Course: CS 1050C – QUIZ 3 (FALL '03)

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Time: 20 minutes **Total Points : 2+2+3+3+2+3 = 15 pts**.

Please write your name and explain all your answers.

Name: Prasad Tetali **Date:** Nov. 24th, 2003.

Throughout you may assume that a relation is a *binary* relation.

Problem 1. (2pts) Let $S = \{1, 2, \ldots, n\}$. Consider the relation $R = \{(1, 1), (2, 2), \ldots, (n, n)\}$ on S. Is R an equivalence relation? Explain your answer.

Soln. Yes, it is an equivalence relation: clearly reflexive; symmetry and transitivity are vacuously true.

Problem 2. (2pts) How many *symmetric* relations can one define on a set of n elements?

Soln. (similar problem done in class, with $n = 8$.) Since a relation on a set S is a subset of $S \times S$, if S has size n, then there are $n \times n = n^2$ possible pairs to choose from in forming a relation. If the relation has to be symmetric, then we only have $n^2 - \binom{n}{2}$ choices, since every time (x, y) is included in the relation (for $x \neq y$), then (y, x) has to be included as well, and there are precisely $\binom{n}{2}$ such pairs.

Thus the answer is $2^{n^2 - \binom{n}{2}}$, since we get to either include or not include for each of the available choices.

Problem 3. (3pts) Find the transitive closure of the following relation, illustrated using a directed graph:

Soln. The only pairs to be included to get the transitive closure are $(1, 3)$ and $(7, 9)$.

Problem 4. (3pts) Answer in True or False:

(a) Reflexivity and Transitivity imply Symmetry of a relation.

False, for example consider the relation $R = \{(1, 1), (1, 2), (2, 2)\}$ on the set $S = \{1, 2\}$. It is reflexive and transitive, but not symmetric.

(b) Given a partition of a set, one can always define an equivalence relation so that the equivalence classes correspond to the parts of the partition.

True, simply define the relation R by xRy if x and y are in the same part of the partition. Can easily be checked that this is well-defined and that it is an equivalence relation.

(c) It takes a polynomial in n steps to move n disks from one peg to another in the Tower of Hanoi problem.

False, since it takes $2^n - 1$ steps, which is exponential in n.

Problem 5. (2pts) A set of blocks contains blocks of heights 1,2, and 4 inches. Let t_n be the number of ways to construct a tower of height n inches using blocks from the set. Find a recurrence relation for t_n, for $n \geq 1$.

Soln. First partition the towers based on the first block: either the first block is of height 1 in. or 2 in. or 4 in. In each case, the problem reduces to building a (shorter) tower of height 1in. or 2 in. or 4 in. less, respectively. Thus we see that, for $n \geq 5$, we have

$$t_n = t_{n-1} + t_{n-2} + t_{n-4},$$

for the number of towers of height n.

The initial conditions can be worked out by explicitly writing down all possible solutions: $t_1 = 1$, $t_2 = 2$ (since we have (1,1) or (2)), $t_3 = 3$ (since we have (2,1) or (1,2) or (1,1,1)), and finally $t_4 = 6$.

Problem 6. (3pts) Solve for C_n in terms of n, given that $C_0 = 1$ and $C_1 = 2$, and for $n \geq 2$,

$$C_n + C_{n-1} - 6C_{n-2} = 0.$$

Soln. As outlined in class, first write down the characteristic equation, $x^2 + x - 6 = 0$, which has the roots $-3, 2$.

Since the roots are distinct, the general solution is $C_n = a(-3)^n + b(2)^n$.

Use the initial conditions to determine a and b:

$$C_0 = 1 = a + b,$$

$$C_1 = 2 = a(-3) + b(2),$$

imply that $a = 0$ and $b = 1$.

Thus the final solution is $C_n = 2^n$. $\square$