# Course: Math 6221 – Test 1, FALL '05

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**Office Hours**: **Mon. 11-12, Tue -Thurs. 2-3pm**

**Due: Solve the first FOUR problems in class and turn in the rest by MONDAY NOON.**

Total : $10 + 10 + 10 + 10 + 10 + 10 = 60$ points

**Problem 1**. For $X$ : a real-valued r.v. on $(\Omega, \mathcal{F}, P)$, let $F(X) := \{X^{-1}(B) : \text{ Borel set } B\}$.

(a) Show that $F(X)$ is a $\sigma$-algebra over $\Omega$.
(b) Let $g$ be a Borel-measurable function, and let $Y = g(X)$. Show that $F(Y) \subseteq F(X)$. Conclude that if r.v.s $X_1$ and $X_2$ are independent then $g(X_1)$ and $g(X_2)$ are independent.

**Problem 2**. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f$ be a bounded nonnegative measurable function. Show that

$$\int_\Omega f d\mu = \lim_{n\to\infty} \sum_{k=1}^{n2^n} \Big(\frac{k-1}{2^n}\Big)\mu\Big(\Big\{x \in \Omega; \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n}\Big\}\Big).$$

(This formula is Lebesgue's original definition of the Lebesgue integral.)

**Problem 3**. We say a r.v. $X$ is *symmetric* if $X$ and $-X$ have the same distribution. Show that $X$ is symmetric if and only if its characteristic function $\Phi(X)$ is real-valued.

**Problem 4**. Let $f_n : \mathbb{R} \to \mathbb{R}$ be $1/n$ times the indicator function of the interval $(0, n)$. Compute $\int \lim f_n \, d\mu_L$ and $\lim \int f_n \, d\mu_L$, where $\mu_L$ denotes the Lebesgue measure. Why isn't this a counterexample to the (Lebesgue) dominated convergence theorem?

**Problem 5**. A real number $m$ is called a *median* of the distribution function $F$ whenever $\lim_{y\uparrow m} F(y) \le 1/2 \le F(m)$. Show that every distribution function has at least one median, and that the set of medians of $F$ is a closed interval of $\mathbb{R}$.

*Hint*: Show that $a := \sup\{x \in \mathbb{R} : \text{F(x)} < 1/2\}$ and $b := \sup\{x \in \mathbb{R} : \text{F(x)} \le 1/2\}$ are both medians.

**Problem 6**. Let $X$ have the Normal distribution, $N(m, \sigma^2)$, with mean $m$ and variance $\sigma^2$. Let $Y = e^X$. Find the mean $(E(Y))$ and variance of $Y$.