GRAPH COMPLEXES AND THE SYMPLECTIC CHARACTER OF THE TORELLI GROUP

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Abstract. The mapping class group of a closed surface of genus g is an extension of the Torelli group by the symplectic group. This leads to two natural problems: (a) compute (stably) the symplectic decomposition of the lower central series of the Torelli group and (b) compute (stably) the Poincaré polynomial of the cohomology of the mapping group with coefficients in a symplectic representation V. Using ideas from graph cohomology, we give an effective computation of the symplectic decomposition of the quadratic dual of the lower central series of the Torelli group, and assuming the later is Kozsul, it provides a solution to the first problem. This, together with Mumford's conjecture, proven by Madsen-Weiss, provides a solution to the second problem. Finally, we present samples of computations, up to degree 13.

Contents

Key words and phrases: mapping class group, Torelli group, symplectic representations, graph homology, Mumford conjecture.

1. Introduction

The mapping class group Γ_g is the group of isotopy classes of diffeomorphisms of a closed oriented surface Σ_g of genus g. A surface diffeomorphism acts on the first homology of the surface $H_1(\Sigma_g,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ preserving the intersection form (a symplectic form defined over the integers). This induces a linearization map $\Gamma_g \to \mathrm{Sp}(2g,\mathbb{Z})$ which turns out to be onto with kernel the Torelli group T_g:

$$1 \longrightarrow \mathrm{T}_g \longrightarrow \Gamma_g \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow 1\,. \tag{1.1}$$

In other words, T_g is the group of surface diffeomorphisms, up to isotopy, that act trivially on the homology of the surface. The lower central series of any group G is a graded Lie algebra, and the action of G on itself by conjugation becomes a trivial action on its graded Lie algebra. Since T_g is a normal subgroup of Γ_g, it follows that the action of Γ_g by conjugation on the lower central series $\mathfrak{t}_g$ of T_g descends to an action of $\mathrm{Sp}(2g,\mathbb{Z})$. Said differently, the lower central series of T_g is a symplectic module, which in fact is finitely generated at each degree. What's more, using Mixed Hodge Structures, R. Hain showed in [Hai97] that (for $g \geq 6$) $\mathfrak{t}_g$ is a quadratic Lie algebra given by the following presentation

$$\mathfrak{t}_g = \mathbb{L}(\langle 1^3\rangle)/(R)$$

where $\mathbb{L}$ is the free Lie algebra, and $R + \langle 2^2\rangle + \langle 0\rangle = \Lambda^2(\langle 1^3\rangle)$. In particular, it follows that the stable algebra $\mathfrak{t} \overset{\mathrm{def}}{=} \lim_{g\to\infty} \mathfrak{t}_g$ exists, and is given by the above presentation. The purpose of our paper is to

(P1) compute the symplectic decomposition of the lower central series of the Torelli group,

(P2) compute (stably) the Poincaré polynomial of the cohomology of the mapping group with coefficients in a symplectic representation V.

Our solution to (P1) uses ideas of graph cohomology to compute the symplectic decomposition of the quadratic dual of $\mathfrak{t}$. Assuming that $\mathfrak{t}$ is Koszul, a symplectic decomposition of $\mathfrak{t}$ follows. The latter, along with Mumford's conjecture (proven by Madsen-Weiss [MW07]), provide a solution to (P2).

Our paper combines the work of Hain, Harer, Kawazumi, Looijenga, Morita Madsen and Weiss, and gives explicit symplectic decomposition formulas, and a sample of computations. Regarding computations, irreducible finite dimensional symplectic representations are parametrized by partitions. The symplectic decomposition of $\mathfrak{t}$ in degree n involves partitions with at most $3n$ parts, and for $n = 13$ (where the answer is given in [Gar]) the number of those is 177970. The article was written in 1997 and a computation of the symplectic character of $\mathfrak{t}$ up to degree 10 was done in Brandeis. The computation was repeated in 2006 in Georgia Tech and reached degree 13. Back in 1997 when the paper was written, it assumed Mumford's conjecture (now proven by Madsen-Weiss [MW07]) and the Kozsulity conjecture for the graded Lie algebra of the

Torelli group (still open), and our results predated the representation-stability ideas of Church-Farb [CF13]. Further work on the Torelli homomorphism and the Goldman-Turaev Lie bialgebra can be found in Kawazumi-Kuno's survey article [KK16].

1.1. **The lower central series of the Torelli group.** Let $\Sigma_{g,s}^r$ be an oriented surface of genus g with r distinct ordered points and s boundary components, and let $\Gamma_{g,s}^r$ (resp. Let $\mathrm{T}_{g,s}^r$) be the mapping class group $\pi_0(\mathrm{Diff}_+(\Sigma_{g,s}^r))$ (resp., the subgroup of the mapping class group that generated by surface diffeomorphisms that act trivially on the homology of the surafce). It is customary to omit the indices s, r when they are zero. Recall that rational irreducible representations of the symplectic group are indexed by partitions. Let $\mathfrak{t}_g$ denote the graded Lie algebra whose degree n part equals (rationally) to the quotient of the nth commutator group of the Torelli group T_g modulo the $(n+1)$th commutator group. The exact sequence (1.1) implies that $\mathfrak{t}_g$ is a graded $R(\mathfrak{sp}_g)$-Lie algebra whose universal enveloping algebra can be identified with the completion $\widehat{(\mathbb{Q}\,\mathrm{T}_g)}_I$ of the group ring $\mathbb{Q}[\mathrm{T}_g]$ with respect to its augmentation ideal I. Using Mixed Hodge Structures, R. Hain showed in [Hai97] that (for $g \geq 6$) $\mathfrak{t}_g$ is a quadratic Lie algebra given by the following presentation

$$\mathfrak{t}_g = \mathbb{L}(\langle 1^3 \rangle)/(R)$$

where $\mathbb{L}$ is the free Lie algebra, and $R + \langle 2^2 \rangle + \langle 0 \rangle = \Lambda^2(\langle 1^3 \rangle)$. In particular, it follows that the stable algebra $\mathfrak{t} \overset{\text{def}}{=} \lim_{g\to\infty} \mathfrak{t}_g$ exists, and is given by the above presentation.

1.2. **The homology of the mapping class group.** Harer [Har85] has proved that the cohomology groups of the mapping class groups stabilize as $g \to \infty$. Let $\gamma_g \in \mathbb{Z}[t]$ be the Poincaré polynomial of Γ_g, and let $\gamma_\infty = \lim_{g\to\infty} \gamma_g \in \mathbb{Z}[\![t]\!]$. Mumford's conjecture, proven by Madsen-Weiss [MW07], amounts to the formula

$$\gamma_\infty = \mathrm{Exp}\left(\frac{1}{1-t^2}\right). \tag{1.2}$$

Let H be the symplectic vector space $H^1(\Sigma_g, \mathbb{C})$. The symplectic action of Γ_g on H induces a homomorphism from the representation ring $R(\mathfrak{sp}_{2g})$ of the symplectic group to that $R(\Gamma_g)$ of the mapping class group. Using Harer's results on stability of the cohomology of the mapping class groups, Looijenga [Loo96] has shown that the for every representation $V \in R(\mathfrak{sp}_{2g})$, the cohomology groups $H^\bullet(\Gamma_g, V)$ stabilize as $g \to \infty$, and will thus be denoted by $H^\bullet(\Gamma, V)$. Assuming Mumford's conjecture, Looijenga further calculated (stably) the Poincaré polynomials for all symplectic representations V.

On the other hand, using an extension due to Morita of the Johnson homomorphism, one can define classes in $H^\bullet(\Gamma, \mathsf{T}(\mathsf{H}))$, where $\mathsf{T}(\mathsf{H}) = \bigoplus_{n=0}^\infty \mathsf{H}^{\otimes n}$. A priori, the number of these classes may differ than the ones counted by Looijenga. It is a purpose of the paper to compare the classes counted by Looijenga with the ones constructed by Morita's extension and to show that they precisely agree, assuming Mumford's conjecture. En route, will also give a computation for the above mentioned Poincaré polynomials using ideas from graph cohomology.

We now briefly recall some homomorphisms studied by Morita. Generalizing results of Johnson [Joh80, Joh83], Morita [Mor93, Mor96] defined group homomorphisms ρ, ρ^1 (over $1/24\mathbb{Z}$, although we will only need their version over $\mathbb{C}$) that are part of a commutative diagram

$$\begin{array}{ccc} \Gamma_g^1 & \xrightarrow{\rho^1} & N^1 \rtimes \mathrm{Sp}(\mathsf{H}) \\ \downarrow & & \downarrow \\ \Gamma_g & \xrightarrow{\rho} & N \rtimes \mathrm{Sp}(\mathsf{H}) \end{array}$$

where $\mathsf{H} = H_1(\Sigma_g, \mathbb{Z})$ and N^1, N are explicit torsion free nilpotent groups, related by an exact sequence $1 \to \langle 1 \rangle + \langle 1^2 \rangle \to N^1 \to N \to 1$, and $\Gamma_g^1 \to \Gamma_g$ is induced by the forgetful map $\Sigma_g^1 \to \Sigma_g$. ρ induces a map on cohomology $\rho^* : H^\bullet(N \rtimes \mathrm{Sp}(\mathsf{H}), \mathsf{T}(\mathsf{H})) \to H^\bullet(\Gamma, \mathsf{T}(\mathsf{H}))$. Since $\mathrm{Sp}(\mathsf{H})$ is reductive, the Lyndon-Hochshild-Serre spectral sequence [HS53] (whose $E^2_{p,q}$-term equals to $H^p(\mathrm{Sp}(\mathsf{H}), H^q(N, \mathsf{T}(\mathsf{H})))$ and vanishes for $p > 0$) collapses, thus implying that $H^\bullet(N \rtimes \mathrm{Sp}(\mathsf{H}), \mathsf{T}(\mathsf{H})) \cong H^\bullet(N, \mathsf{T}(\mathsf{H}))^{\mathrm{Sp}}$.

Since N is a nilpotent Lie group satisfying $[N, [N, N]] = 0$, $N/[N, N] = \langle 1^3 \rangle$, $[N, N] = \langle 2^2 \rangle$, and the natural map $\Lambda^2(N) \to [N, N]$ is onto, it follows that its cohomology $H^\bullet(N, \mathsf{T}(\mathsf{H}))$ can be identified with $\mathsf{A} \otimes \mathsf{T}(\mathsf{H})$, where

$$\mathsf{A} = \Lambda^\bullet(\langle 1^3 \rangle)/(\langle 2^2 \rangle) \tag{1.3}$$

denotes the $R(\mathfrak{sp})$-quadratic algebra with respect to the decomposition

$$\Lambda^2(\langle 1^3 \rangle) = \langle 1^6 \rangle \oplus \langle 1^4 \rangle \oplus \langle 1^2 \rangle \oplus \langle 0 \rangle \oplus \langle 2^2, 1^2 \rangle \oplus \langle 2^2 \rangle.$$

The above discussion implies the existence of an algebra map

$$\mu : \mathsf{A} \longrightarrow H^\bullet(\Gamma, \mathsf{T}(\mathsf{H})),$$

where the product on the right hand side is induced by the shuffle product on $\mathsf{T}(\mathsf{H})$.

