) \to (K, K^{(1)})$ such that the 0-cells are mapped to 0-cells, open i-cells are mapped to open i-cells, for $i = 1, 2$ and ∂D is mapped to the loop representing w in $K^{(1)}$. Such a 2-complex, labelled in the natural way, is called a *Dehn diagram*.

Throughout this text we use two concepts of labels of edge-paths of the standard 2-complex, peripheral complex (introduced below) or Dehn diagram. The label of an edge-path is the sequence of letters determined by the edge-path, where travelling along an edge labelled a contributes the letter a. This is distinct from the word labelling an edge-path, which is the word in the group determined by the path, where travelling along an edge labelled a against the orientation contributes the letter a^{-1}, and travelling with the orientation, the letter a.

A word in a group presentation is said to be *geodesic* if it contains the least number of letters over all representatives of the same word, i.e., w is geodesic if $|w| = \min\{|w'| \mid w =_G w'\}$. A geodesic word represents the identity if and only if it is the empty word. A word in a group presentation is geodesic if and only if it labels a geodesic edge-path in the standard two complex of the presentation.

A key result of small cancellation theory is the following *Geodesic Characterisation Theorem*; see [Joh97, Sec.3] and also [Kap97, Lem.3.2].

Theorem 2.5. *A word in a square presentation is geodesic if and only if it is freely reduced and contains no subword $x_1 \ldots x_n$ which is part of a* chain:
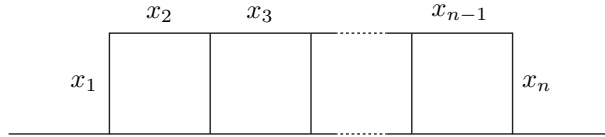
*The word $x_1 \ldots x_n$ is called a* chain word.

Remark 2.6. Observe that the Geodesic Characterisation Theorem immediately provides a quadratic time solution for the word problem in square presentations: Given a word w, freely reduce it to obtain a word w'. If w' is the empty word then $w =_G 1$, otherwise search w' for a chain word. If w' does not contain a chain word then $w' \neq_G 1$. If w' does contain a chain word, replace it with the shorter word which bounds the "other side" of the chain to obtain a shorter word w''. Repeat the above process with the word w'' in place of w.

2.4. **The peripheral complex.** A reduced, prime, alternating, oriented knot diagram gives rise to a grid presentation with solvable word problem; see Theorems 2.4 and 2.5. This grid presentation contains a *peripheral $\mathbb{Z}^2$-subgroup* generated by the meridian m and the longitude l of the knot. Of course, a peripheral subgroup does not exist for a general grid presentation.

Theorem 1.3 requires us to solve the *peripheral word problem*. Following Johnsgard (see [Joh97, Sec.7]), we consider the (rather overlooked) *peripheral complex*, and we discuss how it solves the peripheral word and conjugacy problem.

Let Δ be a reduced, prime, alternating, oriented knot diagram with n crossings, and let $\mathcal{A}_\Delta$ be its augmented Dehn presentation. Each relator of $\mathcal{A}_\Delta$ is a word of length four whose exponents alternate in sign. We may think of the relators as 1×1 Euclidean squares with directed and labelled edges. For convenience, we impose some conventions upon our construction. We discuss the effect of these conventions in Remark 2.9 below.

From the base point of Δ and in the direction of the orientation, walk around the diagram and label the n crossings of Δ with $c_1, c_2, \ldots, c_{2n}$ in the order we meet them and in such a way that the label c_1 is assigned to the first under crossing we meet. For example, for the 5_2 knot we have:
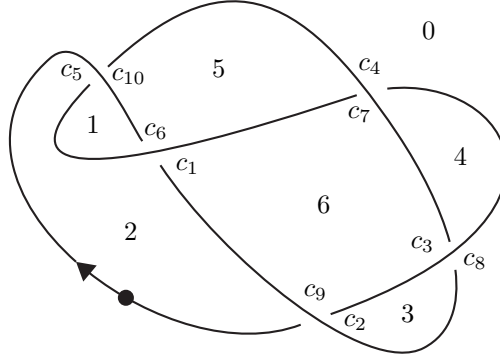
We construct a $2n \times 1$ rectangle made out of $2n$ relator squares inductively as follows. Position the relator square C_1 on the Euclidean plane in such a way that the label of the edge-path from $(0,0)$ to $(1,1)$ describes a loop which follows the knot through the undercrossing at c_1 (on the left is shown the crossing c_1 and on the right is shown the relator square C_1):
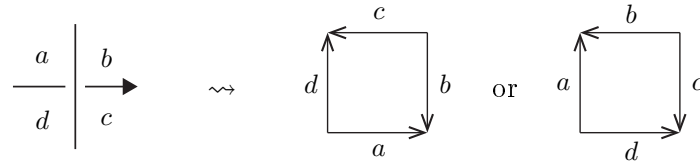
Suppose we have placed a relator square C_k (which arises from the crossing c_k). The relator squares C_k and C_{k+1} have exactly two edge-labels in common (since the diagram Δ is prime and reduced). Identify the right edge of C_k with the unique edge of C_{k+1} which has the same label in a way that preserves the orientation of the edges. This gives a $(k+1) \times 1$ rectangle. Continue this process until we have added the relator square C_{2n}.

