MATH 4022 Intro to Graph Theory (Fall'07) – TEST 2 (Solutions)

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Time: 1 hour 25 minutes **Total Score:** 40 **pts**

Solve the first three in class. Bring the solution to the FOURTH to class on Wednesday. The fifth problem is OPTIONAL (for extra credit); may bring to class on Wednesday.

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1 (10pts). Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z-path through y. (Recall that an x, z-path is a path between vertices x and z.)

Solution. Suppose G is 2-connected. Then given x, y, z, we may use the fan lemma with vertex y and set $U = \{x, z\}$. The resulting fan may be viewed as an x, z path which goes through y.

In the other direction, clearly the graph must be connected; moreover, the removal of any single vertex x still leaves the graph $H := G \setminus \{x\}$ connected : Suppose not. Then consider two vertices y, z from two distinct connected components of H. By the hypothesis in the problem, G contains an x, z path which goes through y. In particular this yields a y, z path which does not involve x (since a path can not repeat vertices), and hence this remains a path y, z path in H. This gives the desired contradiction to the supposition that H is disconnected. We may conclude that G is 2-connected.

2 (5+5pts). (a) Draw a connected graph containing a perfect matching, with vertex connectivity 1, and with maximum degree 3.

(b) [Independent of part (a)]. Construct a graph with vertex connectivity 4, edge connectivity 5 and minimum degree 6.

Solution. (a) Join an edge to a triangle. (For a construction on more vertices, one can repeat the above – by a chain of such triangles, using an edge as a "link," each time.)

(b) This is a special case of a general construction discussed in class: One may start with two disjoint K_7's; choose four vertices from one K_7 and four vertices from the other, and join the two subsets using *five* edges, so that the edges span all four vertices from each set. Each to check that this satisfies the requirements.

3 (10pts). Prove that a 3-regular graph has a perfect matching, if it decomposes into copies of P_4 (which stands for the 4-vertex path.)
(Here "decomposes" refers to the edges of the graph being partitioned into.)

Solution. Let n be the number of vertices and m the number of edges in G. Then $m = 3n/2$, since G is 3-regular. Thus a P_4 decomposition of G consists of $n/2$ edge-disjoint copies, since each copy accounts for 3 edges. Note that the P_4's may share *vertices*, except that the middle two vertices of a P_4 *can not* be shared by other P_4's, since that will create a degree-4 vertex in G!

Thus the set of middle edges gathered from each P_4 constitutes a perfect matching.

Incidentally, the converse is also true; a proof will be covered in class.

4 (10pts). Find the vertex and edge connectivity of the Petersen graph $\mathbf{P_5}$.

Solution. It is easy to see that $\mathbf{P_5}$ is 3-regular and has no cut edge. Hence $2 \leq \kappa(G) = \kappa'(G) \leq 3$. The graph can be viewed as a disjoint union of two five cycles joined together by a perfect matching. So the removal of any two edges (say) can be seen to leave it connected, by considering a couple of cases. This shows that $\kappa'(G) = 3$.

5 (EXTRA CREDIT). Prove the a graph with at least four vertices is 2-connected if and only if for every pair X, Y of disjoint vertex subsets with $|X|, |Y| \geq 2$, there exist two completely disjoint paths P_1, P_2 in G such that each has an endpoint in X and an endpoint in Y and no internal vertex in X or Y.

Solution. For both sides, we may use the fact that G is 2-connected if and only if every pair of *edges* in G lie on a cycle:

Suppose G is 2-connected. Given disjoint sets X and Y (with size at least two), consider an edge in each of X and Y – include a new edge, if X or Y (or both) lack one, which still retains the 2-connected property. Now the (new) graph will have a cycle containing the two edges. This gives rise to two disjoint paths between X and Y with the above property, by simply taking the last entry/exit vertices from X and Y in the cycle.

For the other direction, suppose G has the disjoint path property for every pair of disjoint subsets of size 2. Given any two disjoint edges, define X and Y to be the endpoints of the two edges to obtain two disjoint paths between X and Y. These yield a cycle containing the edges in X and Y. Since this is true for every pair of edges, G is 2-connected. (We may also prove this via a proof by contradiction: if G has connectivity less than 2, then either G is disconnected or has a cut vertex. In either case, we get a contradiction to the above disjoint path property, by choosing X and Y appropriately.)

Note that we implicitly used the fact that G has at least four vertices in the above proofs.