Similarly, we obtain that the cohomology $H^\bullet(N^1 \rtimes \mathrm{Sp}(\mathsf{H}), \mathsf{T}(\mathsf{H}))$ equals to $(\mathsf{A}^1 \otimes \mathsf{T}(\mathsf{H}))^{\mathrm{Sp}}$, where

$$\mathsf{A}^1 = \Lambda^\bullet(\langle 1^3 \rangle + \langle 1 \rangle)/(\langle 2^2 \rangle + \langle 1^2 \rangle)$$

denotes the $R(\mathfrak{sp})$-quadratic algebra with respect to the decomposition

$$\Lambda^2(\langle 1^3 \rangle + \langle 1 \rangle) = \langle 1^6 \rangle \oplus 2\langle 1^4 \rangle \oplus 3\langle 1^2 \rangle \oplus 2\langle 0 \rangle \oplus \langle 2^2, 1^2 \rangle \oplus \langle 2^2 \rangle \oplus \langle 2, 1^2 \rangle,$$

where $\langle 1^2 \rangle \to 3\langle 1^2 \rangle$ is nonzero in all three factors. This induces a map $\mathsf{A}^1 \to \mathsf{A}$ as well as $\mu^1 : \mathsf{A}^1 \to H^\bullet(\Gamma^1, \mathsf{T}(\mathsf{H}))$.

1.3. **Statement of our results.** Our first theorem uses ideas from graph cohomology and its interpretation as symplectic representation theory [Kon94, GN98].

Theorem 1.1. *The character* $\mathrm{c}_t(\mathsf{A})$ *of* A *and* $\mathrm{c}_t(\mathsf{A}^1)$ *of* A^1 *in* $\Lambda[\![t]\!]$ *is given by*

$$\mathrm{c}_t(\mathsf{A}) = \tilde{\omega} \exp(-\mathbb{D}') \operatorname{Exp}\left(\mathrm{ch}_t(\mathcal{V})\right)$$
$$\mathrm{c}_t(\mathsf{A}^1) = \tilde{\omega} \exp(-\mathbb{D}') \operatorname{Exp}\left(h_1 t + \mathrm{ch}_t(\mathcal{V})\right),$$

where

$$\mathrm{ch}_t(\mathcal{V}) = -th_1 - h_2 + \sum_{2k-2+n\geq 0} t^{2k-2+n} h_n = \frac{1}{t^2}\left(\frac{\operatorname{Exp}(th_1)}{1-t^2} - 1 - (t+t^3)h_1 - t^2 h_2\right),$$

$\tilde{\omega} : \Lambda \to \Lambda$ *is the involution on the ring of symmetric functions defined by* $\tilde{\omega}(p_n) = -p_n$ *and*

$$\mathbb{D}' = \sum_{n=1}^{\infty} \left(\frac{n}{2} \frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}}\right).$$

Theorem 1.2. *The maps* μ *and* μ^1 *are algebra isomorphisms that are part of a commutative diagram*

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{\mu} & H^\bullet(\Gamma, \mathsf{T}(\mathsf{H})) \\ \downarrow & & \downarrow \\ \mathsf{A}^1 & \xrightarrow{\mu^1} & H^\bullet(\Gamma^1, \mathsf{T}(\mathsf{H})). \end{array}$$

The rext result assumes a Koszulity conjecture for the quadratic algebra A. For a detailed discussion on Koszul duality, see [PP05].

Theorem 1.3. *Assuming that* $\mathsf{U}(\mathfrak{t})$ *(or equivalently,* A*) is Koszul, the* $\mathfrak{sp}$*-character* $\mathrm{c}_t(\mathfrak{t})$ *of* $\mathfrak{t}$ *in* $\Lambda[\![t]\!]$ *is given by*

$$\mathrm{c}_t(\mathfrak{t}) = t^2 - \operatorname{Log}(\mathrm{c}_{-t}(\mathsf{A})).$$

Remark 1.1. In degree $n \leq 5$, Theorem 1.3 agrees with independent calculations of the character of $\mathfrak{t}$ by S. Morita, giving evidence for the Koszulity of A.

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2. Symmetric functions

2.1. **The ring of symmetric functions.** In this section, we recall some results on symmetric functions and representations of the symmetric, general linear and symplectic groups which we need later. For the proofs of these results, we refer to Macdonald [Mac95].

The ring of symmetric functions is the inverse limit

$$\Lambda = \varprojlim \mathbb{Z}[x_1, \dots, x_k]^{\mathbb{S}_k}.$$

It is a polynomial ring in the complete symmetric functions

$$h_n = \sum_{i_1 \le \dots \le i_n} x_{i_1} \dots x_{i_n}.$$

We may also introduce the elementary symmetric functions

$$e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n},$$

with generating function

$$E(t) = \sum_{n=0}^{\infty} t^n e_n = \prod_i (1 + t x_i) = H(-t)^{-1}.$$

We see that Λ is also the polynomial ring in the e_n. Let $\omega : \Lambda \to \Lambda$ be the involution of Λ which maps h_n to e_n. The power sums (also known as Newton polynomials)

$$p_n = \sum_i x_i^n$$

form a set of generators of the polynomial ring $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$. This is shown by means of the elementary formula

$$P(t) = t \frac{d}{dt} \log H(t), \tag{2.4}$$

where

$$H(t) = \sum_{n=0}^{\infty} t^n h_n = \prod_i (1 - t x_i)^{-1} \quad \text{and} \quad P(t) = \sum_{n=0}^{\infty} t^n p_n = \sum_i (1 - t x_i)^{-1}.$$

Written out explicitly, we obtain Newton's formula relating the two sets of generators:

$$n h_n = p_n + h_1 p_{n-1} + \dots + h_{n-1} p_1.$$

The involution ω acts on the power sums by the formula $\omega p_n = (-1)^{n-1} p_n$.

A partition λ is a decreasing sequence $(\lambda_1 \ge \dots \ge \lambda_\ell)$ of positive integers; we write $|\lambda| = \lambda_1 + \dots + \lambda_\ell$, and denote the length ℓ of λ by $\ell(\lambda)$. Associate to a partition λ the monomial $h_\lambda = h_{\lambda_1} \dots h_{\lambda_\ell}$ in the complete symmetric functions h_n. The subgroup $\Lambda_n \subset \Lambda$ of symmetric functions homogeneous of degree n is a free abelian group of rank $p(n)$, with bases $\{h_\lambda \mid |\lambda| = n\}$ and $\{e_\lambda \mid |\lambda| = n\}$.

We may also associate to a partition λ the monomial $p_\lambda = p_{\lambda_1} \dots p_{\lambda_\ell}$ in the power sums p_n; the vector space $\Lambda_n \otimes \mathbb{Q}$ has basis $\{p_\lambda \mid |\lambda| = n\}$. Inverting (2.4), we obtain the formula

$$H(t) = \exp\Big(\sum_{n=1}^{\infty} t^n \frac{p_n}{n} \Big) = \sum_\lambda t^{|\lambda|} \frac{p_\lambda}{z(\lambda)}. \tag{2.5}$$

The integers $z(\lambda)$ arise in many places: for example, the conjugacy class $\mathcal{O}_\lambda$ of $\mathbb{S}_n$ labelled by the partition λ of n, consisting of those permutations whose cycles have length $(\lambda_1, \ldots, \lambda_\ell)$, has $n!/z(\lambda)$ elements.

2.2. **Schur functions.** There is an identification of Λ with the ring of characters

$$\Lambda = R(\mathfrak{gl}) = \varprojlim R(\mathfrak{gl}_r),$$

where $\mathfrak{gl}_r$ is the Lie algebra of $\mathrm{GL}(r, \mathbb{C})$, obtained by mapping e_λ to the representation

$$\Lambda^{\lambda_1}(\mathbb{C}^\infty) \otimes \ldots \otimes \Lambda^{\lambda_\ell}(\mathbb{C}^\infty) = \varprojlim \Lambda^{\lambda_1}(\mathbb{C}^r) \otimes \ldots \otimes \Lambda^{\lambda_\ell}(\mathbb{C}^r).$$

Thus, Λ has a basis consisting of the characters of the irreducible polynomial representations of $\mathfrak{gl}$. These characters, given by the Jacoby-Trudy formula $s_\lambda = \det\big(h_{\lambda_i - i + j}\big)_{1 \le i,j \le \ell}$, are known as the Schur functions.

2.3. **Symplectic Schur functions.** There is also an identification of Λ with the ring of characters

$$\Lambda = R(\mathfrak{sp}) = \varprojlim R(\mathfrak{sp}_{2g}),$$

where $\mathfrak{sp}_{2g}$ is the Lie algebra of $\mathrm{Sp}(2g, \mathbb{C})$, obtained by mapping e_λ to the representation

$$\Lambda^{\lambda_1}(\mathbb{C}^\infty) \otimes \ldots \otimes \Lambda^{\lambda_\ell}(\mathbb{C}^\infty) = \varprojlim \Lambda^{\lambda_1}(\mathbb{C}^{2g}) \otimes \ldots \otimes \Lambda^{\lambda_\ell}(\mathbb{C}^{2g}).$$

Thus, Λ has a basis consisting of the characters of the irreducible representations $\langle\lambda\rangle$ of $\mathfrak{sp}$. These characters, given by the symplectic Jacoby-Trudy formula

$$s_{\langle\lambda\rangle} = \frac{1}{2} \det\big(h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 2}\big)_{1 \le i \le \ell}, \tag{2.6}$$

are known as the symplectic Schur functions.

2.4. **The Frobenius characteristic.** If V is a finite-dimensional $\mathbb{S}_n$-module V with character $\chi_V : \mathbb{S}_n \to \mathbb{Z}$, its characteristic is the symmetric function

$$\mathrm{ch}_n(V) = \sum_{|\lambda|=n} \chi_V(\mathcal{O}_\lambda) \frac{p_\lambda}{z(\lambda)}.$$

Although it appears from its definition that $\mathrm{ch}_n(V)$ is in $\Lambda \otimes \mathbb{Q}$, it may be proved that it actually lies in Λ.