We call such a $2n \times 1$ rectangle of relator squares a *fundamental block* of Δ. For example, the fundamental block of the 5_2 knot above is
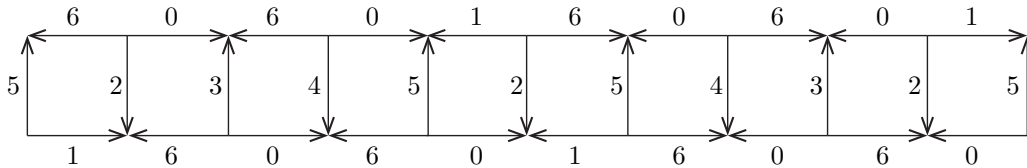
Observe that the fundamental block has oriented edges and its vertices are either sinks or sources. We will often simplify figures by drawing sinks as thickened black vertices. This determines the orientations of the edges.

Notice in the example above that the word labelling the top edge of the fundamental block is a cyclic permutation of the word labelling the bottom edge of the fundamental block, and that the labels and orientations on the left and right edges coincide. This observation holds in general and it allows us to piece together the fundamental blocks in a way that tiles the plane.

Lemma 2.7. *In the fundamental block of a reduced, prime, alternating, oriented knot diagram Δ,*

  (1) the label and orientation of the rightmost and leftmost vertical edges of the fundamental block coincide;
  (2) the label and orientation on the top of the relator square C_i is the same as the label and orientation on the bottom of the relator square C_{i+1}, where the indices are taken modulo $2n$.

We defer the proof of this lemma until the end of Section 2.5.

Using Lemma 2.12 we can piece together together the fundamental blocks according to the following pattern,
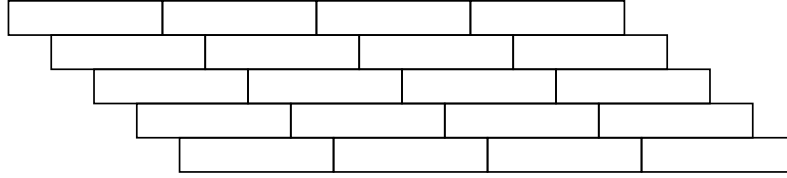
and tile the whole plane by relator squares.

Definition 2.8. We call the resulting 2-dimensional CW complex the *peripheral complex*.

For example, a portion of the peripheral complex for the 5_2 knot is given by:
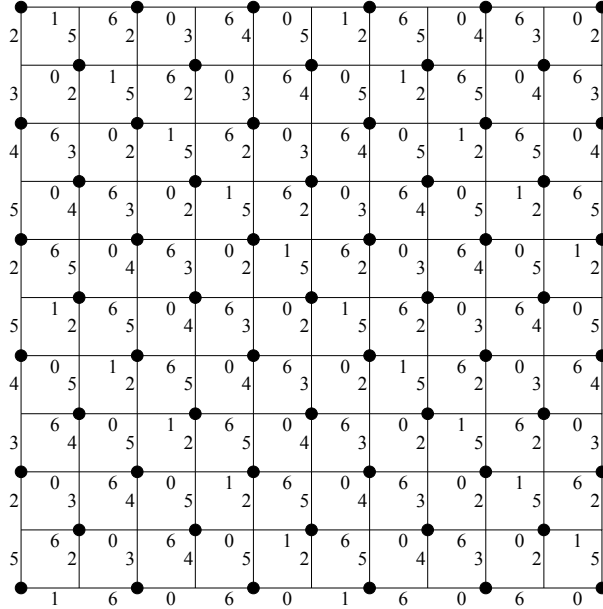
Remark 2.9. Several choices and conventions were made in the construction of the peripheral complex. Namely the choice of base point on Δ, the label c_1 was assigned to the first under crossing we met, and the positioning of the first relator square C_1. It is clear from the construction of the complex that a different choice of base point (as well as orientation) and a different placement of C_1 on the Euclidean plane would result in a peripheral complex which is isometric to the one constructed here. We discuss this in more detail in Section 2.8.

