MATH 3012 Applied Combinatorics (Fall'07) – TEST 1 (Solutions)

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Time: 1 hour 25 minutes **Total Score:** $5 + 10 + 10 + 5 + 10 = 40$ **pts**

1 (5pts). A license plate has 3 letters (from the English alphabet) followed by 3 digits from 0 through 9. How many license plates can be made? (Assume that letters *can not* be repeated, but digits can be repeated.)

Solution. Note that ordering matters, and that letters can not be reused.
So it is $26 \times 25 \times 24 \times 10 \times 10 \times 10$.

2 (10pts). If $n \in \mathbf{Z}^+$ and $n \geq 2$, prove that $2^n < \binom{2n}{n} < 4^n$.

Solution. It is simplest to prove by induction on $n \geq 2$.

Base Case : True, since $2^2 < \binom{4}{2} = 6 < 4^2$.
Induction Hypothesis. Assume that for some $n = k$, $k \geq 2$, the inequalities are true.

Induction Step. Need to prove the case of $n = k+1$, namely that $2^{k+1} < \binom{2(k+1)}{k+1} < 4^{k+1}$.

Observe that

$$\binom{2(k+1)}{k+1} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)(k+1)k!k!} = 2\frac{(2k+1)}{(k+1)}\binom{2k}{k} \qquad (*)$$

Hence we may say,

$$\begin{aligned}
2^{k+1} = 2\ 2^k &< 2\binom{2k}{k} && \text{by the induction hypothesis} \\
&\leq 2\frac{(2k+1)}{k+1}\binom{2k}{k} && \text{since } 1 \leq \tfrac{(2k+1)}{(k+1)} \\
&= \binom{(2k+1)}{k+1} && \text{using the observation (*) and} \\
& && \text{thus proving the first inequality in the induction step} \\
&= 2\frac{(2k+1)}{(k+1)}\binom{2k}{k} && \text{going back one step} \\
&< 4\ \ 4^k && \text{again, by the induction hypothesis, and since } \tfrac{(2k+1)}{k+1} < 2 \\
&= 4^{k+1} && \text{completing the second inequality in the induction step.}
\end{aligned}$$

3 (4+6pts). (a) There are n distinct men and M distinct women, assuming $M \geq n$. In how many ways can we pair them up *simultaneously* (say for ballroom dancing) into man-woman pairs?

Solution. There are M choices for the first man, $M-1$ for the second man, etc., giving the answer: $M(M-1)(M-2)\cdots(M-n+1)$.

Another way to arrive at this is by first choosing a subset of n women, and then (as in the sample quiz problem) observing that there are $n!$ ways to match up the n men and the chosen subset of n women. This results in $\binom{M}{n}n!$, which is the same as the above.

(b) There are $2m$ distinct tennis players, in how many ways can we pair them up *simultaneously* (say for simultaneous tennis matches)?

Solution. There are $(2m-1)$ ways to choose a partner for the first person, then $(2m-3)$ for the next person (since two were matched up), etc.

Another way to arrive at the same answer is by the following:

$$\frac{(2m)!}{m!\ 2 \times 2 \times \cdots \times 2},$$

where there are m 2's in the product in the denominator. Can you see why?

4 (5pts). Give a combinatorial proof for the following identity, for positive integers, k, m, n. Assume k is less than both m and n.

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

Solution. Imagine a set of m men and a set of n women, and say we want to choose a subset of k out of the total of $m+n$ of them. There are clearly $\binom{m+n}{k}$ ways to do this. A complicated way to count the same would be to focus on choosing i men and $k-i$ women for a fixed i (for $0 \leq i \leq k$), and then summing over i. The RHS of the equation corresponds to this count.

5 (10pts). Determine those values of $c \in \mathbf{Z}^+,\ \ 20 \le c \le 25$, for which the (Diophantine) equation $84x + 990y = c$ has no solution. Determine the solutions for the remaining values of c. (Recall that Diophantine simply means we are looking only for *integer* solutions in x and y.)

Solution (i). Recall that there is a solution if and only if gcd(84, 990) divides c. Since

$$\begin{aligned} 990 &= (11)84 + 66 \\ 84 &= (1)66 + 18 \\ 66 &= (3)18 + 12 \\ 18 &= (1)12 + 6 \\ 12 &= (2)6 + 0. \end{aligned}$$

Since the gcd is 6, there is a solution only for $c = 24$ when c is in the range $20 \le c \le 25$.

(ii). First rewrite the gcd 6 in terms of 990 and 84:

$$\begin{aligned} 6 = 18 - 12 &= 18 - [66 - (3)\ 18] = -66 + (4)18 \\ &= -66 + (4)\ [84 - 66] = (4)\ 84 - (5)\ 66 \\ &= (4)\ 84 - (5)[990 - (11)\ 84] \\ &= -(5)\ 990 + (59)\ 84. \\ &=: 990x_0 + 84y_0. \end{aligned}$$

Now 24 can be written in terms of 990 and 84:

$$\mathbf{24} = 990(4x_0) + 84(4y_0) = \mathbf{990} \times (-20) + \mathbf{84} \times 236.$$

Finally, to find *all* solutions, simply add and subtract any multiple of 990×84 to the above solution any number of times. Thus the solutions to $24 = 990X + 84Y$ can be expressed as: $X = (-20) + k \times 84$, and $Y = 236 - k \times 990$, for k: any integer.