Proposition 2.1. *The characteristic map induces an isomorphism of abelian groups*

$$\mathrm{ch}_n : R(\mathbb{S}_n) \longrightarrow \Lambda_n.$$

Let $\mathbb{S}$ be the groupoid formed by taking the union of the symmetric groups $\mathbb{S}_n$, $n \ge 0$. An $\mathbb{S}$-module is a functor $n \to \mathcal{V}(n)$ from $\mathbb{S}$ to the category of $\mathbb{N}$-graded vector spaces, finite dimensional in each degree. If $\mathcal{V}$ is an $\mathbb{S}$-module we define its characteristic to be the sum

$$\mathrm{ch}_t(\mathcal{V}) = \sum_{n=0}^{\infty} \sum_i (-t)^i \, \mathrm{ch}_n(\mathcal{V}_i(n)),$$

where $\mathcal{V}_i(n)$ is the degree i component of $\mathcal{V}(n)$. In our paper, we will consider examples of $\mathbb{S}$-modules that either come from

- geometry, such as $\mathcal{M}$ and $\mathcal{C}_g^n$ given in Section 3, or from
- graph complexes, such as A, $\mathcal{B}_{\mathrm{top}}$, $\mathcal{B}$ and $\mathcal{V}$ given in Section 4, or from
- topology, such as $\mathfrak{t}_g$ and $\mathfrak{t}$ given in Section 1.1.

In summary, we have four different realizations of the same object Λ: Λ itself, $R(\mathfrak{gl})$, $R(\mathfrak{sp})$ and $R(\mathbb{S}) = \bigoplus_{n=0}^{\infty} R(\mathbb{S}_n)$. The induced isomorphisms between $R(\mathbb{S})$, $R(\mathfrak{gl})$ and $R(\mathfrak{sp})$ are induced by the Schur functor, which associates to an $\mathbb{S}_n$-module V the $\mathfrak{gl}$-module

$$\varprojlim \big(V \otimes (\mathbb{C}^r)^{\otimes n}\big)^{\mathbb{S}_n}$$

and the $\mathfrak{sp}$-module

$$\varprojlim \big(V \otimes (\mathbb{C}^{2g})^{\otimes n}\big)^{\mathbb{S}_n}.$$

If λ is a partition of n, we denote by V_λ the irreducible representation of $\mathbb{S}_n$ with characteristic the Schur function $\mathsf{ch}_n(V_\lambda) = s_\lambda$. For λ a partition of n, and for a complex vector space W, the vector space $\big(V_\lambda \otimes W^{\otimes n}\big)^{\mathbb{S}_n}$ is denoted $\mathsf{S}^\lambda(W)$. We have the isomorphism of $\mathbb{S}_n$-modules

$$W^{\otimes n} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes \mathsf{S}^\lambda(W). \tag{2.7}$$

On the other hand, if W is a complex symplectic vector space of dimension $2g$ and λ is a partition with at most g parts, the $\mathfrak{sp}_{2g}$-module $\mathsf{S}^\lambda(W)$ will not in general be irreducible; its submodule of highest weight, which is unique, is denoted $\mathsf{S}^{\langle\lambda\rangle}(W)$; we denote the $\mathfrak{sp}_{2g}$-module $\mathsf{S}^{\langle\lambda\rangle}(\mathbb{C}^{2g})$ by $\langle\lambda\rangle$, generalizing the case where $g \to \infty$. The analogue of (2.7) for symplectic vector spaces is

$$W^{\otimes n} \cong \bigoplus_{k=0}^{[\frac{n}{2}]} \bigoplus_{|\lambda|=n-2k} \mathrm{Ind}_{\mathbb{S}_k \wr \mathbb{S}_2 \times \mathbb{S}_{n-2k}}^{\mathbb{S}_n} \big(\mathbb{1}_-(2)^{\otimes k} \otimes V_\lambda\big) \otimes \mathsf{S}^{\langle\lambda\rangle}(W), \tag{2.8}$$

where $\mathbb{1}_-(2)$ is the alternating character of $\mathbb{S}_2$ and $\wr$ denotes the wreath product.

The products on the rings $R(\mathfrak{gl})$ and $R(\mathfrak{sp})$ correspond to the product on the ring Λ. However, the product on $R(\mathbb{S})$ is perhaps not as familiar: it is given by the formula

$$(\mathcal{V} \times \mathcal{W})(n) = \bigoplus_{k=0}^{n} \mathrm{Ind}_{\mathbb{S}_k \times \mathbb{S}_{n-k}}^{\mathbb{S}_n} \big(\mathcal{V}(k) \otimes \mathcal{W}(n-k)\big).$$

2.5. **The Hall inner product.** There is a non-degenerate integral bilinear form on Λ, denoted $\langle f, g\rangle$, for which the Schur functions s_λ form an orthonormal basis. Cauchy's formula (I.4.2 of Macdonald [Mac95])

$$H(t)(\dots, x_i y_j, \dots) = \prod_{i,j} (1 - t x_i y_j)^{-1} = \sum_{|\lambda|=n} s_\lambda(x) \otimes s_\lambda(y) = \exp\Big(\sum_{k=1}^{\infty} \frac{p_k(x) \otimes p_k(y)}{k}\Big). \tag{2.9}$$

implies that

$$\langle p_\lambda, p_\mu \rangle = z(\lambda)\delta_{\lambda\mu}. \tag{2.10}$$

The adjoint of multiplication by $f \in \Lambda$ with respect to the inner product on Λ is denoted $f^\perp$; in particular, $\langle f, g \rangle = (f^\perp g)(0)$. When written in terms of the power-sums, the operator $f^\perp$ becomes a differential operator (Ex. 5.3 of Macdonald [Mac95]): it follows from (2.10) that

$$f(p_1, p_2, \dots)^\perp = f\big(\tfrac{\partial}{\partial p_1}, 2\tfrac{\partial}{\partial p_2}, 3\tfrac{\partial}{\partial p_3}, \dots\big). \tag{2.11}$$

If $\mathcal{V}$ and $\mathcal{W}$ are $\mathbb{S}$-modules,

$$\operatorname{ch}_t(\mathcal{V})^\perp \operatorname{ch}_t(\mathcal{W}) = \sum_{k=0}^\infty \sum_{n=k}^\infty \operatorname{ch}_{n-k}\Big(\operatorname{Hom}_{\mathbb{S}_k}\Big(\mathcal{V}(k), \operatorname{Res}^{\mathbb{S}_n}_{\mathbb{S}_k \times \mathbb{S}_{n-k}} \mathcal{W}(n)\Big)\Big).$$

Taking the dimension of $\operatorname{ch}_t(\mathcal{V})^\perp \operatorname{ch}_t(\mathcal{W})(0)$, we obtain the formula

$$\langle \operatorname{ch}_t(\mathcal{V}), \operatorname{ch}_t(\mathcal{W}) \rangle = \sum_{n=0}^\infty \sum_i (-t)^i \dim\big(\operatorname{Hom}_{\mathbb{S}_n}(\mathcal{V}(n), \mathcal{W}(n))\big)_i.$$

2.6. **Plethysm.** Aside from the product, there is another associative operation $f \circ g$ on Λ, called plethysm, which is characterized by the formulas

(i) $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$;
(ii) $(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$;
(iii) if $f = f(p_1, p_2, \dots)$, then $p_n \circ f = f(p_n, p_{2n}, \dots)$.

The corresponding operation on $\mathbb{S}$-modules is called composition.

Proposition 2.2. *If $\mathcal{V}$ and $\mathcal{W}$ are $\mathbb{S}$-modules, let*

$$(\mathcal{V} \circ \mathcal{W})(n) = \bigoplus_{k=0}^\infty \Big(\mathcal{V}(k) \otimes \bigoplus_{f:[n]\to[k]} \mathcal{W}(f^{-1}(1)) \otimes \dots \otimes \mathcal{W}(f^{-1}(k))\Big)_{\mathbb{S}_k},$$

where $[n] = \{1, \dots, n\}$ and $[k] = \{1, \dots, k\}$. We have $\operatorname{ch}_t(\mathcal{V} \circ \mathcal{W}) = \operatorname{ch}_t(\mathcal{V}) \circ \operatorname{ch}_t(\mathcal{W})$.

Proof. When $\mathcal{V}$ and $\mathcal{W}$ are ungraded, this is proved in Macdonald [Mac95]. In the general case, the proof depends on an analysis of the interplay between the minus signs in the Euler characteristic and the action of symmetric groups on tensor powers of graded vector spaces. □

The operation

$$\operatorname{Exp}(f) = \sum_{n=0}^\infty h_n \circ f \tag{2.12}$$

plays the role of the exponential in Λ. It takes values in the completion of Λ with respect to the ideal $\ker(\varepsilon)$ where $\varepsilon : \Lambda \to \mathbb{Z}$ is the homomorphism $\varepsilon(f) = f(0)$ which sends h_n to 0 for $n > 0$. We will discuss this completion at greater length in the next section. The operation Exp extends to the λ-ring $\Lambda[\![t]\!]$ by $p_n \circ t = t^n$, and satisfies the property

$$\operatorname{Exp}(f+g) = \operatorname{Exp}(f)\operatorname{Exp}(g) \tag{2.13}$$

Proposition 2.3. *We have the formula* $\operatorname{Exp}(e_2)^\perp = \exp(\mathbb{D})$, *where*

$$\mathbb{D} = \sum_{n=1}^\infty \left(\frac{n}{2}\frac{\partial^2}{\partial p_n^2} - \frac{\partial}{\partial p_{2n}} \right).$$

Proof. By (2.11), it suffices to substitute $n\partial/\partial p_n$ and $2n\partial/\partial p_{2n}$ for p_n and p_{2n} on the right-hand side of

$$\operatorname{Exp}(e_2) = \exp\left(\sum_{n=1}^\infty \frac{p_n}{n}\right) \circ \left(\tfrac12(p_1^2 - p_2)\right) = \exp\left(\sum_{n=1}^\infty \frac{1}{2n}(p_n^2 - p_{2n})\right). \quad \square$$

Using this heat-kernel, we can relate the bases s_λ and $s_{\langle\lambda\rangle}$ of Λ corresponding respectively to the irreducible representations in $R(\mathfrak{gl})$ and $R(\mathfrak{sp})$. The following formula is a consequence of (2.8):

$$s_{\langle\lambda\rangle} = \exp(-\mathbb{D}) s_\lambda. \tag{2.14}$$

3. Stable cohomology of mapping class groups with symplectic coefficients

In this section, we briefly present Looijenga's calculation of the stable cohomology of the mapping class groups with arbitrary symplectic coefficients. Define a linear map $\mathbb{L} : R(\mathfrak{sp}) \to \mathbb{Z}[\![t]\!]$ by the formula

$$\mathbb{L}\big[s_{\langle\lambda\rangle}\big] = \lim_{g\to\infty} \sum_i (-t)^i H^i(\Gamma_g, \mathsf{S}^{\langle\lambda\rangle}(\mathsf{H})).$$

In this section, we calculate the map $\mathbb{L}$ (and its natural extension $\mathbb{L} : R(\mathfrak{sp})[\![t]\!]\hat{\otimes}\Lambda \to \Lambda[\![t]\!]$ on the complete λ-ring) explicitly, by means of a calculation in $R(\mathfrak{sp})[\![t]\!]\hat{\otimes}\Lambda$; our basic ingredients are the results of Looijenga [Loo96].

