Math 7018 (Spring '06) Homework 4

(Due: Wednesday, April 5th)

1. (a) Prove the following version of the local lemma. Let $[n]$ denote the set {1,2, ..., n}. Consider as usual a set of n events, $A_1, A_2, \ldots, A_n$, such that each A_i is mutually independent of events A_j, for $j \in [n] - D_i$ (and $j \neq i$), for some $D_i \subset [n]$. If for each $i \in \{1, 2, \ldots, n\}$,

   (i) $Pr(A_i) \leq 1/8$ and

   (ii) $\sum_{j \in D_i} Pr(A_j) \leq 1/4$.

   then with positive probability, none of the events happen.

   (b) For $\beta \geq 1$, a *proper* coloring of the vertices of a graph is called a β-capped proper coloring, if for each vertex v and color c, the number of times c appears in the neighborhood of v is at most β. If G has maximum degree $\Delta \geq \beta^\beta$ then G has a β-capped proper vertex coloring using at most $16\Delta^{1+1/\beta}$ colors.

2. Let $f(k)$ be the least n so that if the subsets S of $\{1, 2, \ldots, n\}$ of size k are two-colored then there exists a set $T \subset \{1, 2, \ldots, n\}$ of size $k+1$, all of whose k-element subsets are the same color. Use both the basic probabilistic method and the Lovasz local lemma to find lower bounds on $f(k)$ and compare them asymptotically.

3. (Exercise 5.8.3). Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

4. (Exercise 5.8.1). Prove that for every integer $d > 1$ there is a finite $c(d)$ such that the edges of any bipartitie graph with maximum degree d in which every cycle has at least $c(d)$ edges can be colored by $d + 1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle.

Optional Problems.

These can be done using just the basic probabilistic method.
1. (a) Show that there exists an $n \times n$ matrix (for all n) with entries from $\{+1, -1\}$ whose determinant is at least $\sqrt{n!}$.

(b) What is the determinant of a Hadamard matrix of size $n \times n$? [Recall that a *Hadamard matrix* is a square matrix with entries $+1, -1$, and with the property that the rows vectors are mutually orthogonal (and hence also the column vectors).]

2. Show that any 3-satisfiable formula has an assignment which satisfies at least 2/3 of the clauses.