Course: CS 1050C – Test 3 (FALL '03)

Instructor : Prasad Tetali, office: Skiles 126, email: tetali@math.gatech.edu

Time: 50 minutes Total Points : 10 + 10 + 10 = 30

Please explain all your answers.

Name: Prasad Tetali

Problem 1. (4+4+2 pts)

(a) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x/(1+x^2)$. Is f injective (one-to-one)? Is f surjective (onto)?

Solution. Not injective : if $x_1 = 1/x_2$ then $f(x_1) = f(x_2)$, so for example f(2)=f(1/2). (We would come up with this if we tried checking the definition of injective: suppose $f(x_1) = f(x_2)$, then we see that $x_1(1+x_2^2) = x_2(1+x_1^2)$, which implies that $(x_1 - x_2) = x_1x_2(x_1 - x_2)$. So if $x_1 \neq x_2$, then we get that $x_1x_2 = 1$!.

Not surjective: given $y \in \mathbf{R}$, it turns out there is not necessarily an x such that $f(x) = y$. Given y, we need to solve for x in $x/(1+x^2) = y$ which amounts to solving for x in the quadratic equation, $yx^2 - x + y = 0$, (where y is the given real number. The solution of this is $x = \dfrac{1 \pm \sqrt{1-4y^2}}{2y}$, which is not real, if $4y^2 > 1$.

(b) Is the function $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ defined by $f(m,n) = m^2 + n^2$ injective ? Is it surjective?

Solution. Not injective: $f(m,n) = f(n,m)$ (also $= f(m,-n)$.)

Not surjective: no m, n such that $m^2 + n^2 = 3$, for example. Can easily be seen by considering $m, n \in \{-1, 0, 1\}$.

General Info. *A fact from elementary number theory is that no odd prime of the form* $4k+3$ *can be written as a sum of two squares! On the other hand, a famous theorem of Lagrange shows that EVERY positive integer can be written as a sum of FOUR squares (assuming we are allowed to use* 0^2*).*

(c) Write the negation of "$\forall x \in D$, if $P(x)$ then $Q(x)$." Write the contrapositive of the statement in quotation marks.

Solution. The negation is "$\exists x \in D$, such that $P(x)$ and $\neg Q(x)$."

The contrapositive is "$\forall x \in D$, if $\neg Q(x)$ then $\neg P(x)$."

Problem 2. $(2+2+2+2+2$ pts) Briefly explain the following.

(a) Let
$$f(n) = n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \frac{n}{16} + \cdots + 4 + 2 + 1.$$
As n gets large, which grows faster : $f(n)$ or $n \log n$?

Solution. $n \log n$ grows faster: $f(n) \leq n(1 + 1/2 + (1/2)^2 + \cdots) = 2n = o(n \log n)$.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be an increasing function, meaning that, for all $x, y \in \mathbf{R}$, if $x < y$ then $f(x) < f(y)$. Give an example of an increasing function. If f is increasing then is f necessarily injective?

Solution. $f(x) = x$ is increasing. (So is $f(x) = x^2$.)
Necessarily injective: suppose $x_1 \neq x_2$, then without loss of generality assume that $x_1 < x_2$. Now f increasing implies that $f(x_1) < f(x_2)$, in particular, $f(x_1) \neq f(x_2)$.

(c) Give the definition of a countable set.

Solution. A set is countable if (i) it is finite, or if (ii) there is a one-to-one correspondence from $\mathbf{Z}^+$ to the set.

(d) How many integers must you pick from $\{1, 2, \ldots 2n\}$ to make sure you get at least k odd numbers? (You may assume that $1 \leq k \leq n$.)

Solution. Since there are n even numbers, we need to pick at least $n + k$ numbers to guarantee that we get at least k odd numbers.

(e) If $f_1(x) = O(g_1(x))$, and $f_2(x) = O(g_2(x))$, then can we say that $f_1(x) + f_2(x) = O(g_1(x) + g_2(x))$? What can we say about the growth of $f_1(x) \times f_2(x)$ in terms of $g_1(x)$ and $g_2(x)$?

Solution. Yes, for the first part, since
$$f_1(x) \leq M_1 g_1(x), \text{ for } x > x_1,$$
$$f_2(x) \leq M_2 g_2(x), \text{ for } x > x_2,$$
together imply that for the choice $M = \max\{M_1, M_2\}$, and $x_0 = \max\{x_1, x_2\}$, we have
$$f_1(x) + f_2(x) \leq M\left(g_1(x) + g_2(x)\right), \text{ for } x > x_0.$$

Similarly, we can choose $M = M_1 \times M_2$, with x_0 the same as above, and conclude that
$$f_1(x) f_2(x) = O(g_1(x) g_2(x)).$$

Problem 3. (10 pts) Answer the following in True or False.

(a) The set of all functions from the set of positive integers to $\{0, 1\}$ is countable.
False. See Exercise 7.6.28.

(b) $n = O(2^{(\log n)^2})$.
True.

(c) The contrapositive of "all hummingbirds are richly colored" is "every bird that is not richly colored is not a hummingbird."
True. Write the original statement using the universal quantifier: "for every bird, if it is a hummingbird then it is richly colored."

(d) The converse of "all hummingbirds are richly colored" is "all richly colored birds are hummingbirds."
True.

(e) The set of real numbers x such that $2 < x < 5$ is countable.
False : see Exercise 7.6.18.

(f) The set of all subsets of a finite set is countable.
True : The power set of a finite set is finite!

(g) $2^n = O(n^{\sqrt{n}})$.
False : can disprove by taking logarithms on both sides.

(h) n^n grows faster than $n!$ which in turn grows faster than 2^n.
True : as proved in class.

(i) Let $f(n) = 1 + 2 + 3 + \cdots + n$. Then $f(n) = o(n^2)$.
False : $f(n) = n(n+1)/2 = \Theta(n^2)$.

(j) Given any programming language, there exists an uncomputable function (by any program) in that language.
True : as mentioned in class, and also see the reading exercise assigned in class : Example 7.6.6.