## Course: Math 6221 - FINAL EXAM FALL '05

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Office Hours: Mon. Tue. 11-12, Thurs. 2-3pm
Solve any six (out of the seven) problems. Due: Thursday 10am, Dec. 15th, 2005.

Problem 1. Let $X_{1}$ and $X_{2}$ be independent and identically distributed, each being +1 or -1 with equal probability. For $n \geq 3$, let

$$
X_{n}=X_{n-1}+X_{n-2}-E\left[X_{n-2} \mid X_{n-1}\right] .
$$

(a) Is $E\left[X_{n} \mid X_{n-1}\right]=X_{n-1}$, for all $n \geq 3$ ?
(b) Is $E\left[X_{n} \mid X_{n-1}, X_{n-2}, \ldots, X_{1}\right]=X_{n-1}$, for $n=3,4$ ?

Problem 2. Find $E[Y \mid X]$ when

$$
f_{X, Y}(x, y)=\left\{\begin{array}{cc}
4 y(x-y) e^{-(x+y)} & \text { for } 0 \leq y \leq x<\infty  \tag{1}\\
0 & \text { otherwise }
\end{array}\right.
$$

Problem 3. Suppose that 100 balls are to be randomly distributed among 20 urns. Let $X$ denote the number of urns that contain at least five balls. Derive an upper bound for $\operatorname{Pr}\{X \geq 15\}$.

Problem 4. If $\left\{X_{n}\right\}$ and $\left\{Y_{n}\right\}, n \geq 0$, are independent martingales, is $\left\{Z_{n}\right\}, n \geq 0$ a martingale when
(a) $Z_{n}=X_{n}+Y_{n}$ ?
(b) $Z_{n}=X_{n} Y_{n}$ ?

Give examples in each case showing that the result can be false, if the independence assumption is removed.

Problem 5. Let $\left\{X_{n}: n \geq 1\right\}$ be a sequence of i.i.d. r.v.s, each taking +1 or -1 with equal probability. Use the three series theorem (from your previous homework problem) to show that the series $\sum_{r=1}^{n} X_{r} / r$ converges a.s. as $n \rightarrow \infty$.

Problem 6. A gambler playing roulette makes a series of one dollar bets. He has respective probabilities $9 / 19$ and 10/19 of winning and losing each bet. The gambler decides to quit playing as soon as his winnings reach 25 dollars or his net losses reach 10 dollars.
(a) Find the probability that when he quits playing he will have won 25 dollars.
(b) Find his expected loss.

Problem 7. (a) An urn contains $b$ blue and $r$ red balls. Balls are removed uniformly at random until the first blue ball is drawn. Determine the expected number drawn.
(b) The balls are replaced and then removed uniformly at random until all the remaining balls are of the same color. Find the expected number remaining in the urn.

