Math 7018 (Spring '06) Homework 2

(Due: Friday, Feb. 17th)

1. Let X be the number of cycles (not necessarily induced) in $G(n,p)$. Show that if $p = o(n^{-1})$ then $\Pr[X > 0] = o(1)$. Let $c \in (0,1)$ be fixed. Show that with $p = c/n$, $E[X] = O(1)$, where the implicit constant in $O(\cdot)$ only depends on $c > 0$.

2. (Exercise 2.7.7) Let $\mathcal{F}$ be a family of subsets of $N = \{1, 2, \ldots, n\}$, and suppose that there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

$$X = \mid \{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in \mathcal{F}\} \mid .$$

By considering the expectation of X prove that $\mid \mathcal{F} \mid \leq \binom{n}{\lfloor n/2 \rfloor}$.

3. (Exercise 2.7.9) Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

4. (Exercise 3.7.2) Prove that the Ramsey number $R(4,k)$ satisfies

$$R(4,k) \geq \Omega\Big((k/\log k)^2)\Big),$$

and optimize the constant factor in front of the leading term.

5. (Exercise 3.7.3) Prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$.

6. (Exercise 3.7.4) (* This is a starred problem in the book, so might be hard!) Show that there is a finite n_0 such that any directed graph on $n > n_0$ vertices in which each outdegree is at least $log_2 n - \frac{1}{10}\log_2\log_2 n$ contains an even simple directed cycle.

Optional problems. (No need to turn in)

7. Let $k_a(n)$ be the minimal k such that all $n \times n$ 0-1 matrices containing more than k ones contain an $a \times a$ submatrix consisting entirely of ones (the "all-ones" submatrix). Prove that for every constant $a \geq 2$ there is an $\epsilon > 0$ such that $k_a(n) \geq \epsilon n^{2-2/a}$.

8. For a graph G and a subset T (for *TOP*) of the vertices, call an edge *crossing* if it has exactly one vertex in T and let $N(G;T)$ denote the number of such edges.

Prove for an appropriate $\alpha = \alpha(n) = o(n^2)$ that there exists a graph G of $2n$ vertices so that for every set T of n vertices,

$$\mid N(G;T) - \frac{n^2}{2} \mid < \alpha.$$