MATH 3012 Applied Combinatorics (Fall'07) – Quiz 2

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Time: 40 minutes Score: 6 + 6 + 8 = 20 points

1. Let $\mathcal{R}$ denote the set of real numbers. Is the function $f : \mathcal{R} \to \mathcal{R}$ defined as $f(x) = 2x^2 - x$, onto? Is it one-to-one? Explain.

Solution. Given any $y \in \mathcal{R}$, we want an $x \in \mathcal{R}$ such that $2x^2 - x = y$. Choose $y = -1$, then we want a *real* solution to $2x^2 - x + 1 = 0$. But this quadratic equations has the solutions, $x = (1 \pm \sqrt{1-8})/4$, which are complex numbers. Thus the function is not onto.

More generally, for any $y \in \mathcal{R}$, the quadratic equation has the solutions, $x = (1 \pm \sqrt{1+8y})/4$, which are complex, if $1 + 8y < 0$. Thus the range of f is only those y which satisfy $1 + 8y \geq 0$, i.e., $y \geq -1/8$.

2. (a) If we have an injection (i.e., one-to-one function) f from a set A to a set B, what can we say about the relative sizes of A and B? What if there is a bijection instead?

Solution. If $f : A \to B$ one-to-one then $|A| \leq |B|$. If f is a bijection, then $|A| = |B|$.

(b) State the inclusion-exclusion principle for counting the size of the union of four sets.

Let A_1, A_2, A_3, A_4 be the four sets. Then

$$| \cup_{i=1}^4 A_i| = \sum_{i=1}^4 |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4| \,.$$

(c) How long a sequence of distinct numbers must we have, in order to guarantee that there is always either an increasing subsequence (of numbers) of length $n+1$ or a decreasing subsequence of length $n+1$?

Solution. As we saw in class, the Erdós-Szekeres theorem shows that any sequence of $n^2 + 1$ distinct numbers has the desired property.

3. If a fair die is rolled six times, what is the probability that the sum of the six rolls is 19? (Recall that to compute the probability, you need to find out the number of ways of getting a sum of 19 and divide by the total number of outcomes.)

Solution One. First of all the total number of outcomes is 6^6, since each roll results in one of six possible outcomes. Thus the probability is $NUM/6^6$, where N is the number of integer solutions to $x_1+x_2+\cdots+x_6 = 19$, with the restriction that $1 \le x_i \le 6$, corresponding to the outcome of the ith roll.

Now the problem may be solved using inclusion-exclusion. Let c_i be the property that $x_i \ge 7$. Then we are interested in counting the positive integer solutions to $N(\bar{c_1}\bar{c_2}\cdots\bar{c_6})$.

Note that giving 1 away to each of the x_i's, we reduce the total to 13. Then the total number of solutions with the property $x_i \ge 1$ is $N = \binom{13+6-1}{6-1} = \binom{18}{5}$. The number of solutions where $c_i \ge 7$ is $N(c_i) = \binom{(13-6)+6-1}{6-1} = \binom{12}{5}$. The number of solutions where $c_i, c_j \ge 7$ is $N(c_ic_j) = \binom{(13-6-6)+6-1}{6-1} = 6$. And $N(c_ic_jc_k) = 0$, since we can not have three rolls be bigger than seven and still have the sum come to 19.

Hence

$$NUM = \binom{18}{5} - 6\binom{12}{5} + \binom{6}{2}6.$$

Solution Two. This is only a slightly different way to calculate NUM – basically to go ahead and shift the variables so that they range between 0 and 5: We may rewrite the problem as finding the no. of integer solutions to

$$x_1 + x_2 + \cdots + x_6 = 13, \quad \text{with } 0 \le x_i \le 5.$$

Let c_i be the property that $x_i \ge 6$. Then we are interested in counting the positive integer solutions to $N(\bar{c_1}\bar{c_2}\cdots\bar{c_6})$.

Then the total number of solutions with the property $x_i \ge 0$ is $N = \binom{13+6-1}{6-1} = \binom{18}{5}$. The number of solutions where $c_i \ge 6$ is $N(c_i) = \binom{(13-6)+6-1}{6-1} = \binom{12}{5}$. The number of solutions where $c_i, c_j \ge 6$ is $N(c_ic_j) = \binom{(13-6-6)+6-1}{6-1} = 6$. And $N(c_ic_jc_k) = 0$, since we can not have three rolls be bigger than seven and still have the sum come to 19.

Hence

$$NUM = \binom{18}{5} - 6\binom{12}{5} + \binom{6}{2}6,$$

and the probability is $NUM/6^6$.