Math 2406B (Fall 08) – TEST 1 (September 15, 2008)

Instructor : Prasad Tetali, office: Skiles 234, ph: 404-894-9238
Webpage: http://www.math.gatech.edu/∼tetali **email**: tetali@math.gatech.edu

Closed Book, Closed Notes; single 2-sided help sheet allowed
Please EXPLAIN All Answers. No Calculators Allowed

Time : 50 minutes Total Points : $10 + 10 + 10 = 30$

1. (5+5 points) Let $\mathcal{P} = \{p(x) :$ polynomial in $x\}$ denote the set of polynomials in x of arbitrary degree. Let $D : \mathcal{P} \to \mathcal{P}$ be the map that takes a polynomial $p(x) = \sum_{i=0}^{k} a_i x^i$ to the polynomial $D(p(x)) = \sum_{i=1}^{k} a_i\ i\ x^{i-1}$. (In other words, $D : \mathcal{P} \to \mathcal{P}$ is the differentiation operator from calculus.)

(a) Is D an injection?

(b) Is D a surjection (onto)?

Solution.

2. (5+5 points) Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbf{R}\}$. Define *vector addition* and *scalar multiplication* as follows. Determine in each case whether V is a linear space over $\mathbf{R}$? If not, show at least one axiom that does not hold:

(a) for $A = (a_1, a_2),\ B = (b_1, b_2)$, and $c \in \mathbf{R}$,

$$A + B = (a_1 + b_1, a_2 b_2), \qquad \text{and} \qquad cA = (ca_1, a_2)\,.$$

(b) for $A = (a_1, a_2),\ B = (b_1, b_2)$, and $c \in \mathbf{R}$,

$$A + B = (a_1 + b_1, a_2 + b_2), \qquad \text{and} \qquad cA = (ca_1, 0)\,.$$

Solution.

3. (2+2+6 points) Briefly explain answers, when appropriate.

(a) Define a *basis* of a linear space. When is a basis *orthogonal*?

Solution.

(b) Let V be a Euclidean space, with an inner product $(\cdot, \cdot)$. For $\mathbf{x}, \mathbf{y} \in V$, does $(\mathbf{x}, \mathbf{y}) = 0$ imply that $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$? Prove or give an example to illustrate the point.

Solution.

(c) In the linear space P_n of all polynomials of degree $\leq n$, define the inner product

$$(f, g) = \sum_{k=0}^{n} f(\frac{k}{n})\ g(\frac{k}{n})\,.$$

If $f(t) = t$, find all linear (that is, degree one) polynomials g orthogonal to f.

(If need be, you may use the fact that $\sum_{i=1}^{n} i = n(n+1)/2$ and that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.)

Solution.

Extra Space