Math 2406 Homework 7

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(6.4.2) x is an eigenvector of T^2 belonging to λ^2.

$T(x) = \lambda x$

$T^2(x) = T(T(x)) = T(\lambda x) = \lambda T(x) = \lambda(\lambda x) = \lambda^2 x$ □

x is an eigenvector of T^n belonging to λ^n.

Induction hypothesis

For $n = k$, $T^n(x) = \lambda^n x$

Basis: $n = 1$

$T^1(x) = \lambda^1 x$ by definition

Step: $n = k + 1$

$$\begin{aligned} T^{k+1}(x) &= T(T^k(x)) \\ &= T(\lambda^k x) \text{ (by induction hypothesis)} \\ &= \lambda^k T(x) = \lambda^k(\lambda x) = \lambda^{k+1} x1 \end{aligned}$$ □

x is an eigenvector of $P(T)$ belonging to $P(\lambda)$.

Exercise 1 gives that if x is an eigenvector for T_1 and T_2, then x is an eigenvector for $aT_1 + bT_2$ belonging to $a\lambda_1 + b\lambda_2$.

This can be trivially extended to conclude that if x is an eigenvector for $T_1, T_2, \ldots, T_n$, then for some sequence of scalars $\{c_i\}_{i=1}^n$, x is an eigenvector for $\Sigma_{i=1}^n c_i T_i$ belonging to the eigenvalue $\Sigma_{i=1}^n c_i \lambda_i$.

Let $T_1, T_2, \ldots, T_n = T, T^2, \ldots, T^n$. This is possible because if x is an eigenvector of T, then x is also an eigenvector for T_k, with eigenvalue λ^k. The summation is now an n-degree polynomial in terms of T.

Let $P(x) = \sum_i c_i x^i$. Then x is an eigenvector of $\sum_i c_i T^i = P(T)$, belonging to the eigenvalue $\sum_i c_i \lambda^i = P(\lambda)$. □

Equivalently, Let $P(t) = \sum_i c_i t^i$, poly. in t.

Then $P(T) = \sum_i c_i T^i$, where $T^0 = I$.

Let x be the eigenvector : $Tx = \lambda x$.

Then $p(T)[x] = \sum_i c_i T^i [x] = \sum_i c_i \lambda^i x = (\sum_i c_i \lambda^i) x$

$= p(\lambda) \cdot x \Rightarrow x$: eigen vector of $p(T)$ with eigenvalue $p(\lambda)$. □

(6.4.6) Suppose, for contradition, T has two distinct eigenvalues λ and μ corresponding to nonzero eigenvectors x and y.

Choose $a \neq 0$ and $b \neq 0$ such that $ax + by$ is nonzero.

Every nonzero element of T is an eigenvector, so $ax + by$ is an eigenvector of T.

Exercise 5 tells us that $a = 0$ or $b = 0$ (contradiction). T does not have two distict eigenvalues.

Let λ be the single eigenvalue of T.

$\forall\ x,\ T(x) = \lambda x = \lambda I(x) \quad \Rightarrow \quad T = \lambda I$ $\square$

(6.4.7) Let k_p be the largest i such that the coefficient of t^i in $p(t)$ is nonzero. ✓

Looking for eigenvalues...

$T(p) = \lambda p$

$p(t+1) = \lambda p(t)$

$\sum_{i=0}^{k_p} c_i(t+1)^i = \lambda \sum_{i=0}^{k_p} c_i t^i$

Coefficient of t^{k_p}: $c_k = \lambda c_{k_p}$

If T has an eigenvalue, 1 is its only eigenvalue.

If p is an eigenfunction, then k_p is 0:

$\sum_{i=0}^{k_p} c_i(t+1)^i = \sum_{i=0}^{k_p} c_i t^i$

$\sum_{i=0}^{k_p} c_i \sum_{j=0}^{i} \binom{i}{j} t^{i-j} = \sum_{i=0}^{k_p} c_i t^i$

Coefficient of t^{k_p-1}: $(c_{k_p})(k_p) + c_{k_p-1} = c_{k_p-1}$

$c_{k_p} \neq 0$, so $k_p = 0$ (note: $c_{k_p} \neq 0$, by defn. of k_p.)

