MATH 3012 Applied Combinatorics (Fall'07) – Comments on HWs 2 and 3

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Time: 30 minutes

Homework 2

Exercise 1.4.4, **Exercise 1.4.10** are both straightforward, distribution of identical soup cans to distinct families type problems.

Exercise 1.4.12. The somewhat clever way is to introduce an auxiliary variable, x_6 and turn the *inequality* into an equality:

(a) It is equivalent to

$$x_1 + x_2 + \cdots + x_5 + \mathbf{x_6} \; = 40,$$

where $x_i \geq 0$, for $1 \leq i \leq 5$, and $x_6 \geq 1$.

(b) It is equivalent to counting the integer solutions to

$$(x_1 + 3) + (x_2 + 3) + \cdots + (x_5 + 3) < 40 + 3 + 3 + 3 + 3 + 3 = 55,$$

where $x_i + 3 \geq 0$. This in turn is equivalent to counting the integer solutions to

$$y_1 + y_2 + \cdots + y_5 + y_6 = 55,$$

where $y_i \geq 0$, for $1 \leq i \leq 5$, and $y_6 \geq 1$. Note that from a solution in y's, we get back the solution in x's by doing $x_i = y_i - 3$, for $1 \leq i \leq 5$.

Exercise 1.5.10. Done in class. For (b), using the swapping of U's and R's beyond the first place where the number of U's dominate the number of R's idea, one should get the answer: $\binom{m+n}{m} - \binom{m+n}{m+1}$. The first term counts the total number of sequences, and the second counts the invalid sequences obtained after the swapping – each such sequence will have precisely $m+1$ of one symbol and $n-1$ of the other, post swapping.

Exercise 4.1.2. Straightforward. Simply follow the standard three-step procedure of induction (as in the solution of Exercise 4.1.14 below.)

Exercise 4.1.4. Done in class. The idea was to consider the average of the 25 sums of consecutive triples, $s_1 = a_1 + a_2 + a_3$, $s_2 = a_2 + a_3 + a_4$, $\ldots$, $s_{25} = a_{25} + a_1 + a_2$, and to show that the average was 39. Hence there must be a consecutive triple whose sum s_i (for some i) is at least the average.

Exercise 4.1.14. (First of all notice that the statement is NOT true for $n = 3$. Secondly, the stmt makes sense for large n, since the $\log_2$ of the lhs is n whereas the log of $n!$ is roughly $n \log n$, which grows faster than n. But the point of the exercise is to verify *for every* n, beyond some small integer.)

We may prove the assertion by induction on $n > 3$. Base case: $n = 4$ is true, since $16 < 24$. Step 2 (Induction Hypothesis): Assume that it is true for SOME $n = k > 3$.

Step 3 (Induction Step): Need to prove for $k+1$: Note $2^{k+1} = 2(2^k) < 2(k!)$, by the induction hypothesis, valid as long as $k > 3$. Since $k > 3$, it is safe to say, $2 < k + 1$. Hence $2(k!) < (k+1)k! = (k+1)!$ completing the induction step.

Homework 3

Exercise 4.3.8. Done in class: the point is that each of the 12 numbers is divisible by 6, and so the sum of any subset of them will also be divisible by 6. But 500 is *not* divisible by 6. Hence Eleanor has not won.

Exercise 4.4.14. Also done in class: use the extended GCD algorithm to find X_0 and Y_0 so that $1 = gcd(33, 29) = 33X_0 + 29Y_0$. Then write $2490 = 33(2490X_0) + 29(2490Y_0)$. Then observe that adding and subtracting any multiple of 33 times 29 also gives other solutions (in integers) to the equation $2490 = 33X + 29Y$. So need to find an appropriate multiple so that X and Y are both positive integers: That is, find k so that

$$X = (2490X_0 - 29k) \text{ and } Y = (2490Y_0 + 33k)$$

are both positive integers. (Note that you may choose k to be positive or negative!)

Exercise 20 on Page 245. This is tricky, as you will see – such is the case unfortunately, whenever a solution involves the word "consider ..."

The present solution uses the Pigeon-Hole Principle (PHP) and elementary divisibility property, besides some cleverness. Recall that when we try to divide an integer by 5, the remainder is one of the five integers, 0, 1, 2, 3, or 4. Now **consider** the five sets, $\{a_1\}$, $\{a_1, a_2\}$, $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_3, a_4\}$, and $\{a_1, a_2, a_3, a_4, a_5\}$.

Let s_1, s_2, s_3, s_4, s_5 denote the sum of the elements in each of the above sets, respectively. If any of these five sums is divisible by 5, then we are done – we have found a subset S of the type that the exercise asks for. If not, then by the PHP, at least two of the sums must yield the *same* remainder when divided by 5. Suppose these are sets S_i and S_j. With foresight we chose the sets as nested – so we may subtract the smaller set (say S_i) from the larger, so that the set we seek is $S = S_j \setminus S_i$. Clearly, the corresponding sum will have the remainder 0, when divided by 5.