Let $\mathcal{M}_g$ be the moduli stack of genus g curves (that is, the homotopy quotient of Teichmüller space by Γ_g), and for $n > 0$, let $\mathcal{M}_g^n$ be the moduli stack of genus g curves with a configuration of n ordered distinct points. Let $\mathcal{M}_g$ be the $\mathbb{S}$-module $\mathcal{M}_g(n) = H^\bullet(\mathcal{M}_g^n, \mathbb{C})$, and $\mathcal{M}$ its stable version.

Theorem 3.1.

$$\operatorname{ch}_t(\mathcal{M}) = \gamma_\infty \operatorname{Exp}\Big(\frac{h_1}{1-t^2}\Big)$$

Proof. In [Loo96, Prop.2.2] Looijenga proves that in degree less than g,

$$H^\bullet(\mathcal{M}_g^n, \mathbb{C}) \cong H^\bullet(\mathcal{M}_g, \mathbb{C}) \otimes \mathbb{C}[u_1, \dots, u_n],$$

where the classes u_i of degree 2 are the Chern classes of the line bundles whose fibre at $[\Sigma, z_1, \dots, z_n]$ is $T^*_{z_i}\Sigma$. If $\mathbb{1}(n)$ is the trivial $\mathbb{S}_n$-module and $\mathbb{C}[u]$ is the $\mathbb{S}$-module such that $\mathbb{C}[u](1) = \mathbb{C}[u]$ and $\mathbb{C}[u](n) = 0$ for $n \neq 1$, we have

$$\mathbb{1}(n) \circ \mathbb{C}[u] \cong \mathbb{C}[u_1, \dots, u_n].$$

Taking the characteristic of both sides, we see that

$$\operatorname{ch}_t(\mathbb{C}[u_1, \dots, u_n]) = h_n \circ \frac{h_1}{1-t^2}.$$

Summing over n and using Equation (2.12) concludes the proof. □

Let $\mathcal{C}^n_g$ be the nth fibred product of the universal curve $\mathcal{M}^1_g \to \mathcal{M}_g$. There is an open embedding $\mathcal{M}^n_g \hookrightarrow \mathcal{C}^n_g$, whose image is the complement of a divisor with normal crossings. Let $\mathcal{C}_g$ be the $\mathbb{S}$-module $\mathcal{C}_g(n) = H^\bullet(\mathcal{C}^n_g, \mathbb{C})$, and $\mathcal{C}$ its stable version.

Theorem 3.2.

$$\operatorname{ch}_t(\mathcal{C}) = \mathbb{L}\big[\operatorname{Exp}\big((1 - t\mathsf{H} + t^2)h_1\big)\big]$$

Proof. Since π is a smooth projective morphism, the Leray-Serre spectral sequence for the projection $\pi : \mathcal{C}^n_g \to \mathcal{M}_g$ collapses at its E_2-term $H^p(\mathcal{M}_g, R^q\pi_*\mathbb{C})$. Observe that

$$R^q\pi_*\mathbb{C} \cong \bigoplus_{\substack{j+k+\ell=n \\ k+2\ell=q}} \operatorname{Ind}^{\mathbb{S}_n}_{\mathbb{S}_j\times\mathbb{S}_k\times\mathbb{S}_\ell}\big(\mathbb{C}^{\otimes j} \otimes \mathsf{H}[-1]^{\otimes k} \otimes \mathbb{C}[-2]^{\otimes \ell}\big),$$

and hence that

$$H^i(\mathcal{C}^n_g, \mathbb{C}) \cong \bigoplus_{p=0}^{i} \bigoplus_{\substack{j+k+\ell=n \\ k+2\ell=q}} \operatorname{Ind}^{\mathbb{S}_n}_{\mathbb{S}_j\times\mathbb{S}_k\times\mathbb{S}_\ell}\big(\mathbb{C}^{\otimes j} \otimes H^p(\mathcal{M}_g, \mathsf{H}[-1]^{\otimes k}) \otimes \mathbb{C}[-2]^{\otimes \ell}\big).$$

By the main theorem of Looijenga [Loo96], the right-hand side stabilizes for $p < g/2+k$. Taking characteristics, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}(-t)^i \operatorname{ch}_n(H^i(\mathcal{C}^n_g, \mathbb{C})) &= \sum_{j+k+\ell=n} \sigma_j(p_1)\cdot \mathbb{L}\big[\sigma_k(-t\mathsf{H}h_1)\big]\cdot \sigma_\ell(t^2h_1) \\ &= \mathbb{L}\big[\sigma_k((1-t\mathsf{H}+t^2)h_1)\big]. \quad \square\end{aligned}$$

Theorem 3.3. $\operatorname{ch}_t(\mathcal{C}) = \operatorname{ch}_t(\mathcal{M}) \circ t^{-2}\big(\operatorname{Exp}(t^2h_1) - 1\big)$

Proof. Fill in ... Easy, given results in Looijenga. □

Corollary 3.1. $\operatorname{ch}_t(\mathcal{C}) = \gamma_\infty \operatorname{Exp} \dfrac{1}{t^2}\left(\dfrac{\operatorname{Exp}(t^2h_1)-1}{1-t^2}\right)$

Combining all of these results, we obtain a formula for the operation $\mathbb{L}$.

Theorem 3.4.

$$\mathbb{L}\big[\mathrm{Exp}\big(\mathsf{H}h_1\big)\big] = \frac{\gamma_\infty}{\mathrm{Exp}\big((1-t^2)^{-1}\big)} \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(-th_1)}{1-t^2} - 1 + (t+t^3)e_1\right)$$

Proof. The above theorems imply that

$$\mathbb{L}\big[\mathrm{Exp}\big((1-t\mathsf{H}+t^2)h_1\big)\big] = \gamma_\infty \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(t^2h_1)-1}{1-t^2}\right).$$

Multiplying both sides by $\mathrm{Exp}\big(-(1+t^2)h_1\big)$ gives

$$\mathbb{L}\big[\mathrm{Exp}\big(-t\mathsf{H}h_1\big)\big] = \gamma_\infty \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(t^2h_1)-1}{1-t^2} - t^2(1+t^2)h_1\right).$$

Replacing h_1 by $-h_1/t$, we obtain

$$\mathbb{L}\big[\mathrm{Exp}\big(\mathsf{H}h_1\big)\big] = \gamma_\infty \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(-th_1)-1}{1-t^2} + t(1+t^2)h_1\right),$$

and the result follows. □

Corollary 3.2.

$$\begin{aligned}\sum_\lambda \mathbb{L}\big[s_{\langle\lambda\rangle}\big]s_\lambda &= \mathbb{L}\big[\exp(-\mathbb{D})\,\mathrm{Exp}\big(\mathsf{H}h_1\big)\big] \\ &= \frac{\gamma_\infty}{\mathrm{Exp}\big((1-t^2)^{-1}\big)} \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(-th_1)}{1-t^2} - 1 + (t+t^3)e_1 - t^2e_2\right) \\ &= \frac{\gamma_\infty}{\mathrm{Exp}\big((1-t^2)^{-1}\big)}\tilde{\omega} \,\mathrm{Exp}\, \frac{1}{t^2}\left(\frac{\mathrm{Exp}(th_1)}{1-t^2} - 1 - (t+t^3)h_1 - t^2h_2\right) \\ &= \frac{\gamma_\infty}{\mathrm{Exp}\big((1-t^2)^{-1}\big)}\tilde{\omega} \,\mathrm{Exp}\left(\mathrm{ch}_t \mathcal{V}\right).\end{aligned}$$

4. Graph cohomology and the $\mathfrak{sp}$-character of A

In this section, we will calculate the $\mathfrak{sp}$-character of A using the ideas of graph cohomology, which interprets classical invariant theory of Lie groups in terms of graphs. For similar applications of graph cohomology see [Kon94, GN98].

Let $\mathrm{c}_t(\mathsf{A}) = \sum_{k=0}^\infty (-t)^k \mathsf{A}_k$ denote the character of A in $\Lambda[\![t]\!]$. Let $\mathcal{B}_{\mathrm{top}}$ be the $\mathbb{S}$-module given by

$$\mathcal{B}_{\mathrm{top}}(n) = (\mathsf{A}^\bullet \otimes \mathsf{T}^n_{\mathrm{top}}(\mathsf{H}))^{\mathfrak{sp}}, \tag{4.15}$$

where $\mathsf{T}_{\mathrm{top}}(\mathsf{H})$ is the quotient of the algebra $\mathsf{T}(\mathsf{H})$ modulo the two sided ideal generated by $\omega_{\mathrm{sympl}} \in \mathsf{T}^2(\mathsf{H})$, where ω_{sympl} is the symplectic form on H.

Proposition 4.1.

$$\mathrm{ch}_t(\mathcal{B}_{\mathrm{top}}) = \mathrm{Exp}\left(\mathrm{ch}_t(\mathcal{V})\right),$$

where $\mathrm{ch}_t(\mathcal{V})$ *is as in the statement of theorem 1.1.*

Corollary 4.2.

$$\mathrm{c}_t(\mathsf{A}) = \tilde{\omega}\exp(-\mathbb{D}')\,\mathrm{ch}_t(\mathcal{B}_{\mathrm{top}}), \tag{4.16}$$

which implies part of theorem 1.1.

Proof. It follows from the fact that

$$\mathsf{T}_{\mathrm{top}}(\mathsf{H}) = \bigoplus_{\lambda} \mathsf{S}^{\langle\lambda\rangle}(\mathsf{H}) \otimes \mathsf{S}^{\lambda}(\mathsf{H}),$$

as well as equation (2.14) and $\exp(-\mathbb{D})\tilde{\omega} = \tilde{\omega}\exp(-\mathbb{D}')$. □

Corollary 4.3.

$$\frac{\gamma_\infty}{\mathrm{Exp}((1-t^2)^{-1})}\,\mathrm{c}_t(\mathsf{A}) = \sum_{\lambda} \mathbb{L}\big[s_{\langle\lambda\rangle}\big]\,s_{\langle\lambda\rangle}$$

Proof. Compare corollaries 3.2 and 4.2. □

Proof. (of proposition 4.1) Recall, from [KM94] and [GK98, Section 2.5], that a graph G is a finite set $\mathrm{Flag}(G)$ (whose elements are called flags) together with an involution σ and a partition λ. (By a partition of a set, we mean a disjoint decomposition into several unordered, possibly empty, subsets called blocks).