If p is an eigenfunction of T, then the degree of p is 0.

Any polynomial $p(t) = c$, where $c \in \mathbb{R}$, is an eigenfunction corresponding to eigenvalue 1: $T(p) = p(t+1) = c = (1)c = \lambda c = \lambda p(t)$ $\square$

(6.10.4) $\det(\lambda I - P_1) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda)(\lambda) - (-1)(-1) = \lambda^2 - 1$

$\det(\lambda I - P_2) = \begin{vmatrix} \lambda & i \\ -i & \lambda \end{vmatrix} = (\lambda)(\lambda) - (i)(-i) = \lambda^2 - 1$

$\det(\lambda I - P_3) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1) = \lambda^2 - 1$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues -1 and 1 if its characteristic polynomial is $\lambda^2 - 1$.

$(\lambda - a)(\lambda - d) - (b)(c) = \lambda^2 - 1$

$\lambda^2 - a\lambda - d\lambda + ad - bc = \lambda^2 - 1$

$\lambda(-a - d) + ad - bc = -1$

Solvable if $a = -d$:

$-a^2 - bc = -1$

$a^2 = 1 - bc$

(6.10.8) c. $0 = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix}$

$$= [(\lambda)(\lambda) - (-1)(-1)]^2$$
$$= (\lambda + 1)(\lambda + 1)(\lambda - 1)(\lambda - 1)$$

$\lambda = \pm 1$

d. $0 = \begin{vmatrix} \lambda & i & 0 & 0 \\ -i & \lambda & 0 & 0 \\ 0 & 0 & \lambda & i \\ 0 & 0 & -i & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & i \\ -i & \lambda \end{vmatrix} \begin{vmatrix} \lambda & i \\ -i & \lambda \end{vmatrix}$

$$= [(\lambda)(\lambda) - (i)(-i)]^2$$
$$= (\lambda + 1)(\lambda + 1)(\lambda - 1)(\lambda - 1)$$

$\lambda = \pm 1$

e. $0 = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda + 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 0 \\ 0 & 0 & 0 & \lambda + 1 \end{vmatrix}$

$$= (\lambda - 1)(\lambda + 1)(\lambda - 1)(\lambda + 1)$$

$\lambda = \pm 1$

(6.10.13) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - (-b)(-c) \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (\operatorname{tr} A)\,\lambda + (\det A) \end{aligned}$$

$p_B(\lambda) = \lambda^2 - (\operatorname{tr} B)\,\lambda + (\det B)$

If $\operatorname{tr} A = \operatorname{tr} B$ and $\det A = \det B$, then:

$p_B(\lambda) = \lambda^2 - (\operatorname{tr} A)\,\lambda + (\det A) = p_A(\lambda)$

Counterexample for $n = 3$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$\operatorname{tr} A = \operatorname{tr} B = 3$

$$\det B = 1 \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} + 0 = 1(-3) - 2(-2) = 1 = \det A$$

$$\begin{aligned} p_A = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} &= (\lambda - 1)^3 \\ &= \lambda^3 - 3\lambda^2 + 3\lambda - 1 \end{aligned}$$

$$\begin{aligned} p_B = \begin{vmatrix} \lambda - 1 & -2 & 0 \\ 0 & \lambda - 3 & -1 \\ -2 & 0 & \lambda + 1 \end{vmatrix} &= (\lambda - 1)(\lambda - 3)(\lambda + 1) + (-2)(-1)(-2) \\ &= (\lambda^2 - 1)(\lambda - 3) - 4 \\ &= \lambda^3 - 3\lambda^2 - \lambda - 1 \end{aligned}$$