Course: CS1050c (Fall '03) Homework2 Solutions

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Section 3.7

Problem 10: *Prove that $\sqrt[3]{2}$ is irrational*
Solution:
Note that if n^3 is even, then n is even, for otherwise, n would be odd, making n^3, the product of odd integers, odd.

Suppose $\sqrt[3]{2}$ is rational. Thus $\sqrt[3]{2} = a/b$, where a and b are integers and $b \neq 0$. Also assume that they don't have any common factors.Cubing both sides we get,
$2 = a^3/b^3$ or $a^3 = 2b^3$. Which implies a^3 is even. Thus a is even, and so a^3 is divisible by 8. Thus, b^3 is divisible by 4, making b^3, and thus b even. Which implies a and b do have a common factor leading to a contradiction. Hence, $\sqrt[3]{2}$ is irrational.

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Problem 13: *Prove that $\sqrt{2}$ is irrational, using the unique prime factorization theorem*
Solution:
Suppose $\sqrt{2}$ is rational.Thus $\sqrt{2} = a/b$, where a and b are integers and $b \neq 0$. Also $a^2 = 2b^2$. Now consider the unique prime factorization of a^2. Since every prime factor of a occurs twice in the upf (unique prime factorization) of a^2, the upf of a^2 must contain an *even* number of 2's. If 2—a, then this even number is positive, else zero. Similarly, upf of b^2 also has an even number of 2's thus upf of $2b^2$ has an odd number of $2's$, leading to a contradiction. Hence, $\sqrt{2}$ is irrational.

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Problem 22: *Prove that there is a unique prime number of the form $n^2 + 2n - 3$, where n is a positive integer.*
Solution:
Its not difficult to see that $n^2 + 2n - 3 = (n + 3)(n - 1)$. Now if this were to be a prime, then one of the two of $(n+3)$ or $(n-1)$ must be 1 and the other must be prime, else they will be the two factors. Since $n > 0$, $n + 3 > 1$. Thus only $n - 1 = 1$ is possible, which means $n = 2$ is the only case when

$n^2 + 2n - 3$ will be a prime. Indeed, $n = 2$ gives us $n^2 + 2n - 3 = 5$ which is a prime. For other values of n, $n^2 + 2n - 3$ is composite. Hence there is a unique prime of the form $n^2 + 2n - 3$.

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Section 3.8

Problem 14: *Using Euclidean method, find the GCD of* $3,510$ *and* 672
Solution:
$3510 = 672 \cdot 5 + 150$
$672 = 150 \cdot 4 + 72$
$150 = 72 \cdot 2 + 6$
$72 = 6 \cdot 12$

Thus $gcd(3510, 672) = 6$.

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Problem 24a: *Find* $lcm(12, 18)$
Solution:
lcm(12,18) $=$ lcm($2^2 \cdot 3^1$, $2^1 \cdot 3^2$)
Thus lcm(12,18) $= 2^2 \cdot 3^2 = 36$.

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Problem 24b: *Find* $lcm(12, 18)$
Solution:
lcm($2^1 \cdot 3^2 \cdot 5$, $2^3 \cdot 3^1$)
Thus lcm(12,18) $= 2^3 \cdot 3^2 \cdot 5 = 360$.

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Problem 24c: *Find* $lcm(3500, 1960)$
Solution:
lcm(3500,1960) $=$ lcm($2^2 \cdot 5^3 \cdot 7$, $2^3 \cdot 5 \cdot 7^2$)
Thus lcm(12,18) $= 2^3 \cdot 5^3 \cdot 7^2 =$ 49,000.

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Problem 28: *Prove that for any two numbers* a *and* b, $gcd(a, b) \cdot lcm(a, b) = a \cdot b$
Solution:
Claim: $gcd(a, b) \cdot lcm(a, b) \leq a \cdot b$

Proof: Since $gcd(a,b)|a$, $\frac{a}{gcd(a,b)}$ is an integer. Thus b divides $\frac{ab}{gcd(a,b)}$. Similarly, a divides $\frac{ba}{gcd(a,b)}$. Thus $\frac{a}{gcd(a,b)}$ is a common multiple of a and b, implying $lcm(a,b) \leq \frac{a}{gcd(a,b)}$, or, $gcd(a,b) \cdot lcm(a,b) \leq a \cdot b$

Claim: $gcd(a,b) \cdot lcm(a,b) \geq a \cdot b$
Proof: By definition, $a|lcm(a,b)$. Thus $lcm(a,b) = ak$, for some integer k. Thus $b \cdot lcm(a,b) = abk$ which implies $b = \frac{ab}{lcm(a,b)} \cdot k$, which implies $b|\frac{ab}{lcm(a,b)}$. Similarly $a|\frac{ab}{lcm(a,b)}$. Thus $\frac{ab}{lcm(a,b)}$ is a common divisor. Hence, $\frac{ab}{lcm(a,b)} \leq gcd(a,b)$ implying $gcd(a,b) \cdot lcm(a,b) \geq a \cdot b$

By the two claims, $gcd(a,b) \cdot lcm(a,b) = a \cdot b$

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Section 4.2

Problem 2: *Using mathematical Induction, show that any postage of denomination greater that 8c can got from stamps of 3c and 5c*
Solution:
Base Case: 8c = 1 3c stamp + 1 5c stamp
Induction Hypothesis: Any postage of denomination k can be got from stamps of 3c and 5c, when $k \geq 8$.
Claim: A stamp of denomination $k+1$ can be got from 3c and 5c stamps.
Proof:
Suppose there is a 5c stamp used to make the k cents postage. Remove it and put 2 3c stamps, we get a postage of $k+1$ stamps.