The vertices of G are the blocks of the partition λ, and the set of them is denoted by $\mathrm{Vert}(G)$. The subset of $\mathrm{Flag}(G)$ corresponding to a vertex v is denoted by $\mathrm{Leg}(v)$. Its cardinality is called the valence of v, and denoted by $n(v)$. The degree of a graph with k trivalent vertices and n legs equals to $2k+n$. This agrees with the grading of stable graphs given in [GK98], if the genus label of every trivalent vertex is zero.

The edges of G are the pairs of flags forming a two-cycle of σ, and the set of them is denoted by $\mathrm{Edge}(G)$. The legs of G are the fixed points of σ, and the set of them is denoted by $\mathrm{Leg}(G)$.

Following [Wey39], we review a description of the $\mathfrak{sp}$-invariant algebra $\mathsf{T}(\mathsf{H})^{\mathfrak{sp}}$ in terms of an algebra of graphs. A chord diagram of degree m is a graph (σ, λ^1_{2m}) where σ is an involution of the set $[2m] = \{1,\dots,2m\}$ without fixed points and λ^1_{2m} is the partition of the set $[2m]$ in all one-element subsets. We consider the two flags in each chord as ordered. Given a chord diagram of degree $2m$, one can associate to it a $\mathfrak{sp}$-invariant tensor in $\mathsf{T}^{2m}(\mathsf{H})^{\mathfrak{sp}}$ by placing a copy of the symplectic form on each edge. Upon changing the order of the two flags in a chord, the associated invariant tensor changes sign. Thus, we get a map

$$\mathcal{CD} \longrightarrow \mathsf{T}(\mathsf{H})^{\mathfrak{sp}} \tag{4.17}$$

where $\mathcal{CD}$ is the quotient of the vector space $\mathbb{C}\langle\text{chord diagrams}\rangle$ over $\mathbb{C}$ spanned by all chord diagrams modulo the relation O_1 shown in figure 1. The above map is a stable algebra isomorphism, where the product of chord diagrams is the disjoint union.

The above map can describe the $\mathfrak{sp}$-invariant part of several quotients of the tensor algebra $\mathsf{T}(\mathsf{H})$. For every quotient $\mathcal{A}$ of the tensor algebra $\mathsf{T}(\mathsf{H})$ which we consider below, there is an onto map from the algebra $\mathcal{CD}$ to a combinatorial algebra $\mathcal{CA}$, together with a commutative diagram

$$\begin{array}{ccc} \mathcal{CD} & \longrightarrow & \mathsf{T}(\mathsf{H})^{\mathfrak{sp}} \\ \downarrow & & \downarrow \\ \mathcal{CA} & \longrightarrow & \mathcal{A}^{\mathfrak{sp}} \end{array}$$

such that the map $\mathcal{CA} \to \mathcal{A}^{\mathfrak{sp}}$ is a stable isomorphism of graded algebras (which multiplies degrees by 2). We now discuss some quotients of the tensor algebra.

The natural projection $\mathsf{T}^3(\mathsf{H}) \to \Lambda^3(\mathsf{H})$ induces a map $\mathsf{T}(\mathsf{T}^3(\mathsf{H})) \to \mathsf{T}(\Lambda^3(\mathsf{H}))$. The corresponding quotient of $\mathcal{CD}$ is the algebra $\mathbb{C}\langle\text{T-graphs-no legs}'\rangle$ of trivalent graphs (without legs, equipped with an ordering of the set of their vertices, a cyclic ordering of the three flags around each vertex, as well as an orientation on each edge), modulo the relations (O_1, O_2), where O_2 is shown in figure 1. The map $\mathcal{CD} \to \mathbb{C}\langle\text{T-graphs-no legs}'\rangle/(O_1, O_2)$ sends a chord diagram (σ, λ^1_{6m}) of degree $6m$ to the trivalent graph (σ, λ^3_{6m}) of degree $6m$, where $\lambda^3_{6m} = \{\{1,2,3\},\{4,5,6,\},\dots,\{6m-2, 6m-1, 6m\}\}$, thus inducing a stable isomorphism of algebras

$$\mathbb{C}\langle\text{T-graphs-no legs}'\rangle/(O_1, O_2) \longrightarrow \mathsf{T}(\Lambda^3(\mathsf{H}))^{\mathfrak{sp}}.$$

The projection $\Lambda^3(\mathsf{H}) \to \langle 1^3\rangle$, induces a map $\mathsf{T}(\Lambda^3(\mathsf{H})) \to \mathsf{T}(\langle 1^3\rangle)$, where throughout the text, all projections of $\mathfrak{sp}$-modules will be well defined up to a nonzero scalar. The corresponding quotient of $\mathcal{CD}$ is the algebra $\mathbb{C}\langle\text{T-graphs-no legs}'\rangle/(O_1, O_2, \text{loop})$, where loop is the relation shown in figure 1, thus inducing a stable isomorphism of algebras

$$\mathbb{C}\langle\text{T-graphs-no legs}'\rangle(O_1, O_2, \text{loop}) \longrightarrow \mathsf{T}(\langle 1^3\rangle)^{\mathfrak{sp}}.$$

Consider the projection $\mathsf{T}(\langle 1^3\rangle) \to \Lambda(\langle 1^3\rangle)$. The corresponding quotient of $\mathcal{CD}$ is the algebra of trivalent graphs (without legs, equipped with a sign ordering of the set of their vertices, a cyclic ordering of the three flags around each vertex, and an orientation on each edge), modulo the relations $(O_1, O_2, O_3, \text{loop})$, where O_3 is shown in figure 1. It is easy to see that the above algebra is isomorphic to the algebra $\mathbb{C}\langle\text{T-graphs-no legs}\rangle$ of trivalent graphs (without orientations or legs) modulo the relation loop, thus inducing a stable isomorphism of algebras

$$\mathbb{C}\langle\text{T-graphs-no legs}\rangle/(\text{loop}) \longrightarrow \Lambda(\langle 1^3\rangle)^{\mathfrak{sp}}.$$

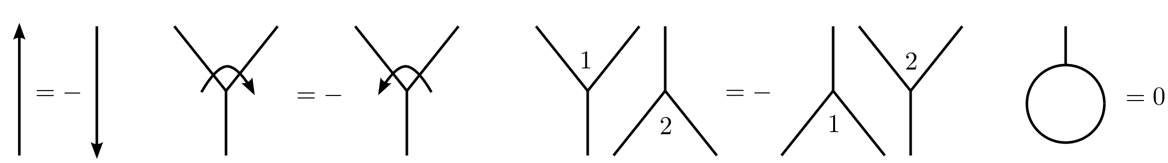

FIGURE 1. The antisymmetry relations O_1, O_2, O_3 and the loop relation.

Consider the projection $\Lambda(\langle 1^3 \rangle) \to \mathsf{A}$. It was shown in [GN98] that the corresponding quotient of the algebra $\mathcal{CD}$ is the algebra of trivalent graphs (without any orientations) modulo the relations (IH, loop), where IH is shown in figure 2. In addition, it was shown that the algebra $\mathbb{C}\langle\text{T-graphs}\rangle/(\text{IH}, \text{loop})$ is isomorphic to a free polynomial algebra $\mathbb{Q}[e_{2,0}, e_{3,0}, \dots]$ (where $e_{k,0}$, shown in figure 3, is of degree $2k-2$), thus inducing a stable isomorphism of algebras

$$\mathbb{Q}[e_{2,0}, e_{3,0}, \dots] \longrightarrow \mathsf{A}^{\mathsf{sp}}. \tag{4.18}$$

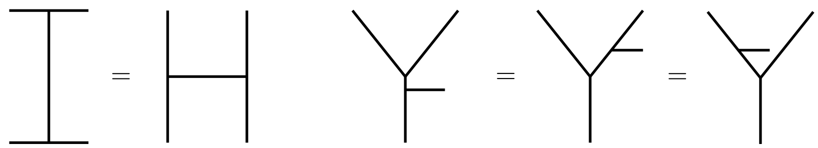

FIGURE 2. On the left, the IH relation, with the understanding that the flags of I and H are not part of an edge. On the right, a consequence of the IH relation.

Consider the projection $\Lambda(\langle 1^3 \rangle) \otimes \mathsf{T}(\mathsf{H}) \to \mathsf{A} \otimes \mathsf{T}(\mathsf{H})$. The above discussion implies that the corresponding quotient of $\mathcal{CD}$ is the algebra $\mathbb{C}\langle\text{T-graphs}\rangle$ of trivalent graphs with ordered legs (equipped with an orientation on each edge that connects two legs), modulo the relations IH, loop. We claim that every connected such graph with $2k-2+n$ trivalent vertices and n legs, equals, modulo the IH relation to the graph $e_{k,n}$ shown in figure 3. Indeed, using the IH relation as in figure 3, we can move edges touching a leg anywhere around the graph, thus we can assume that there are no legs (i.e., $n=0$), in which case the result follows from a previous discussion. Notice that $e_{k,n}$ is a trivial representation of $\mathbb{S}_n$.

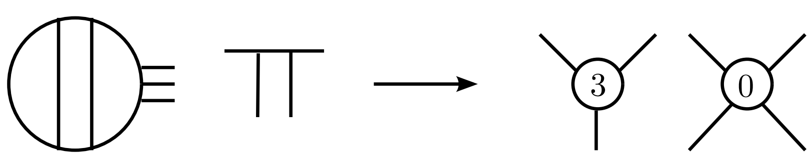

FIGURE 3. The T-graph $e_{k,n}$ for $k \geq 1$ has $k-1$ vertical edges on a circle and n horizontal legs. The T-graph $e_{0,n}$ for $n \geq 2$ is a tree with $n-2$ vertical legs. On the left, the graphs $e_{3,3}$ and $e_{0,4}$, and on the right their images.

