Course: Math 6221 – Homework 2 (Fall 2005)

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Office Hours: **Mon. Tue. 11-12, Thurs. 2-3pm**

Due: Tuesday, Sept. 20th

Problem 1 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For measurable subsets A and B of Ω, let $S(A, B)$ denote the symmetric difference of A and B. Show that, if $\mu(S(A, B)) = 0$, then, for every nonnegative measurable function f,

$$\int_A f\, d\mu = \int_B f\, d\mu.$$

Solve the next two problems WITHOUT using any convergence theorems – monotone convergence, Fatou's lemma , or Dominated convergence.

Problem 2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f and g nonnegative measurable functions. Show that, if g is simple, then, for all $E \in \mathcal{F}$,

$$\int_E (f+g)\, d\mu = \int_E f\, d\mu + \int_E g\, d\mu.$$

Problem 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let f be a bounded nonnegative function and let s_n be the sequence of simple functions contructed in Theorem 6 (of the handout from Adams-Guillemin book). Show that for $E \in \mathcal{F}$,

$$\int_E s_n\, d\mu \to \int_E f\, d\mu.$$

(*Hint: The sequence s_n converges uniformly to f. Moreover,*

$$\int_E (f - s_n)\, d\mu + \int_E s_n\, d\mu = \int_E f\, d\mu$$

by Problem 2.)

Problem 4. For $n = 1, 3, 5, \ldots$ let f_n be the characteristic function of the interval $(0, 1/2)$, and for $n = 2, 4, 6, \ldots$ let f_n be the characteristic function of the interval $(1/2, 1)$. Compare $\int \liminf f_n d\mu_L$ and $\liminf \int f_n d\mu_L$, where μ_L is the Lebesgue measure on $[0, 1]$.

Problem 5. Let F be a distribution function and r a positive integer. Show that the following are distribution functions:

1. $F(x)^r$,
2. $F(x) + (1 - F(x)) \log(1 - F(x))$,
3. $(F(x) - 1)e + \exp(1 - F(x))$.

Problem 6. Which of the following are density functions? Find c and the corresponding distribution function F for those that are.

1. $f(x) = \{\, cx^{-d}, x > 1$
   0, otherwise.
2. $f(x) = ce^x(1 + e^x)^{-2}, x \in \mathbb{R}$.

Problem 7. Express the distribution functions of

$$X^+ = \max\{0, X\}, \quad X^- = -\min\{0, X\}, \quad |X| = X^+ + X^-, \quad -X,$$

in terms of the distribution function F of the random variable X.