Course: Math 6221 Fall'05 – Homework 5

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu
Office Hours: **Mon. Tue. 11-12am, Thur. 2-3pm**

Due: Thursday, Dec. 8th, 2005

(**Note: Take-home final will be posted on Dec. 8th.**)

Problem 1. Let h be a function such that if the vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ differ in at most one coordinate (that is, for some k, $x_i = y_i$ for all $i \neq k$) then $|h(X) - h(Y)| \leq 1$. Let $X_1, \dots, X_n$ be independent random variables. Then, with $\mathbf{X} = (X_1, \dots, X_n)$, we have for $a > 0$ that

(i) $\Pr\{h(\mathbf{X}) - E[h(\mathbf{X}] \geq a\} \leq e^{-a^2/2n}$,
(ii) $\Pr\{h(\mathbf{X}) - E[h(\mathbf{X}] \leq -a\} \leq e^{-a^2/2n}$,

Problem 2. If T_1 and T_2 are stopping times with respect to the filtration $\mathcal{F}$, show that $T_1 + T_2$ and $\max\{T_1, T_2\}$ are stopping times also.

Problem 3. Let $X_1, X_2, \dots$ be a sequence of non-negative independent r.v.s and let $N(t) = \max\{n : X_1 + X_2 + \cdots + X_n \leq t\}$. Show that $N(t) + 1$ is a stopping time with respect to a suitable filtration to be specified.

Problem 4. If X_i, $i \geq 1$ are i.i.d. with $E[|X_1|] < \infty$ and if T is a stopping time for $X_1, X_2, \dots$ with $E[T] < \infty$, then

$$E[\sum_{i=1}^{T} X_i] = E[T]E[X_1].$$

Hint: Use the OST: if $\{Z_n\}$, $n \geq 1$, is a martingale sequence, and T: a stopping time with $E[T] < \infty$, and there is a $K < \infty$ such that

$$E[|Z_n - Z_{n-1}| \mid Z_1, \dots, Z_{n-1}] < K,$$

then $E[Z_T] = E[Z_1]$.

Problem 5. Suppose that T is a stopping time such that for some $N \in \mathbb{N}$ and some $\epsilon > 0$, we have, for every $n \in \mathbb{N}$:

$$\Pr(T \leq n + N | \mathcal{F}_n) > \epsilon, \quad \text{a.s.}$$

Then $E(T) < \infty$.

Hint: Prove by induction that for $k = 1, 2, 3, \dots$

$$\Pr(T > kN) \leq (1 - \epsilon)^k.$$

Problem 6. Prove that if $\mathcal{C}$ and $\mathcal{D}$ are UI classes of r.v.s, and if we define

$$\mathcal{C} + \mathcal{D} := \{X + Y : X \in \mathcal{C}, Y \in \mathcal{D}\},$$

then $\mathcal{C} + \mathcal{D}$ is UI.