Course: Math 6221 Fall'05 – Homework 4

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Office Hours: **Mon. 11-12am, Tue., Thur. 2-3pm**

Due: Tuesday, Nov. 1st

Problem 1. Let $X_1, X_2, \ldots$ be i.i.d. random variables. Show that

$$\mathbb{P}[|X_n| \geq n \text{ i.o.}] = 0,$$

if and only if $\mathbb{E}[|X_1|] < \infty$.

Problem 2. Let $\mathcal{T}$ be the *tail* σ-algebra generated by the r.v.s $X_1, X_2, \ldots$ Let $S_n = X_1 + X_2 + \cdots + X_n, n \geq 1$.

(a) Show that the following events are in $\mathcal{T}$:
$A_1 = \{\limsup_n X_n < \infty\}$.
$A_2 = \{\limsup \frac{S_n}{n} < c\}$.

(b) Show that the following events are NOT tail events:
$B_1 = \{\lim_n S_n$ exists and is less than $c\}$
$B_2 = \{S_n = 0 \text{ i.o.}\}$, where, say, X_i is ± 1 with equal probability, as in the one-dimensional simple random walk.

Problem 3. Let X_n be independent random variables. Show that $\sum_{n=1}^{\infty} X_n$ converges almost surely if, for some $a > 0$, the following three series all converge.

1. $\sum_n \mathbb{P}(|X_n| > a)$,
2. $\sum_n \operatorname{Var}(X_n \mathbf{1}_{\{|X_n| \leq a\}})$,
3. $\sum_n \mathbb{E}(X_n \mathbf{1}_{\{|X_n| \leq a\}})$.

(The converse is also true, but harder to prove.)
Problem 4. Define the *total variation distance* $d_{tv}(X, Y)$ between two random variables X and Y to be

$$d_{tv}(X, Y) = \sup |\mathbb{E}(u(X)) - \mathbb{E}(u(Y))|,$$

where the supremum is over all (measurable) functions $u : \mathbb{R} \to \mathbb{R}$ such that $\|u\|_\infty = 1$. (Recall $\|u\| = \sup_x |u(x)|$.)

(a) If X and Y are continuous with respective density functions f and g, show that

$$d_{tv}(X, Y) = \int_{-\infty}^{\infty} |f(x) - g(x)| \, dx = 2 \sup_{A \subseteq \mathbb{R}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

(b) Show that $d_{tv}(X_n, X) \to 0$ implies that $X_n \to X$ in distribution, but the converse is false.
Problem 5. Let $X_1, X_2, \ldots$ be independent random variables with common density function

$$f(x) = \{ 0 \text{ if } |x| \leq 2, \frac{c}{x^2 \log |x|} \text{ if } |x| > 2,$$

where c is a constant. Show that the X_i have no mean, but $(1/n) \sum_{i=1}^{n} X_i$ goes to 0 in probability. Show that the convergence does not take place almost surely.