All of the arguments presented here can be made with any construction of the peripheral complex, however the directions specified in the statements of results and proofs in this paper may change.

2.5. **Some properties of the peripheral complex.** By construction, the peripheral complex embeds in the standard 2-complex of the augmented Dehn presentation. In fact it embeds geodesically:

Lemma 2.10. *The peripheral complex of Δ embeds geodesically in the standard 2-complex of the augmented Dehn presentation $\mathcal{A}_\Delta$. In particular, the word labelling any geodesic edge-path in the peripheral complex is a geodesic word in the augmented Dehn presentation.*

Proof. The proof uses the Geodesic Characterisation Theorem (Theorem 2.5). Since any two paths in the peripheral complex with common beginning and ending represent the same word in $\mathcal{A}_\Delta$, it suffices to show that a path p that goes horizontally q steps and then vertically r steps in the peripheral complex is geodesic in the standard 2-complex.

Consider an edge-path p in the peripheral complex that goes horizontally q steps and then vertically r steps. Such a path has at most one pair subword, since a sister-set is never a pair by the

$T(4)$ condition. Thus we see that the label of the edge-path cannot contain a chain word, as this requires two pairs. (Recall the definitions of pairs and sister sets from Section 2.2.)

It remains to show that the label of the edge-path is freely reduced. To see why this is we begin by observing that since Δ is reduced, four distinct regions of Δ meet at every crossing and therefore every relator square has four distinct labels. Now suppose that ab is a subword of the word labelling the edge-path p. If the subword belongs to the horizontal path, then it labels the bottom of two relator squares D_i and D_{i+1} in the peripheral complex. By Lemma 2.7, the bottom label of D_{i+1} is also a label of the top of the relator square D_i. This means that b cannot label the bottom of D_i and therefore $a \neq b$ and ab is freely reduced.

If the letter a comes from a horizontal edge and b from a vertical edge of p, then the subword ab labels two sides of a relator square and is therefore freely reduced.

Finally, If the subword belongs to the vertical path, then it labels the right hand side of two relator squares D_i and D_{i+1} in the peripheral complex. By the periodicity of the peripheral complex, the right hand label of D_i is also the label of the left hand side of the relator square D_{i+1}. This means that b cannot label the right of D_{i+1} and therefore $a \neq b$ and ab is freely reduced. $\qquad \square$

Proof of Lemma 2.7. Since the exponents of the relators of the augmented Dehn presentation alternate in sign, all of the orientations of the edges of the fundamental block are of the form required by the lemma.

It remains to show that the edge labels are of the required form. First we show that the label on the top of the relator square C_i is the same as the label on the bottom of the relator square C_{i+1} for $i = 1, \ldots, n$.

Consider the relator square C_1 positioned as
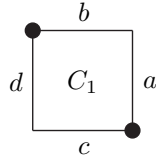
By convention the labels a and b also appear in C_2 (since the regions a and b of Δ are incident with the crossings c_1 and c_2). Therefore C_2 has one of the following forms
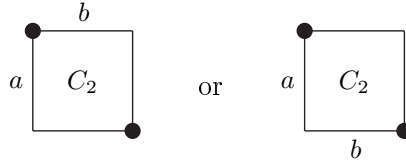
These two relators have edge-paths $b^{-1}a$ and ab^{-1} respectively. By the small cancellation conditions, a pair uniquely determines a relator, so the word ab^{-1} cannot appear in C_2 (as it appears in C_1 as $(ab^{-1})^{-1}$). Therefore b must be the label on the bottom of C_2.

We proceed inductively. Suppose that we have shown that the label on the top of C_{k-1} coincides with the label on the bottom of C_k. Since the crossing c_k shares two incident regions with c_{k-1} and two with c_{k+1}, the relator square C_k share two labels with C_{k-1} and two with C_{k+1}. By hypothesis, C_k shares the labels on the bottom and left-hand edges with C_{k-1}, so it shares the labels on the top and right-hand edges with C_{k+1}.

Suppose the word on the edge-path which follows the right-hand and then top edge of C_k is rs^{-1} or $r^{-1}s$. We will deal with each case separately.

