MATH 3012 (Fall'07) – Solutions to HW 4

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu

Homework 4

Exercise 5.1.10. The number of people who know the rumor is $1+2+2(3)+2(3)(5) = 39$. The number of calls is (one less) = 38, since everyone but the first person received a call.

Exercise 5.2.2. The set of reals can not be the domain of f, since $x = \sqrt{2}$ would imply, $f(x) = 1/0$, which is undefined. But f can be a function from the set of integers to the set of reals, without a problem.

Exercise 5.2.22. Done in class. In brief, part (c) has the answer $\binom{m+n-1}{m}$: the way to see this is that every monotone increasing function from a set X_m of size m to a set X_n of size n can be thought of as choosing m objects from a set of size n *with repetition*! This is the tricky part of this exercise. Once you see this, Chapter 1 tells us the answer, namely that this is the same as a soup can type problem.

What matters is *how many* of each distinct object from X_n are chosen, so that we have a total of m of them. Once we choose these, we order them and assign them to the elements of X_m in the obvious increasing way. Suppose $m = 7$ and $n = 5$, and suppose the seven objects we chose from $\{1, 2, 3, 4, 5\}$ with repetition are 1, 1, 1, 2, 2, 4, 4. Then the (unique) monotone increasing function corresponding to this choice is :

$$f(1) = 1,\ f(2) = 1,\ f(3) = 1,\ f(4) = 2,\ f(5) = 2,\ f(6) = 4,\ f(7) = 4\,.$$

For part (d), the answer is $\binom{3+4-1}{3} \times \binom{6+5-1}{6}$, since once $f(4) = 4$, we see that we need to map $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ (in a monotone way), and $\{5, 6, 7, 8, 9, 10\}$ to $\{4, 5, 6, 7, 8\}$ in a monotone way. And any choice of the top mapping can be put together with any choice of the bottom mapping to get the final mapping of all the elements. (That is why we multiply the two partial answers above.)

Exercise 5.2.26. (a) Only one function in S_1, namely $f(a) = f(b) = 1$, and $f(c) = 2$.

(b) There are $2 \times 2 = 4$ functions, since each of $f(a)$ and $f(b)$ can be 1 or 2.

(c) Similarly there are i^2 such functions.

(d) $|T_1| = \binom{n+1}{2}$, since any function here can be determined by two choices out of the $n+1$ elements: choose x, y so that $1 \le x < y \le n+1$, where $f(a) = f(b) = x$ and $f(c) = y$.

(e) Here we choose three elements $x < y < z$ so that $f(a) = x, f(b) = y, f(c) = z$.

(f) We can say S is the union of the S_i's where every pair of S_i's are disjoint, and T is the union of the T_j's with pairs of T_j being disjoint.

(g) From part (f), we have

$$|S| = \sum_i |S_i| = \sum_{i=1}^{n} i^2 = \sum_{j=1}^{3} |T_j| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

This gives the formula for the sum of the first n squares.

Exercise 5.3.2. To prove a function f is one-to-one, we need to show that $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Or equivalently that $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, FOR ALL pairs x_1, x_2 from the domain. See (b) for such a proof.

To show that f is NOT one-to-one, it suffices to produce SOME pair x_1, x_2 from the domain so that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. To *find* such a pair, we could proceed in a systematic way, rather than randomly guessing. See (e) for such a proof.

To prove a function f is onto, we need to show that FOR EVERY y from the codomain, there is SOME x from the domain so that $f(x) = y$.

To show that f is NOT onto, it suffices to produce SOME y from the codomain for which there is NO x in the domain so that $f(x) = y$. See (e) for such a proof.

(a) one-to-one and onto.

(b) one-to-one but not onto. The range consists of all the odd integers.

To show one-to-one: suppose $f(x_1) = f(x_2)$, meaning that $2x_1 - 3 = 2x_2 - 3$, then clearly $2x_1 = 2x_2$, and hence $x_1 = x_2$.

To show NOT onto: given y, we want x so that $f(x) = y$. This gives the equation, $2x - 3 = y$, which means, $x = (y + 3)/2$. But the domain is the set of integers (and not the reals), so if y is even, then x is NOT an integer, and hence there is no solution. So not onto. But for every odd integer y, there is an integer solution x. Hence the range is the set of all odd integers.

(c) one-to-one and onto.

(d) Not one-to-one, since $f(1) = f(-1)$. Also not onto, since the range is only the set of squares.

(e) Suppose $f(x_1) = f(x_2)$, meaning that $x_1^2 + x_1 = x_2^2 + x_2$. This gives, $(x_1 - x_2)(x_1 + x_2) = (x_2 - x_1)$. Assuming $x_1 \neq x_2$, we can cancel $(x_1 - x_2)$ from both sides, and obtain $(x_1 + x_2) = -1$. This certainly has solutions such as $x_1 = 0, x_2 = -1$ or $x_1 = 1, x_2 = -1$, thus the function is NOT one-to-one.

Similarly to prove that it is not onto, we want to show there is a y so that $x^2 + x = y$ has no solution in integer x. Choose $y = 1$, for example. Since x takes integer values, $x^2 + x$ is either 0 or $1^2 + 1 = 2$ or $2^2 + 2 = 6$ etc. So no x so that $f(x) = 1$. The range is $\{0, 2, 6, 12, \ldots\}$.

(f) one-to-one but not onto.

Exercise 5.3.4. (a) 6^4; $6!/2!$; 0. (b) 4^6; (4!)S(6,4); 0.

Exercise 5.3.6. (b) m^n counts the number of ways to distribute n distinct objects among m distinct containers.

The right hand side also counts the same: let us first decide how many containers we actually use, as in nonempty containers. There can be i nonempty, which can be chosen in $\binom{m}{i}$ ways; if the container were identical, we could distribute the n objects into the i containers in $S(n, i)$ ways. Since the containers are distinct (and not identical), we get $i!S(n, i)$ ways. Now we simply sum over i.

Exercise 5.3.10. (a) $4!S(7, 4)$; (b) When Container II contains only the blue ball, it is $(3!)S(6, 3)$; when Container II contains more than just the blue ball, it is $(4!)S(6, 4)$. So the answer is the sum of the two.

(c) $S(7, 4) + S(7, 3) + S(7, 2) + S(7, 1)$.

Exercise 5.3.18. Done in class.