MATH 2601 - FoMP - Homework 4

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Due: **Wednesday, October 3, in class**

Problem 1. Recall that a tournament is an oriented complete graph and a Hamilton path is a directed path consisting of all the vertices in the graph. Compute the expected number of Hamilton paths in a *random* tournament on 3 players (vertices) in two ways:

i) Compute the expectation by writing it as a sum of 0-1 random variables and using the linearity of expectation, as done in class.

ii) Now compute the expectation as an average over all possible tournaments: First enumerate the 8 possible (labeled) tournaments on 3 vertices. Write the number of Hamilton paths in each of the tournaments. Take the average.

Problem 2. Determine $R(k, 2) = ?$ for $k \geq 2$. (Recall that you need to argue both the upper bound and the lower bound.)

Problem 3. Show that $R(4, 3) \leq 10$.
Hint: Given an arbitrary Red-Blue coloring of the edges of K_{10}, consider whether vertex v_1 has at least 6 red edges incident to it or not.

Problem 4. (I will illustrate the following with examples for $k = 2, 3$ in class). Let $k \geq 2$. Let Ω be a finite set of elements and let $S_1, S_2, \ldots, S_m \subseteq \Omega$ be m subsets, each of cardinality k. We say such a collection of sets admits a *proper Red-Blue coloring* if there exists an assignment of Red/Blue color to the elements of Ω so that none of the S_i is monochromatic. In other words :

$$\exists \chi : \Omega \to \{Red, Blue\} : \forall i \in \{1, 2, \ldots, m\},\ \exists x, y \in S_i,\ \text{with } \chi(x) = Red \text{ and } \chi(y) = Blue\,.$$

Show that whenever $m < 2^{n-1}$, such a collection always admits a proper Red-Blue coloring.
Hint: Consider a *random* Red-Blue coloring and estimate the expected number of monochromatic sets.

Additionally, the following problems from Hammack's book:

Ch.7: 6, 8, 18
Ch.10: 2, 10, 24

Note. For Problem 10.24 above, feel free to use the following identities, which are true for all integers k, n, with $1 \leq k \leq n$:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} = 2^n\,.$$