If the path is rs^{-1}. Then C_{k+1} also has edges labelled r and s and must contain the word sr^{-1} or $r^{-1}s$. Since a pair determines a relator and $C_k \neq C_{k+1}$, we have that C_{k+1} must contain the word $r^{-1}s$ (since $sr^{-1} = (rs^{-1})^{-1}$). The only way this can happen is if the letter s is on the bottom of C_{k+1}.

Similarly, if the path is $r^{-1}s$. Then C_{k+1} also has edges labelled r and s and must contain the word $s^{-1}r$ or rs^{-1}. Since a pair determines a relator and $C_k \neq C_{k+1}$, we have that C_{k+1} must contain the word rs^{-1}. The only way this can happen is if the letter s^{-1} is on the bottom of C_{k+1}.

We have shown that the label on the top of the relator square C_i is the same as the label on the bottom of the relator square C_{i+1} for $i = 1, \ldots, n$.

To complete the proof, consider the relator square C_{2n} in the fundamental block. C_{2n} is of the form
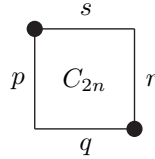

where the labels p and q are shared with C_{2n-1} and r and s are shared with C_1 (since c_1 and c_{2n} share incident regions of Δ). But again, the small cancellation conditions say that a pair uniquely determines a relator and $C_{2n} \neq C_1$, therefore we must have $s = c$ and $r = d$, where c and d are the labels of C_1 as shown above. This completes the proof of the lemma. $\qquad\square$

Lemma 2.11. *In the fundamental block of a reduced, prime, alternating, oriented knot diagram Δ,*

(1) the label of an edge-path from the bottom-right to top-left corner of C_{2n} describes a curve homotopic to a meridional loop, or its inverse, of the knot through the base point of Δ;

(2) the label of an edge-path from the bottom left to top right corner of C_{2i-1}, $i = 1, \ldots, n$ describes a loop which follows the under-crossing of the knot at c_{2i-1};

Proof. The relator square C_{2n} comes from a crossing of the form
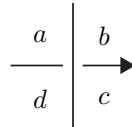
and, since b and c are also labels of C_1, it must appear in the fundamental block in one of the following forms
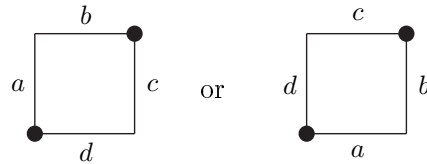
In either case we see that the edge-path bottom-right to top-left corner describes a meridian or its inverse. This proves the first statement of the lemma.
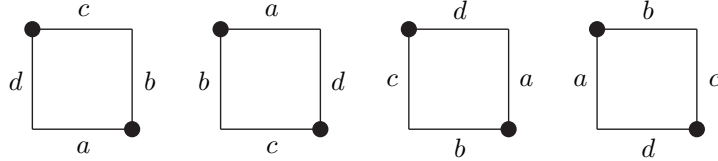
We now prove the second statement. The relator square C_{2i-1} appears in the fundamental block with orientation
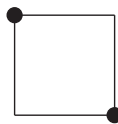
and at c_{2i-1} we travel along an undercrossing of the form

$$\frac{a}{d} \quad \bigg| \quad \frac{b}{c} \rightarrow$$

which contributes the relator $ab^{-1}cd^{-1}$. Therefore C_{2i-1} is of one of the four forms

(these are all possible ways that the relator can fit the orientation of C_{2i-1}). But, by the construction of the fundamental block, a or d must label the vertical left edge of C_{2i-1}. This eliminates two of the four possible labellings of C_{2i-1} above, and it is easily seen that in the remaining two possibilities, the label of an edge-path from the bottom left to top right corner describes a loop which follows the under-crossing of the knot at c_{2i-1}, as required.   □

Let n denote the number of crossings of Δ. Further, let λ denote the double of the diagram Δ determined by the blackboard framing, and based at a point x_0. Let μ be the meridian of Δ based at x_0. The orientations of μ and λ are determined by the orientation of Δ. A peripheral element of the knot group $\pi_1(S^3 - K)$ is then a product $\lambda^a\mu^b$, $a, b \in \mathbb{Z}$. The curves λ and μ in Δ also determine canonical elements $\phi^{-1}(\iota(\lambda))$ and $\phi^{-1}(\iota(\mu))$ of the augmented knot group $G = \pi_1(S^3 - (K \cup \mathcal{O}))$. We abuse notation and also denote these elements by λ and μ respectively. We say that an element of the (augmented) knot group is *peripheral* if it represents the element $\lambda^a\mu^b$ for some $a, b \in \mathbb{Z}$.