The stable isomorphism of algebras

$$\mathbb{C}\langle\text{T-graphs}\rangle/(\text{IH}, \text{loop}) \longrightarrow (\mathsf{A} \otimes \mathsf{T}(\mathsf{H}))^{\mathsf{sp}},$$

implies that the $\mathbb{S}$-module $\mathcal{B}$ defined by

$$\mathcal{B}(n) = (\mathsf{A}^{\bullet} \otimes \mathsf{T}^n(\mathsf{H}))^{\mathsf{sp}}$$

equals to $\mathcal{E} \circ (e_{0,2} + \mathcal{V})$, where $\mathcal{E}$ is the $\mathbb{S}$-module whose characteristic $\mathrm{ch}_t(\mathcal{E})$ equals to $\sum_{h\geq 0} h_n$, and $\mathcal{V}$ is the $\mathbb{S}$-module with basis $e_{k,n} \in \mathcal{V}_{2k-2+n}(n)$, $k \geq 0$, where $e_{k,n}$ spans a copy of the trivial representation of $\mathbb{S}_n$, excluding $e_{0,0}, e_{1,0}, e_{1,1}$ and $e_{0,2}$. It follows that the characteristic of $\mathcal{V}$ is given by the statement of theorem 1.1, and that $\mathrm{ch}_t(\mathcal{B}) = \mathrm{Exp}(h_2 + \mathrm{ch}_t(\mathcal{V}))$.

Finally, consider the projection $\Lambda(\langle 1^3\rangle) \otimes \mathsf{T}(\mathsf{H}) \to \mathsf{A} \otimes \mathsf{T}_{\mathrm{top}}(\mathsf{H})$. The above discussion implies that the corresponding quotient of $\mathcal{CD}$ is the quotient of the algebra $\mathbb{C}\langle\text{T-graphs}\rangle$ modulo graphs some component of which contains an edge connecting two legs; thus obtaining that $\mathcal{B}_{\mathrm{top}} = \mathcal{E} \circ \mathcal{V}$, which concludes the proof of proposition 4.1. □

Similarly, we have a stable isomorphism of algebras

$$\mathbb{C}\langle\text{T-graphs}\rangle/(\mathrm{IH}^1) \longrightarrow (\mathsf{A}^1 \otimes \mathsf{T}(\mathsf{H}))^{\mathfrak{sp}},$$

(where IH^1 is the relation of figure 2, assuming that at most two of the four flags of the graphs I and H belong to the same edge), which implies that the $\mathbb{S}$-module $\mathcal{B}^1$ defined by

$$\mathcal{B}^1(n) = (\mathsf{A}^\bullet \otimes \mathsf{T}^n(\mathsf{H}))^{\mathfrak{sp}}$$

equals to $\mathcal{E} \circ (e_{0,2} + e_{1,1} + \mathcal{V})$, and that the $\mathbb{S}$-module $\mathcal{B}^1_{\mathrm{top}}$ defined by

$$\mathcal{B}^1_{\mathrm{top}}(n) = (\mathsf{A}^{1,\bullet} \otimes \mathsf{T}^n_{\mathrm{top}}(\mathsf{H}))^{\mathfrak{sp}}$$

equals to $\mathcal{E} \circ (e_{1,1} + \mathcal{V})$, thus

$$\mathrm{c}_t(\mathsf{A}^1) = \tilde{\omega} \exp(-\mathbb{D}') \,\mathrm{ch}_t(\mathcal{B}^1_{\mathrm{top}}) = \mathrm{Exp}(h_1 t + \mathrm{ch}_t(\mathcal{V}))$$

which concludes the proof of theorem 1.1.

Remark 4.4. There is a curious degree preserving map from t-graphs $e_{k,n}$ of degree $2k - 2 + n > 0$ with n legs to stable graphs (in the terminology of [GK98]) with one vertex of genus k and n legs, see figure 3. There is also a similarity between the character $\mathrm{ch}_t(\mathcal{V})$ of $\mathcal{V}$ and the Feynman transform of [GK98]. Notice also that the $\mathbb{S}$-module $\mathcal{V}$ has an additional multiplication

$$\diamond : \mathcal{V}(n) \otimes \mathcal{V}(m) \longrightarrow \mathcal{V}(n + m - 2)$$

defined by joining a leg of $e_{k,n}$ with one of $e_{l,m}$, in other words by $e_{k,n} \diamond e_{l,m} = e_{k+l,n+m-2}$. Under the isomorphism $\mathcal{B}_{\mathrm{top}} \cong \mathcal{E} \circ \mathcal{V}$, $\diamond$ corresponds to a map $\diamond : \mathcal{B}_{\mathrm{top}}(n) \otimes \mathcal{B}_{\mathrm{top}}(m) \to \mathcal{B}_{\mathrm{top}}(n + m - 2)$ (denoted by the same name) defined using the product in A and the contraction map

$$\diamond : \mathsf{T}(\mathsf{H}) \otimes \mathsf{T}(\mathsf{H}) \longrightarrow \mathsf{T}(\mathsf{H})$$

given by

$$(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_m) \longrightarrow \sum_{i,j} (a_i \cdot b_j) a_1 \otimes \ldots \hat{a}_i \ldots \otimes a_n \otimes b_1 \otimes \ldots \hat{b}_j \ldots \otimes b_m,$$

where $a_i, b_j \in \mathsf{H}$ and $\cdot : \mathsf{H} \otimes \mathsf{H} \to \mathbb{C}$ is given by the symplectic form.

5. Proof of theorem 1.2

Let $\mathcal{N}$ denote the $\mathbb{S}$-module $\mathcal{N}(n) = H^\bullet(\Gamma, \mathsf{T}^n(\mathsf{H}))$. Under the isomorphism $\mathcal{B} \cong \mathcal{E} \circ \mathcal{V}$, the discussion in the introduction implies the existence of a map

$$\mu_F : \mathcal{B}(n) \longrightarrow \mathcal{N}(n)$$

which is part of a commutative diagram:

$$\begin{array}{ccc} \mathcal{B}(n) \otimes \mathcal{B}(m) & \xrightarrow{\mu_F(n) \otimes \mu_F(m)} & \mathcal{N}(n) \otimes \mathcal{N}(m) \\ \diamond \downarrow & & \downarrow \diamond \\ \mathcal{B}(n+m-2) & \xrightarrow{\mu_F(n+m-2)} & \mathcal{N}(n+m-2) \end{array}$$

where the vertical map on the right hand side is induced by the the cup product follows by a contraction on $\mathsf{T}(\mathsf{H})$.

Note that the left vertical map is onto. We claim that assuming Mumford's conjecture,

- The right vertical map is onto
- $\mu_F(0)$ is onto (follows from Morita's papers, assuming Mumford's conjecture. We could give a proof here anyway.)
- $\mu_F(1)$ is onto.

Then, μ_F is onto and since $\mathrm{ch}_t(\mathcal{B}) = \mathrm{ch}_t(\mathcal{N})$, μ_F is an isomorphism, thus proving theorem 1.2.

Let us do as an example the following special case for $R = \mathsf{T}^3(\mathsf{H})$. Assuming Mumford's conjecture it follows from corollary 3.2 that $\dim H^1(\Gamma, \mathsf{T}^3(\mathsf{H})) = 1$, and that $\dim H^1(\Gamma^1, \mathsf{T}^3(\mathsf{H})) = 4$. With the notation of the introduction, after projecting $N \to N/[N,N] = \langle 1^3 \rangle$, ρ can be thought of as a crossed homomorphism $\Gamma_g \to \langle 1^3 \rangle \hookrightarrow R$, in other words as a 1-cocycle of Γ_g with values in R; let $[\rho] \in H^1(\Gamma_g, R)$ denote its cohomology class. Then $\mu_F(1)(e_{0,3}) = [\rho]$, and Morita proves that $[\rho]$ is nonzero, by an explicit cocycle computation. Similarly, $\mu^1_F(1)(e_{0,3}) = [\rho^1] \in H^1(\Gamma^1_g, R)$ is nonzero too.

$$\begin{array}{ccccccc} (A_1 \otimes R)^{\mathfrak{sp}} & \longrightarrow & H^1(\Gamma, R) & & e_{0,3} & \longrightarrow & [\rho] \\ \downarrow & & \downarrow & \text{where} & \downarrow & & \downarrow \\ (A^1_1 \otimes R)^{\mathfrak{sp}} & \longrightarrow & H^1(\Gamma^1, R) & & 3e_{1,1}e_{0,2} + e_{0,3} & \longrightarrow & [\rho^1] \end{array}$$

where $2e_{1,1}e_{0,2}$ correspond to the three labelings of the legs of the graph $e_{1,1}e_{0,2}$, and these maps are up to scalar multiples that are missing.

Note that Morita's map k_0 [KM96] is simply $\mu_F(2)(e_{0,2}) \in H^1(\mathcal{M}^2_g, \mathsf{H})$, and that the contraction formula that they have must be included in our commutative diagram above.

6. The symplectic character of the Torelli Lie algebra

In this section we give a proof of Theorem 1.3 using properties of quadratic algebras and Koszul duality, explained in detail in [PP05]. Recall from Section 1.1 the quadratic presentation of the $R(\mathfrak{sp})$-Lie algebra $\mathfrak{t}_g$. Hain also established a central extension of graded Lie algebras

$$1 \longrightarrow \mathbb{Q}(2) \longrightarrow \mathfrak{t}_g \longrightarrow \mathfrak{u}_g \longrightarrow 1,$$

where $\mathbb{Q}(2)$ is a one dimensional abelian subalgebra of $\mathfrak{t}_g$ in degree 2. The characteristic of the Lie algebra $\mathfrak{t}_g$ and its universal enveloping algebra $\mathsf{U}(\mathfrak{t}_g)$ are related by

$$\mathrm{c}_t(\mathfrak{t}_g) = \mathrm{Log}(\mathrm{c}_t(\mathsf{U}(\mathfrak{t}_g)) = t^2 + \mathrm{Log}(\mathrm{c}_t(\mathsf{U}(\mathfrak{u}_g))$$

On the other hand, the quadratic dual $\mathsf{U}(\mathfrak{u}_g)^!$ of $\mathsf{U}(\mathfrak{t}_g)$ equals, by definition, to A; see [PP05, Chpt.1.2]. It follows that that A is Koszul if and only if $\mathfrak{u}$ is Koszul if and only if $\mathfrak{t}$ is Koszul. Since Koszul dual algebras $(B, B^!)$ satisfy $\mathrm{c}_t(B)\,\mathrm{c}_{-t}(B^!) = 1$ (see [PP05, Cor.2.2]), this concludes the proof of Theorem 1.3.

7. Computations

Theorem 1.1 implies that the character of A_n is a linear combination of symplectic Schur functions $s_{\langle\lambda\rangle}$ for $|\lambda| \leq 3n$, and can thus be calculated by truncating Exp in degrees at most $3n$. Note that partitions with $|\lambda| = 3n$ appear in A_n.

Assuming that A is Koszul, Theorem 1.3 implies that the character of $\mathfrak{t}_n$ is a linear combination of symplectic Schur functions $s_{\langle\lambda\rangle}$ for $|\lambda| \leq n+2$, and can be calculated by truncating Exp in degrees at most $3n$.

