MATH 4022 (Intro to Graph Theory) Homework 2

Due: Wednesday, Sept. 28, 2016 (in class)

- **Instructor:** Prasad Tetali, tetali-at-math-dot-gatech-dot-edu; 404-894-9238 (o)
- **Class Location and Time:** Skiles 268, MW 3:05–4:25pm
- **Office hours:** Skiles 118B, Monday 4:30-5:30, Tuesday, Friday 2:00–3:00pm

Most of the following problems are from the textbook by Doug West.

1. Let G_n be the graph whose vertices are the permutations of $\{1, 2, \dots, n\}$, with two permutations adjacent if they differ by interchanging *any pair* of entries. Prove that G_n is bipartite.

Hint. An *inversion* in a permutation $a = (a_1, a_2, \dots, a_n)$ is a pair a_i and a_j such that $i < j$ but $a_i > a_j$. Consider the bipartition of permutations using the parity (even or odd) of the number of inversions in the permutation.

2. Determine which of the following sequences are graphic - that is, form the degree sequence of a simple graph: i) (5,5,4,4,2,2,1,1) ; ii) (5,5,5,3,2,2,1,1) ; iii) (5,5,5,4,2,1,1,1).

3. Show using induction $n \geq 2$ that every tournament on n vertices has a (directed) Hamilton path. [If a tournament on $(n-1)$ vertices has such a path, it is not difficult to see that it can always be extended to work for the tournament on n vertices.]

4. (A bit tricky) Use the Prüfer correspondence to show that the number of spanning trees of K_n which don't use the edge $\{n-1, n\}$ is $(n-2)n^{n-3}$.

Hint. Show that a spanning tree doesn't contain the edge $\{n-1, n\}$ if and only if the Prüfer sequence doesn't end with the label $n-1$ or n. (First observe that n never gets deleted during the construction of the Prüfer sequence.)

5. (Warning: annoyingly, we switch the ordering of the degrees in this exercise.) Show that a simple graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ is *connected* if $d_k \geq k$ for all k such that $k \leq n - 1 - d_n$.

Hint. To prove by contradiction, consider the size of the component containing the vertex with degree d_n, and that of another component (not connected to this).

$*\quad *\quad *\quad *$ Reading Homework $*\quad *\quad *\quad *$

I. Nice application of Eulerian digraphs; you can read about it in Section 1.4 of Doug West's book or in other sources: the notion of **de Bruijn sequences/cycles**.
There are 2^n binary strings of length n. A de Bruijn sequence is a cyclic arrangement of the 2^n binary strings such that the 2^n strings of n consecutive digits are all distinct! For $n = 4$, for example, (0000111101100101) works.

For each $n \geq 2$, the solution comes by constructing a suitable digraph (with equal indegree and outdegree) on the $(n-1)$-tuples and following an Eulerian circuit in such a graph.

II. Useful to know, although doesn't produce a graph, unlike Havel-Hakimi; the Wiki has proof sketches and references for further reading:

Theorem (Erdös-Gallai). A sequence of non-negative integers $d_1 \geq d_2 \geq \cdots d_n$ is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)\,,$$

holds for every k, $1 \leq k \leq n$.