Lemma 2.12. *Let w be a word which labels an edge-path from the point $(0,0)$ to the point $(an - b, an + b)$ in the peripheral complex of Δ, where $a, b \in \mathbb{Z}$ and n is the number of crossings of Δ. Then w is peripheral and represents the element $\lambda^a\mu^b$. Conversely, every peripheral element $\lambda^a\mu^b$ has a representative as the label of an edge-path from the point $(0,0)$ to the point $(an - b, an + b)$ in the peripheral complex.*

Proof. We show that there exists one word $l^a m^b$ labelling the edge-path from $(0,0)$ to $(an-b, an+b)$ in the peripheral complex which represents $\lambda^a\mu^b$, for each choice of a and b. Since the peripheral complex complex embeds in the standard 2-complex of the augmented Dehn presentation, it follows that any word which labels a edge-path from the point $(0,0)$ to the point $(an - b, an + b)$ represents the peripheral element $l^a m^b$.

Label the crossings of Δ by $c_1, \ldots, c_{2n}$ according to the conventions in Subsection 2.4. We can find a representative l of λ in the augmented Dehn presentation $\mathcal{A}_\Delta$ as follows: take a framed double λ of Δ. Begin by taking l to be the empty word. Walk once around λ and concatenate a subword $X_a X_b^{-1}$ to the right of l whenever we pass under an arc of Δ from a region labelled a to a region labelled b. The word l obtained clearly represents λ.

Since the knot is alternating and, by our convention on the labelling of the crossing, the double λ of Δ passes under an arc of Δ at the crossings c_{2i-1}, for $i = 1, \ldots, n$. By Lemma 2.11, the two letter subword contributed to l at the crossing c_{2i-1} is exactly the label of an edge-path from the bottom left to top right corner of a relator square C_{2i-1} in the peripheral complex. Therefore l can be described as an edge-path from the bottom left to the top right of the following complex:
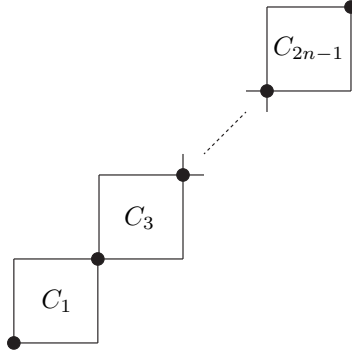
Clearly, such a complex is embedded in the peripheral complex and the edge-path is a path from $(0,0)$ to (n,n). By the periodicity of the peripheral complex, it follows that the word l^a is an edge-path in the peripheral complex from $(0,0)$ to (an,an) for all $a \in \mathbb{Z}$.

We have shown that powers of the longitude are contained in the peripheral complex. We now show that powers of meridians and peripheral elements are also contained in the peripheral complex.

By Lemma 2.11 the label m of an edge-path from $(0,0)$ to $(-1,1)$ represents the meridian μ or its inverse μ^{-1}. By the periodicity of the peripheral complex the label m^b of an edge path from $(0,0)$ to $(-b,b)$, $b \in \mathbb{Z}$, represents a power of the meridian and, again by the periodicity of the peripheral complex (an,an) to $(an-b,an+b)$, $b \in \mathbb{Z}$ represents a power of the meridian for each $a,b \in \mathbb{Z}$.

Therefore the label of an edge-path from the point $(0,0)$ to the point $(an-b,an+b)$ in the peripheral complex is $l^a m^b$ and is peripheral. $\qquad\square$

2.6. The peripheral word problem.
The aim of this section is to solve the peripheral word problem using the peripheral complex.

Theorem 2.13. *Let w be a geodesic word in the augmented Dehn presentation of the n crossing diagram Δ. Then w is peripheral and represents the element $\lambda^a \mu^b$ if and only if it labels a geodesic edge-path from $(0,0)$ to $(an-b,an+b)$ in the peripheral complex of Δ for some $a,b \in \mathbb{Z}$.*

To prove the theorem we need the following result from [Joh00] and [Kra94].

Theorem 2.14. *Let w be a geodesic word in a square presentation of a group G all of whose relators are of length four. Then w uniquely determines a tiling of relator squares bounded by (but not necessarily filling) a rectangle in the Euclidean plane such that:*

(1) the tiling embeds in the standard 2-complex of the group, i.e. it is a Dehn diagram;

(2) the word labels a geodesic edge-path from one corner of the rectangle to the opposite corner; and

(3) if w' is a geodesic word then $w' =_G w$ if and only if w' labels a geodesic edge-path from one corner of the rectangle to the opposite corner path homotopic to w.

The tiling produced by the theorem for a geodesic word w is called the *geodesic completion* of w.

