MATH 4022 (Intro to Graph Theory) Quiz 1

Sept. 14, 2016 (in class, closed book, closed notes); duration: 40 minutes

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Please provide a proof whenever appropriate; do not simply write an answer without due explanation or justification. Total score: 30 points.

1. (5 pts) State the matrix-tree theorem.

Answer. Given a graph G, let $Q = D - A$, where D is the diagonal matrix of degrees and A is the adjacency matrix of G. So the diagonal entry Q_{ii} is the degree of the vertex v_i and $Q_{ij} = -1$ or 0, depending on whether vertices v_i and v_j are adjacent.

Let Q_i^* be the matrix obtained by removing the ith row and ith column of Q. The matrix-tree theorem asserts that the determinant of Q_i^* is the number of the spanning trees of G (irrespective of which i we choose).

A more general version removes the i row and the jth column and asserts that the determinant of the matrix thus obtained equals $(-1)^{i+j}$ times the number of spanning trees.

2. (5 pts) What is the most number of edges a bipartite graph can have on n vertices? (No need to prove it.) Is there a graph achieving the bound?

Answer. At most $\lfloor n/2 \rfloor \lceil n/2 \rceil$, and is achieved by a bipartite graph with the n vertices partitioned into the corresponding (nearly equal) numbers of vertices. (This is Mantel's theorem.)

3. (1+1+3 = 5 pts) For $d \geq 1$, the discrete *d-cube* is a graph on 2^d vertices corresponding to all binary sequences of length d; two vertices in the d-cube are adjacent if they differ in precisely one position. (That is, $u, v \in \{0,1\}^d$ are adjacent iff $u_i \neq v_i$, for some i, and $u_j = v_j$, for $j \neq i$.)

(a) Draw the 3-cube.

Answer. Easy enough. One way is to draw a small square inside another square and join the corresponding corners with 4 matching edges.

(b) Is the 3-cube bipartite?

Answer. Yes, see below.

(c) Is the d-cube bipartite (for $d \geq 1$)?

Answer. OK, this was not so great as a Quiz question, perhaps better suited for a Test.

Each vertex can be thought of as a binary vector of length d. *Consider* separting the vertices into the sets V_1 and V_2 based on whether the vertex has odd number of 1's (in its vector) or an even number of 1's.

This is a valid bipartition of the vertices with no edge between any two in the same V_i, because that would mean that two vertices which differ by a single 1 have the same parity – even or odd number of 1's – which is impossible.

4. (2+3 = 5 pts) (a) Define the Ramsey number $R(s,t)$ in your own words. (Assume $s, t \geq 2$ are integers.)

Answer. $R(s,t)$ is the smallest n so that no matter how the edges of K_n are Red/Blue colored, there exists a K_s all of whose edges are Red or a K_t all of whose edges are Blue.

(b) Show that $R(4, 2) = 4$.

Answer. $R(4, 2) > 3$, since one can color the edges of K_3 all Red.
$R(4, 2) \leq 4$, since if no edge is colored Blue, then all edges must be Red giving a Red K_4. Else there is a Blue K_2.

5. (5 pts) Compute the Prüfer sequence corresponding to the following spanning tree on 7 vertices: the spanning tree has the edges $(1, 2), (3, 2), (2, 4), (4, 7), (4, 5)$, and $(5, 6)$.

Answer. The order of the removal of leaves is $1, 3, 2, 6$ and 5. This gives rise to the sequence: $(2, 2, 4, 5, 4)$.

6. (2+3 =5 pts) (a) What is the sum of the degrees of any tree on n vertices?

Answer. The sum of the degrees equals twice the number of edges, so the answer is $2(n - 1)$, since every tree has $n - 1$ edges.

(b) Show that every tree has at least two leaves. *Hint: You may use (a).*

Answer. Note that every vertex must have degree at least 1, otherwise, the graph is not connected. If there was at most one vertex with degree 1, then the sum of the degrees would be at least $2 \times (n - 1) + 1$, which is more than what Part (a) dictates. Hence there must be at least two.