MATH 4022 Intro to Graph Theory (Fall'07) – TEST 3

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu

Time: 1 hour 25 minutes **Total Score:** 30 **pts**

Name : Prasad Tetali

1 (10pts). (a) What is the largest that the chromatic number and the list-chromatic number can be over all planar graphs?

Solution. The chromatic number is at most 4 and the list-chromatic number is at most 5 (as discussed in class).

(b) The chromatic number of a graph is equal to the largest clique size in the graph. True of False.

Solution. False. The odd cycle is an example.

(c) If the capacities in a network are all integers, is there necessarily a maximum flow which assigns integer-valued flows on every edge?

Solution. Yes, as discussed in class (See e.g., Corollary 4.3.12, Page 181.)

(d) What is the chromatic polynomial of a tree on n vertices?

Solution. As discussed in class, $\chi(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}$, for $\lambda \geq 1$, and $n \geq 1$.

(e) What is the chromatic number of the Petersen graph $\mathbf{P_5}$? (Show a proper coloring with that many colors.)

Solution. Easy to color with at most three colors. This is tight, since an odd cycle requires three colors, and the Petersen graph contains an odd cycle.

2 (10 pts). Find the maximum flow in the following network. Justify why that is the maximum flow.

Solution. The max-flow is 6. One way is to send 3 units out of each of the two edges out of s and saturating the two horizontal edges, and then receiving 3 units each from the two edges incident to t.

The optimality follows from the fact that there is a cut with capacity equalling 6 – for example, choose s and its two neighbors to define the cut.

3 (5+5pts). (a) Prove that the chromatic polynomial of the cycle C_n (for $n \geq 3$) is

$$\chi(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)\,.$$

(You may use k in place of λ, as in the book, if you prefer.)

Solution. (a) We may use induction on $n \geq 3$ and the recursive formula (discussed in class, see also Theorem 5.3.6) for the chromatic polynomial, using deletion and contraction of any edge of the graph: for G a simple graph, and e any edge, we have

$$\chi(G, \lambda) = \chi(G \setminus e, \lambda) - \chi(G \cdot e, \lambda).$$

Note that when we delete an edge, we get a path (which is also a tree), and so we may use the formula for $\chi(T_n, \lambda)$; and when we contract an edge, we get C_{n-1}, which can be handled using the induction hypothesis. Here is the math:

$$\begin{aligned}\chi(T_n, \lambda) - \chi(C_{n-1}, \lambda) &= \lambda(\lambda - 1)^{n-1} - (\lambda - 1)^{n-1} - (-1)^{n-1}(\lambda - 1)\\ &= (\lambda - 1)^n + (-1)^n(\lambda - 1) = \chi(G, \lambda).\end{aligned}$$

(b) [independent of Part (a)]. Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$. (Hint: The chromatic number of a bipartite graph is at most 2.)

Solution. Remove any odd cycle from G. This destroys *all* odd cycles in G, since every odd cycle intersects this cycle! The new graph thus obtained can be 2-colored, since it is bipartite. Now we may color the vertices of the original odd cycle using 3 more colors, which we can choose to be completely new colors. Thus 5 colors are sufficient.

(Technically speaking, we need not remove any cycle, but instead claim that G minus any odd cycle can be 2-colored, and hence G can be colored with at most $2 + 3$ colors.) $\square$