Proof of Theorem 2.13. Let R_{ab} be the rectangle in the peripheral complex determined by the points $(0,0)$ and $(an-b,an+b)$ for some integers a and b, and let w_{ab} be the label of any geodesic edge-path between these two points (for example the edge-path from $(0,0)$ to $(an-b,0)$ to $(an-b,an+b)$ will do). Then since the words labelling geodesic edge-paths in the peripheral complex are geodesic words in the augmented Dehn presentation (by Lemma 2.10), w_{ab} is a geodesic word.

Therefore, w_{ab} is a geodesic word in a grid presentation which labels a geodesic edge-path between two opposite corners of the rectangle R_{ab}. By Theorem 2.14, a geodesic word in the augmented Dehn presentation represents the word w_{ab} if and only if it is the label of a geodesic edge-path between

$(0,0)$ and $(an - b, an + b)$. So all geodesic representatives of w_{ab} are words labelling geodesic edge-paths from $(0,0)$ and $(an - b, an + b)$ in the peripheral complex. Finally, by Lemma 2.12, every peripheral element is presented by a word w_{ab} for some $a, b \in \mathbb{Z}$ and the result follows. $\square$

2.7. **Proof of Theorem 1.3.** Let Δ be a prime, reduced alternating projection of a knot K in S^3. Theorem 1.3 follows from the following lemma.

Lemma 2.15. (a) The Wirtinger arcs, Wirtinger loops, and Dehn arcs and conjugates of the short arcs of Δ have explicit geodesic representatives in the peripheral complex.
(b) The above geodesic representatives of Wirtinger arcs, Wirtinger loops and Dehn arcs are non-peripheral, and the above geodesic representatives of the short arcs are not conjugate to a peripheral element.

Proof. We apply the notation and discussion from the last three paragraphs of Subsection 2.1.

Let w be a word representing a Wirtinger arc, Wirtinger loop, short arc or Dehn arc of Δ. Recall the map ϕ from Equation (2) and the method for reading the a representative word $\phi^{-1}(\iota(w))$ in the augmented Dehn presentation described in Remark 2.1. We use the peripheral complex to show that its image $\phi^{-1}(\iota(w))$ is non-peripheral.

We deal with each type of loop separately. Throughout we let $l = l_1 l_2 \cdots l_{2n}$ be a geodesic representative of λ which was constructed in the proof of Lemma 2.12. It is given by an edge-path following the sequence of relator squares $C_1, C_2, \ldots, C_{2n-1}$. Each two letter subword $l_{2i-1} l_{2i}$ of l labels an edge path on C_{2i-1}. Also let $m = m_1 m_2$ be a geodesic representative of μ in the augmented Dehn presentation. By Theorem 2.13, l and m are labels of edge paths from $(0,0)$ to (n,n) and $(0,0)$ to $(-1,1)$ respectively, in the peripheral complex.
Wirtinger arcs: Wirtinger arcs are loops which follow the double λ of Δ returning to the base-point after passing through fewer than $2n$ crossings. Therefore a Wirtinger loop is represented by a subword $l_1 l_2 \cdots l_{2p}$, wher $2p < 2n$, of l. Moreover, $l_1 l_2 \cdots l_{2p}$ is represented by the label of any edge-path from $(0,0)$ to (p,p) for $p < n$. By Theorem 2.13, it follows that $l_1 l_2 \cdots l_{2p}$ is non-peripheral as it does not label a path between $(0,0)$ and $(an - b, an + b)$.
Wirtinger loops: We may move a Wirtinger loop close to some crossing c_i. By the way that the relators of the augmented Dehn presentation are read from Δ (see Subsection 2.1), we see that the Wirtinger loop can be described by a geodesic edge-path between two opposite corners of D_i (which two opposite corners depends on the given Wirtinger loop). Let the label of this edge-path be w.

The word w is geodesic of length two. Therefore, if w is peripheral it must represent $\mu^{\pm 1}$. However, by Lemma 2.11, the only geodesic words which represent $\mu^{\pm 1}$ arise as a path from $(0,0)$ to $(\mp 1, \pm 1)$ in the peripheral complex, so w cannot be the label of such an edge-path since by the definition of Wirtinger loops, w is not a representative of the meridian, which is described by a path from $(0,0)$ and $(-1,1)$.
Dehn arcs: Suppose that a given Dehn arc intersects the bounded region a of Δ. Then it is represented by $X_a X_0^{-1}$ in the augmented Dehn presentation. The word $X_a X_0^{-1}$ is geodesic since it is freely reduced ($a \neq 0$) and clearly does not contain a chain subword (see Theorem 2.5).