A computation of the character of A and (assuming Koszulity) of $\mathfrak{t}$ was obtained up to degree 12, using J. Stembridge's [Ste95] symmetric function package `SF` for `maple`. Below, we present the results for A_n for $n = 1, \dots, 4$ and for $\mathfrak{t}_n$ for $n = 1, \dots, 8$. A table for $n \leq 13$ is available in [Gar].

To explain our computations, we add a few comments on some of our equations.

Equation (2.14) states the following: if $f = \sum_\lambda a_\lambda s_\lambda$, then $\exp(-\mathbb{D})f = \sum_\lambda a_\lambda s_{\langle\lambda\rangle}$.

Using $\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g)$ and $h_n(t) = t^n$ and $\mathrm{Exp}(t) = \sum_{n=0}^\infty h_n(t) = (1-t)^{-1}$, we see that the right hand side of Equation (1.2) is given by:

$$\mathrm{Exp}\Big(\frac{1}{1-t^2}\Big) = \mathrm{Exp}\Big(\sum_{n=0}^\infty t^{2n}\Big) = \prod_{n=0}^\infty \mathrm{Exp}(t^{2n}) = \frac{1}{\prod_{n=1}^\infty (1-t^{2n})}$$

Corollary 4.2 states that if $\tilde{\omega}\ \mathrm{ch}_t(\mathcal{B}_{\mathrm{top}}) = \sum_\lambda a_\lambda s_\lambda$, then $\mathrm{c}_t(\mathsf{A}) = \sum_\lambda a_\lambda s_{\langle\lambda\rangle}$.

The next lemma is an effective way to compute the character $\mathrm{Exp}(\mathrm{ch}_t(\mathcal{V}))$ using the `SF` package.

Lemma 7.1. We have:

$$\mathrm{Exp}(\mathrm{ch}_t(\mathcal{V})) = \sum_\lambda \gamma_\lambda \tag{7.19}$$

where

$$\gamma_{(1^{l_1}2^{l_2}\dots)} = \prod_n h_{l_n} \circ \frac{t^{c_n} h_n}{1-t^2} = t^{\delta(\lambda)} \prod_n h_{l_n} \circ \frac{h_n}{1-t^2} \tag{7.20}$$

and $c_n = n-2$ (resp., 0, 3, 2) when $n \geq 3$ (resp., $n=0$, $n=1$, $n=2$) and $(1^{l_1}2^{l_2}\dots) = \sum_n c_n l_n$.

Note that to calculate the $\Lambda_n[\![t]\!]$ part of $\mathrm{Exp}(\mathrm{ch}_t(\mathcal{V}))$, we can truncate Equation (7.19) to partitions λ with $|\lambda| \leq 3n$. Indeed, the worst case occures when $\lambda = (3^n)$ when $|\lambda| = 3n$ and $(\lambda) = n$. Compare also with [Loo96, Eqn.(1)].

Proof. Using Equations (2.12) and (2.13), we have

$$\begin{aligned} \mathrm{Exp}(\mathrm{ch}_t(\mathcal{V})) &= \mathrm{Exp}\Big(\frac{1}{1-t^2} + \frac{t^3 h_1}{1-t^2} + \frac{t^2 h_2}{1-t^2} + \sum_{n=3}^{\infty} \frac{t^{n-2} h_n}{1-t^2}\Big) \\ &= \prod_{n=0}^{\infty} \mathrm{Exp}\Big(\frac{t^{c_n} h_n}{1-t^2}\Big) \\ &= \prod_{n=0}^{\infty} \sum_{l_n=0}^{\infty} h_{l_n} \circ \frac{t^{c_n} h_n}{1-t^2} \end{aligned}$$

Encoding partitions $\lambda = (1^{l_1}2^{l_2}\dots)$ by their number of parts, the first result follows. Since h_{l_n} is homogeneous of degree l_n, we have $h_{l_n} \circ t^{c_n} h_n = t^{l_n c_n} h_{l_n} \circ h_n$. The result follows. □

n	A_n
1	$\langle 1^3 \rangle$
2	$\langle 0 \rangle + \langle 1^4 \rangle + \langle 1^2 \rangle + \langle 1^6 \rangle + \langle 2^2, 1^2 \rangle$
3	$\langle 1^9 \rangle + \langle 2^2, 1^5 \rangle + \langle 2^3, 1^3 \rangle + \langle 3^2, 1^3 \rangle + \langle 3, 2^3 \rangle + 2\langle 1^5 \rangle + \langle 2, 1^3 \rangle + \langle 2^2, 1 \rangle + \langle 1^7 \rangle + \langle 2, 1^5 \rangle$ $+\langle 2^2, 1^3 \rangle + \langle 2^3, 1 \rangle + 2\langle 1^3 \rangle + \langle 1 \rangle$
4	$2\langle 0 \rangle + 4\langle 2^2, 1^4 \rangle + \langle 3, 2^3, 1 \rangle + 4\langle 1^6 \rangle + 4\langle 1^4 \rangle + 2\langle 1^2 \rangle + 3\langle 2^2, 1^2 \rangle + \langle 2^2 \rangle + \langle 3^4 \rangle + 2\langle 2^3, 1^2 \rangle$ $+3\langle 1^8 \rangle + 2\langle 2^4 \rangle + \langle 3^2, 1^2 \rangle + \langle 2^2, 1^8 \rangle + \langle 3, 2^2, 1 \rangle + 2\langle 2, 1^6 \rangle + \langle 1^{12} \rangle + \langle 3, 2^3, 1^3 \rangle + \langle 2^3, 1^6 \rangle$ $+\langle 3^2, 1^6 \rangle + \langle 2^4, 1^4 \rangle + 2\langle 2^4, 1^2 \rangle + 2\langle 2^3, 1^4 \rangle + \langle 3, 2, 1^5 \rangle + \langle 3, 2^2, 1^3 \rangle + \langle 3^2, 1^4 \rangle + \langle 3^2, 2^2 \rangle$ $+\langle 3^2, 2, 1^2 \rangle + \langle 2, 1^2 \rangle + \langle 2, 1^8 \rangle + 2\langle 2^2, 1^6 \rangle + \langle 1^{10} \rangle + \langle 3, 2, 1^3 \rangle + \langle 2^3 \rangle + 2\langle 2, 1^4 \rangle + \langle 3^2, 2, 1^4 \rangle$ $+\langle 2^6 \rangle + \langle 3^2, 2^2, 1^2 \rangle + \langle 4^2, 1^4 \rangle + \langle 4, 3, 2^2, 1 \rangle$

n	Degree n part of $\mathfrak{t}$
1	$\langle 1^3\rangle$
2	$\langle 0\rangle + \langle 2^2\rangle$
3	$\langle 3,1^2\rangle$
4	$\langle 2\rangle + \langle 3,1\rangle + \langle 4,2\rangle + \langle 2^3\rangle + \langle 3,1^3\rangle$
5	$\langle 2,1\rangle + \langle 1^3\rangle + \langle 4,1\rangle + \langle 3,2\rangle + \langle 3,1^2\rangle + \langle 2^2,1\rangle + \langle 2,1^3\rangle + \langle 5,1^2\rangle + \langle 4,2,1\rangle + \langle 3^2,1\rangle$ $+\langle 3,2,1^2\rangle + \langle 2^2,1^3\rangle$
6	$\langle 2\rangle + \langle 0\rangle + \langle 2^2,1^4\rangle + \langle 2^4\rangle + \langle 3,2,1^3\rangle + \langle 3,2^2,1\rangle + 2\langle 3^2,1^2\rangle + \langle 4,1^4\rangle + \langle 4,2,1^2\rangle + 2\langle 4,2^2\rangle$ $+\langle 4,3,1\rangle + \langle 4^2\rangle + \langle 5,1^3\rangle + \langle 5,2,1\rangle + \langle 6,2\rangle + \langle 1^6\rangle + \langle 2,1^4\rangle + 3\langle 2^2,1^2\rangle + \langle 2^3\rangle + 2\langle 3,1^3\rangle$ $+4\langle 3,2,1\rangle + 2\langle 3^2\rangle + 2\langle 4,1^2\rangle + 2\langle 4,2\rangle + 2\langle 5,1\rangle + 2\langle 1^4\rangle + 2\langle 2,1^2\rangle + 5\langle 2^2\rangle + 2\langle 3,1\rangle + 2\langle 4\rangle$ $+2\langle 1^2\rangle$
7	$\langle 7,1^2\rangle + 6\langle 4,1^3\rangle + 10\langle 4,2,1\rangle + 4\langle 4,3\rangle + 4\langle 5,1^2\rangle + 4\langle 5,2\rangle + 2\langle 6,1\rangle + 13\langle 3,1^2\rangle + 7\langle 4,1\rangle$ $+\langle 5\rangle + 4\langle 3\rangle + 2\langle 3^3\rangle + \langle 4,1^5\rangle + 2\langle 4,3,2\rangle + 3\langle 5,2,1,1\rangle + \langle 5,2^2\rangle + 3\langle 5,3,1\rangle + \langle 6,1^3\rangle$ $+\langle 6,2,1\rangle + \langle 6,3\rangle + 3\langle 3,2^2,1^2\rangle + \langle 3,1^6\rangle + 2\langle 3^2,2,1\rangle + 3\langle 4,3,1^2\rangle + 3\langle 4,2^2,1\rangle + 3\langle 4,2,1^3\rangle$ $+\langle 4^2,1\rangle + \langle 3,2,1^4\rangle + \langle 2^3,1^3\rangle + \langle 3^2,1^3\rangle + \langle 3,2^3\rangle + 7\langle 2,1\rangle + 5\langle 1^3\rangle + 2\langle 2,1^5\rangle + 4\langle 2^2,1^3\rangle$ $+4\langle 2^3,1\rangle + 4\langle 3,1^4\rangle + 10\langle 3,2,1^2\rangle + 6\langle 3,2^2\rangle + 6\langle 3^2,1\rangle + \langle 1\rangle + 8\langle 2^2,1\rangle + 8\langle 2,1^3\rangle + \langle 1^5\rangle$ $+8\langle 3,2\rangle$
8	$15\langle 2\rangle + 5\langle 2^2,1^4\rangle + 3\langle 2^4\rangle + 20\langle 3,2,1^3\rangle + 23\langle 3,2^2,1\rangle + 15\langle 3^2,1^2\rangle + 11\langle 4,1^4\rangle + 29\langle 4,2,1^2\rangle$ $+17\langle 4,2^2\rangle + 22\langle 4,3,1\rangle + 3\langle 4^2\rangle + 10\langle 5,1^3\rangle + 20\langle 5,2,1\rangle + 6\langle 6,2\rangle + \langle 1^6\rangle + 16\langle 2,1^4\rangle$ $+20\langle 2^2,1^2\rangle + 20\langle 2^3\rangle + 32\langle 3,1^3\rangle + 45\langle 3,2,1\rangle + 9\langle 3^2\rangle + 25\langle 4,1^2\rangle + 31\langle 4,2\rangle + 12\langle 5,1\rangle$ $+6\langle 1^4\rangle + 29\langle 2,1^2\rangle + 16\langle 2^2\rangle + 31\langle 3,1\rangle + 8\langle 4\rangle + 5\langle 1^2\rangle + 3\langle 7,1\rangle + 5\langle 6\rangle + 8\langle 5,3,1^2\rangle$ $+4\langle 5,3,2\rangle + 3\langle 5,4,1\rangle + 2\langle 6,1^4\rangle + 4\langle 6,2^2\rangle + 3\langle 6,3,1\rangle + 2\langle 6,4\rangle + \langle 7,1^3\rangle + 2\langle 7,2,1\rangle$ $+\langle 8,2\rangle + 10\langle 5,3\rangle + \langle 4,3^2\rangle + \langle 5,1^5\rangle + 5\langle 5,2,1^3\rangle + 5\langle 5,2^2,1\rangle + 3\langle 6,2,1^2\rangle + 6\langle 6,1^2\rangle$ $+3\langle 3^3,1\rangle + \langle 3^2,2^2\rangle + 5\langle 4,2^2,1^2\rangle + 2\langle 2,1^6\rangle + 5\langle 4,2,1^4\rangle + \langle 2^3,1^4\rangle + 9\langle 4,3,2,1\rangle + 5\langle 4,2^3\rangle$ $+6\langle 4,3,1^3\rangle + \langle 4^2,1^2\rangle + 5\langle 4^2,2\rangle + 14\langle 3^2,2\rangle + 9\langle 2^3,1^2\rangle + 8\langle 3,1^5\rangle + 4\langle 3,2^2,1^3\rangle + \langle 3,1^7\rangle$ $+2\langle 3,2,1^5\rangle + 2\langle 2^5\rangle + 6\langle 3^2,2,1^2\rangle + \langle 3^2,1^4\rangle + 3\langle 3,2^3,1\rangle$