If no 5c stamps were used, then atleast 3 3c stamps must have been used to make the k cent amount as $k \geq 8$. Thus remove 3 3c stamps and put in 2 5c stamps to get $k+1$c worth of postage.
hence proved

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Problem 7: *Use Mathematical Induction to show that 1+5+9+* $\cdots$ *+* $4n-3$ $= n(2n-1)$
Solution:
Base Case: n=1.

$$LHS = 1$$

$$\begin{aligned} RHS &= 1(2-1) \\ &= 1 \end{aligned}$$

Induction Hypothesis: Let 1+5+9+ $\cdots + 4n - 3 = n(2n-1)$ for all $n \leq k$.
Claim: 1+5+9+ $\cdots + 4(k+1) - 3 = (k+1)(2(k+1)-1)$
Proof:

$$\begin{aligned} LHS &= 1+5+9+\cdots+4(k+1)-3 \\ &= 1+5+9+\cdots+4k-3+4(k+1)-3 \\ &= k(2k-1)+(4k+1) && \text{By Induction Hypothesis} \\ &= 2k^2+3k+1 \\ RHS &= (k+1)(2(k+1)-1) \\ &= (k+1)(2k+1) \\ &= 2k^2+3k+1 \end{aligned}$$

Hence Proved

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Problem 13: *Using Mathematical Induction, prove* $\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$ *for all* $n \geq 0$
Solution:
Base Case: n=0.

$$\begin{aligned} LHS &= 1.2^1 \\ &= 2 \\ RHS &= 0+2 \\ &= 2 \end{aligned}$$

Induction Hypothesis:$\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$ for all $n \leq k$
Claim: $\sum_{i=1}^{k+2} i2^i = (k+1)2^{k+3} + 2$
Proof:

$$\begin{aligned} LHS &= \sum_{i=1}^{k+2} i2^i \\ &= \sum_{i=1}^{k+1} i2^i + (k+2)2^{k+2} \end{aligned}$$

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\begin{aligned}
&= k.2^{k+2} + 2 + (k+2)2^{k+2} && \text{By Induction Hypothesis} \\
&= 2^{k+2}(2k+2) + 2 \\
&= (k+1)2^{k+3} + 2 \\
&= RHS
\end{aligned}
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Hence Proved

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Additional problem

Problem : *Find the GCD of 34709 and 100313; also express the GCD as an integer combination of the two numbers*
Solution:
Using Euclid's algorithm,
100313 = 34709 . 2 + 30895
34709 = 30895 . 1 + 3814
30895 = 3814 . 8 + 383
3814 = 383 . 9 + 367
383 = 367 . 1 + 16
367 = 16 . 22+ 15
16 = 15 . 1 + 1
Thus gcd(100313, 34709) =1

Backtracking we get the following.
16 - 15.1 = 1
16 - (367 - 16 . 22) . 1 = 1
16.23 - 367.1 = 1
(383 - 367).23 - 367 . 1 = 1
383 . 23 - 367 . 24 = 1
383 . 23 - (3814 - 383 . 9) .24 = 1
383 . 239 - 3814 . 24 = 1
Continuing thus we shall finally get, 100313 . 2175 + 34709 . (-6286) = 1.

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A few Optional Problems

Problem 3.7.15: *Prove that* $log_2 3$ *is irrational*
Solution:
Suppose not. Then $log_2 3 = a/b$ for some integers a and b. Thus $2^a = 3^b$.

Now the LHS has the prime factors 2 while the RHS has prime factors 3. Since $b \neq 0$, there is atleast one 3 in the RHS, but none in the LHS. Thus there is a contradiction.

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Problem 3.7.18: *Suppose for any odd prime p, there exists no solutions to the equation $x^p + y^p = z^p$. Then show that for every integer which is not a power of 2, $x^n + y^n = z^n$ has no solutions*
Solution:
Since n is not a power of two, there exists an odd prime factor p of n. Suppose $n = pk$. Then $x^n = (X^k)^p$. Thus if the equation $x^n + y^n = z^n$ had solutions x_0, y_0, z_0 then x_0^k, y_0^k, z_0^k form the solutions to the equation $x^p + y^p = z^p$, contadicting the premise of the problem. Hence proved

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Problem 3.8.23.a: *Prove: $gcd(a, b) = gcd(b, a - b)$ for any two integers $a \geq b > 0$.*
Solution:
Suppose $d|a$ and $d|b$. Then $d|(a - b)$ Thus every common divisor of a and b is a common divisor of b,$a - b$. Suppose $d|b$ and $d|a - b$, then $d|b + (a - b)$ that is $d|a$. Thus every common divisor of b and $b - a$ is a common divisor of a and b. In other words all common divisors are same, or more specifically the Greatest common divisors are same.

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Problem 4.2.8: *Using Mathematical Induction , prove:* $1+2+2^2+\cdots+2^n = 2^{n+1} - 1$ *for all* $n \geq 0$
Solution:
Base Case: n=0.

$$\begin{aligned} LHS &= 1 \\ RHS &= 2^1 - 1 \\ &= 1 \end{aligned}$$

Induction Hypothesis: $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \leq k$
Claim: $1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$
Proof

$$LHS = 1 + 2 + 2^2 + \cdots + 2^{k+1}$$

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\begin{aligned}
&= 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2.2^{k+1} - 1 \\
&= 2^{k+2} - 1 \\
&= RHS
\end{aligned}
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Problems 3.7.10, 3.8.14, 4.2.7 and the Additional problem were graded. These solutions were prepared by Deeparnab Chakrabarty(deepc@cc) Please report bugs to the same