MATH 4022 (Intro to Graph Theory) Homework 3

Due: Wednesday, October 19, 2016 (in class)

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- **Office hours:** Skiles 118B, Monday 4:30-5:30, Tuesday, Friday 2:00–3:00pm

Most of the following problems are from the textbook by Doug West.

1. (Problem 3.1.1) Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting a vertex cover of the same size.

2. (Problem 3.1.9) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

3. Show that Tutte's theorem implies Hall's theorem.

Hint. Given a bipartite graph $G = (X \cup Y, E)$, assume for simplicity that G has an even number of vertices. Build a new graph H using G: keep the edges of G and make Y into a complete graph in H. Show that G has a matching saturating X if and only if H has a perfect matching.

4. Show that if every row and column of a matrix A (consisting of non-negative real entries) sums to t (for $t > 0$), then it can be written as a linear combination of permutation matrices with non-negative coefficients summing to t.

Hint. As explained in class, prove by induction on the number of nonzero entries in the matrix. As also mentioned in class, this is a bit more general (and more convenient to prove) than the fact that every doubly stochastic matrix can be written as a convex combination of permutation matrices.

5. (Problem 3.1.32) In a bipartite graph $G = (X \cup Y, E)$, the *deficiency* of a set S is $\text{def}(S) = |S| - |N(S)|$, with the understanding that def(empty set) is zero. Prove that the size of the largest matching $\alpha'(G)$ equals $|X| - \max_{S \subset X} \text{def}(S)$.

Hint. Let $d = \max_{S \subset X} \text{def}(S)$. To prove G has a matching as large as $|X| - d$, form a new graph G' by adding d new vertices to Y and connecting each of the new vertices to all vertices in X.

6. (Problem 3.1.28) Exhibit a perfect matching in the graph shown or give a short proof that it has none.

Hint. Find a vertex cover of size 20.