Appendix A. λ-rings

Using the ring Λ, it is possible to give a very direct definition of Grothendieck's λ- and special λ-rings. We follow more recent usage in referring to special λ-rings as λ-rings, and to λ-rings as pre-λ-rings.

A.1. **Pre-λ-rings.** A pre-λ-ring is a commutative ring R, together with a morphism of commutative rings $\sigma_t : R \to R[\![t]\!]$ such that $\sigma_t(a) = 1 + ta + O(t^2)$. Expanding σ_t in a power series

$$\sigma_t(a) = \sum_{n=0}^{\infty} t^n \sigma_n(a),$$

we obtain endomorphisms σ_n of R such that $\sigma_0(a) = 1$, $\sigma_1(a) = a$, and

$$\sigma_n(a+b) = \sum_{k=0}^{n} \sigma_{n-k}(a)\sigma_k(b).$$

There are also operations $\lambda_k(a) = (-1)^k\sigma_k(-a)$, with generating function

$$\lambda_t(a) = \sum_{n=0}^{\infty} t^n \lambda_n(a) = \sigma_{-t}(a)^{-1}. \tag{A.21}$$

The λ-operations are polynomials in the σ-operations with integral coefficients, and vice versa. In this paper, we take the σ-operations to be more fundamental; nevertheless, following custom, the structure they define is called a pre-λ-ring.

Given a pre-λ-ring R, there is a bilinear map $\Lambda \otimes R \to R$, which we denote $f \circ a$, defined by the formula

$$(h_{n_1} \dots h_{n_k}) \circ a = \sigma_{n_1}(a) \dots \sigma_{n_k}(a).$$

The image of the power sum p_n under this map is the operation on R known as the Adams operation ψ_n. We denote the operation corresponding to the Schur function s_λ by σ_λ. Note that (2.5) implies the relation

$$\sigma_t(a) = \exp\Bigl(\sum_{n=1}^{\infty} \frac{t^n\psi_n(a)}{n}\Bigr),$$

from which the following result is immediate.

Proposition A.1. *If R and S are pre-λ-rings, their tensor product $R \otimes S$ is a pre-λ-ring, with σ-operations*

$$\sigma_n(a \otimes b) = \sum_{|\lambda|=n} \sigma_\lambda(a) \otimes \sigma_\lambda(b),$$

and Adams operations $\psi_n(a \otimes b) = \psi_n(a) \otimes \psi_n(b)$.

For example, $\sigma_2(a \otimes b) = \sigma_2(a) \otimes \sigma_2(b) + \lambda_2(a) \otimes \lambda_2(b)$.

A.2. **λ-rings.** The polynomial ring $\mathbb{Z}[x]$ is a pre-λ-ring, with σ-operations characterized by the formula $\sigma_n(x^i) = x^{ni}$. Taking tensor powers of this pre-λ-ring with itself, we see that the polynomial ring $\mathbb{Z}[x_1, \dots, x_k]$ is a pre-λ-ring. The λ-operations on this ring are equivariant with respect to the permutation action of the symmetric group $\mathbb{S}_k$ on the generators, hence the ring of symmetric functions $\mathbb{Z}[x_1, \dots, x_k]^{\mathbb{S}_k}$ is a pre-λ-ring. Taking the limit $k \to \infty$, we obtain a pre-λ-ring structure on Λ.

Definition A.2. A λ-ring is pre-λ-ring such that if $f, g \in \Lambda$ and $x \in R$,

$$f \circ (g \circ x) = (f \circ g) \circ x. \tag{A.22}$$

By definition, the pre-λ-ring Λ is a λ-ring; in particular, the operation $f \circ g$, called plethysm, is associative.

The following result (see Knutson [Knu73]) is the chief result in the theory of λ-rings.

Theorem A.1. *Λ is the universal λ-ring on a single generator h_1.*

This theorem makes it straightforward to verify identities in λ-rings: it suffices to verify them in Λ. As an application, we have the following corollary.

Corollary A.3. The tensor product of two λ-rings is a λ-ring.

Proof. We need only verify this for $R = \Lambda$. A torsion-free pre-λ-ring whose Adams operations are ring homomorphisms which satisfy $\psi_m(\psi_n(a)) = \psi_{mn}(a)$ is a λ-ring. It is easy to verify these conditions for $\Lambda \otimes \Lambda$, since $\psi_n(a \otimes b) = \psi_n(a) \otimes \psi_n(b)$. □

In the definition of a λ-ring, it is usual to adjoin the axiom

$$\sigma_n(xy) = \sum_{|\lambda|=n} \sigma_\lambda(a) \otimes \sigma_\lambda(y).$$

However, this formula follows from our definition of a λ-ring: by universality, it suffices to check it for $R = \Lambda \otimes \Lambda$, $x = h_1 \otimes 1$ and $y = 1 \otimes h_1$, for which it is evident.

A.3. **Complete λ-rings.** A filtered λ-ring R is a λ-ring with decreasing filtration

$$R = F^0 R \supset F^1 R \supset \dots,$$

such that

(i) $\bigcap_k F^k R = 0$ (the filtration is discrete);
(ii) $F^m R F^n R \subset F^{m+n} R$ (the filtration is compatible with the product);
(iii) $\sigma_m(F^n R) \subset F^{mn} R$ (the filtration is compatible with the λ-ring structure).

The completion of a filtered λ-ring is again a λ-ring; define a complete λ-ring to be a λ-ring equal to its completion. For example, the universal λ-ring Λ is filtered by the subspaces $F^n\Lambda$ of polynomials vanishing to order $n-1$, and its completion is the λ-ring of symmetric power series, whose underlying ring is the power series ring $\mathbb{Z}[\![h_1, h_2, h_3, \dots]\!]$.

The tensor product of two filtered λ-rings is again a filtered λ-ring, when furnished with the filtration

$$F^n(R \otimes S) = \sum_{k=0}^{n} F^{n-k} R \otimes F^k S.$$

If R and S are filtered λ-rings, denote by $R \hat{\otimes} S$ the completion of $R \otimes S$.

If R is a complete λ-ring, the operation

$$\operatorname{Exp}(a) = \sum_{n=0}^{\infty} \sigma_n(a) : F_1 R \longrightarrow 1 + F_1 R$$

is an analogue of exponentiation, in the sense that $\operatorname{Exp}(f+g) = \operatorname{Exp}(f)\operatorname{Exp}(g)$. Its logarithm is given by a formula of Cadogan [Cad71].

Proposition A.4. *On a complete filtered λ-ring R, the operation $\operatorname{Exp} : F_1R \to 1+F_1R$ has inverse*

$$\operatorname{Log}(1+a) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+\psi_n(a)).$$

Proof. Expanding $\operatorname{Log}(1+a)$, we obtain

$$\operatorname{Log}(1+a) = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d)\psi_d(-a)^{n/d} = \sum_{n=1}^{\infty} \operatorname{Log}_n(a).$$

Let χ_n be the character of the cyclic group C_n equalling $e^{2\pi i/n}$ on the generator of C_n. The characteristic of the $\mathbb{S}_n$-module $\operatorname{Ind}_{C_n}^{\mathbb{S}_n} \chi_n$ equals

$$\frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi ik/n} p_{(k,n)}^{n/(k,n)} = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d},$$

while the characteristic of the $\mathbb{S}_n$-module $\operatorname{Ind}_{C_n}^{\mathbb{S}_n} \chi_n \otimes \varepsilon_n$, where ε_n is the sign representation of $\mathbb{S}_n$, equals

$$\frac{1}{n} \sum_{d|n} \mu(d)\big((-1)^{d-1}p_d\big)^{n/d} = \frac{(-1)^n}{n} \sum_{d|n} \mu(d)(-p_d)^{n/d}.$$

It follows that $(-1)^{n-1}\operatorname{Log}_n$ is the operation associated to the $\mathbb{S}_n$-module $\operatorname{Ind}_{C_n}^{\mathbb{S}_n} \chi_n \otimes \varepsilon_n$, and hence defines a map from F_1R to F_nR.

To prove that Log is the inverse of Exp, it suffices to check this for $R = \Lambda$ and $x = h_1$. We must prove that

$$\operatorname{Exp}\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+p_n)\right) = 1 + h_1.$$

The logarithm of the expression on the left-hand side equals

$$\exp\Big(\sum_{k=1}^{\infty} \frac{p_k}{k}\Big) \circ \Big(\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+p_n)\Big) = \sum_{n=1}^{\infty} \sum_{d|n} \mu(d) \frac{\log(1+p_n)}{n} = \log(1+p_1),$$

and the formula follows. □

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