The meridian μ has exactly two geodesic representatives which label the edge-path $(0,0)$ to $(-1,1)$ of C_{2n}. It is easily seen from the definition of Dehn arcs that neither of these words can be $X_a X_0^{-1}$.
Short arcs: Short arcs are found by walking around the double λ of Δ, and at some point, jumping to an adjacent arc of λ and walking back to the base point in one of two ways. Short arcs are then represented by words of the form

$$(3) \qquad\qquad l_1 l_2 \cdots l_k l_p \cdots l_{2n}$$

and

$$(4) \qquad\qquad l_1 l_2 \cdots l_k l_p \cdots l_1.$$

These representatives of short arcs do not necessarily embed as edge-paths in the peripheral complex. However, since l^2 embeds in the complex as a path from $(-n, -n)$ to (n, n), the conjugates

$$(5) \qquad\qquad l_p \cdots l_{2n} l_1 l_2 \cdots l_k$$

and

$$(6) \qquad\qquad l_p \cdots l_1 l_1 l_2 \cdots l_k$$

both embed in the peripheral complex as paths from $(-p, -p)$ to (k, k), for (5), and (q, q) to (r, r) or (r, r) to (q, q), for (6), where $q = \min\{k, p\}$ and $r = max\{k, p\}$. These paths travel along the South-West to North-East axis.

Since these words in (3) and (4) are of length less that $2n$, if they are peripheral, then they must be equal to a power of the meridian m^b. This will happen if and only if (3) and (4) represent

$$(7) \qquad\qquad (l_1 \cdots l_k)^{-1} m^b (l_1 \cdots l_k),$$

for some integer b. This conjugate of m^b embeds into the peripheral complex as a path from (k, k) to $(0, 0)$ to $(-b, b)$ to $(k - b, k + b)$ which has a geodesic representative as a path from (k, k) to $(k - b, k + b)$. But this path travels along the South-East to North-West axis. Therefore, by Theorem 2.14, the geodesic words in (3) and (4) cannot be equal to the words of the form in (7) and the short arcs are non-peripheral. $\qquad\qquad\square$

We conclude this subsection with three remarks on short arcs.

Remark 2.16. The argument above showing that short arcs are non-peripheral in fact proves a stronger result. It shows that any loop in the knot complement that follows the double of Δ from the basepoint, then at some point jumps (above the projection plane) to any other point on the double, then follows it back to the basepoint in either direction is non-peripheral.

Remark 2.17. Notice that in the proof for the non-peripherality of short arcs we actually solved the conjugacy problem. We could also have shown that these elements were non-peripheral by using Johnsgard's solution to the conjugacy problem [Joh97]: using Johnsgard's algorithm, the fact that the peripheral complex contains the geodesic completion of l^m, and the periodicity of the peripheral complex, it is straight-forward to show that a geodesic word in the augmented Dehn presentation is conjugate to a peripheral element $l^a m^b$ if and only if it embeds as a geodesic path from $(0, k)$ to $(an + k - b, an + b)$, for some integer k. It is easy to see that two words in (5) and (6) are not of this form. We can use a similar argument for Wirtinger loops.

Also note that this characterisation of conjugates of peripheral elements as paths in the peripheral complex provides a method for solving the peripheral conjugacy problem.

Remark 2.18. A different approach and proof to the non-peripherality of short arcs, as noted by M. Thistlethwaite and A. Tsvietkova in [TT14], that uses the incompressibility of checkerboard surfaces can be found in C. Adams [Ada07], and D. Futer, E. Kalfagianni and Efstratia and J.S. Purcell [FKP14]. (In light of this alternative approach and the previous two remarks, it is notable that short arcs are so amenable to both geometric and algorithmic approaches.) It might be that a proof of the whole of Theorem 1.3 can be obtained by this approach. It would be interesting to see such an argument and its connections with the more algorithmic approach taken here.

2.8. **The peripheral complex and the Gauss code of an alternating knot.** In our proof of Theorem 1.3, the peripheral complex plays a key role, and encodes the peripheral structure of a prime, reduced, alternating projection of a knot. In this section we discuss additional properties of the peripheral complex and its relation with the Gauss code.

In Subsection 2.4, we constructed the peripheral complex by placing relator squares of the augmented Dehn presentation on the plane in a way determined by the oriented knot diagram. As previously noted, some conventions were used in this construction. There is a way to construct the peripheral complex directly from the relators of the augmented Dehn presentation without any reference to the knot diagram:

(1) Choose any relator square from the augmented Dehn presentation of a prime, reduced, alternating knot diagram, and place it in the Euclidean plane.
(2) Choose two diagonally opposite vertices of this relator square, call them a and b. Form a "diagonal line" of relator squares by placing copies of the relator square in such a way that each vertex a is identified with a vertex b and all of the relator squares are translations of the first.
(3) Complete the tiling by adding relator squares from the augmented Dehn presentation in a way consistent with the words labelling edge-paths. (Proposition 2.19 tells us that this can be done in a unique way.)

The construction is indicated in Figure 2.8. Throughout this section we call this the *unoriented construction* of the peripheral complex, and we refer to the complex constructed in Subsection 2.4 as the *oriented construction*. We will also refer to the resulting complexes as the unoriented and oriented peripheral complexes respectively.
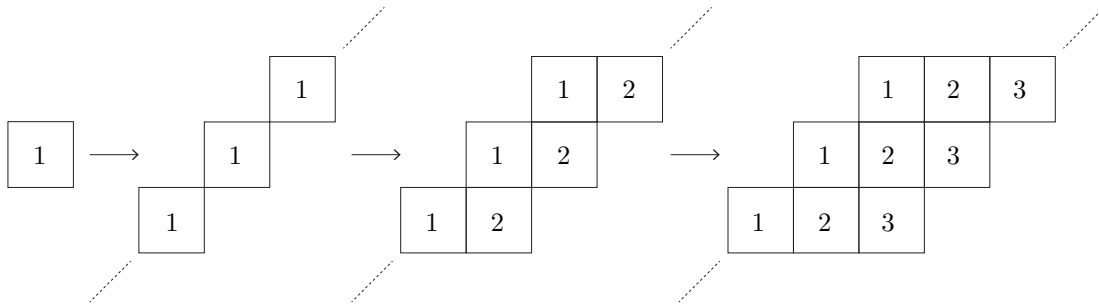
The following proposition tells us that the complex just described exists and is the peripheral complex.

Proposition 2.19. *The unoriented construction of the peripheral complex described above, produces a unique plane tiling of relator squares. Moreover, the resulting complex is isometric to the oriented peripheral complex constructed in Subsection 2.4.*

Proof. First of all, we note that if the complex exists, then it must be unique, since the corners between pairs of relator squares in the "diagonals" used in the construction are labelled by pairs and, in a grid presentation, a pair uniquely determines a relator.

To show existence, let T be the relator square in the plane from the first step of the unoriented construction above. Suppose also that T has the vertices a and b specified. Since every relator square and the reflection of every relator square of the augmented Dehn presentation appears in the fundamental block of the oriented peripheral complex, there is an isometry taking T to a relator square of the peripheral complex which sends the vertices a and b to the top-left and bottom-right vertices of that relator square. By uniqueness, this extends to an isometry of the complexes.   □

The following proposition tells us that an unoriented knot can be recovered from its unoriented peripheral complex, and an oriented knot from its oriented peripheral complex.

Proposition 2.20. *Let Δ be a prime, reduced, alternating, oriented knot diagram. The Gauss code of D can be recovered from the oriented peripheral complex; and the Gauss code of Δ or its inverse $-\Delta$ can be recovered from the unoriented peripheral complex.*

Proof. We first prove the result for the oriented complex. Choose any $2n \times 1$ horizontal block of the complex. Every relator square appears exactly twice in this block. By the construction of the complex, this block is a cyclic permutation of a fundamental block, and therefore the order of the relator squares in the block is precisely the order we meet the crossings as we travel around the knot in the direction of the orientation from some base point. With this observation, it is straight-forward to recover the Gauss code: label the relator squares $S_1, S_2, \ldots, S_{2n}$ by reading along the strip from left to right. Assign the number -1 to S_1 if it has orientation
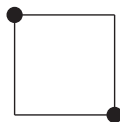
otherwise assign the number $+1$ to S_1. Suppose you have assigned the number $\pm j$ to the relator square S_i. If the relator square S_{i+1} has not been encountered previously assign the number $\mp(j+1)$ to it, if the relator square has been encountered previously and has been assigned the number $\pm p$, then assign the number $\mp p$ to this square. The resulting sequence is the Gauss code.

To recover a Gauss code from an unoriented peripheral complex, we can use the same method. However, since the $2n \times 1$ horizontal strip of the unoriented complex can be a reflection of a $2n \times 1$ horizontal strip of the oriented complex, we are unable to determine if the Gauss code obtained is that of the knot diagram or its inverse. □

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