Math 7018 (Spring '06)

(Due: Wednesday, March 1st)

1. (a) Let $A \subset \{1, 2, \ldots, 4n\}$, $|A| = n$. Let T be uniformly selected from among the subsets of $\{1, 2, \ldots, 4n\}$ containing exactly $2n$ elements. Set $B = \{1, 2, \ldots, 4n\} - T$. Find a precise expression, in terms of binomial coefficients for

$$\Pr[A \subset T \text{ or } A \subset B].$$

Using Stirling's formula, find an asymptotic value for this probability.

(b) Using the above result prove a theorem of the following form for α as big as possible. Let $A_1, A_2, \ldots, A_m \subset \{1, 2, \ldots, 4n\}$, $|A_i| = n$, with $m < \alpha$. Then there exists a two-coloring of $\{1, 2, \ldots, 4n\}$ so that none of the A_i are monochromatic.

Hint. Take a random *equicoloring.*

2. Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i^2 = 1$, if $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ is a $\{-1, +1\}$ random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$\Pr[|\sum_i \epsilon_i a_i \leq 1|] \geq c.$$

3. An (n, k, l) *covering* is a family $\mathcal{F}$ of k-subsets of an n-element set such that every l-subset is contained in at least one of $A \in \mathcal{F}$. Let $M(n, k, l)$ denote the minimal cardinality of such a design.

(a) Observe that $M(n, k, l) \geq \binom{n}{l}/\binom{k}{l}$.

(b) Show that

$$M(n, k, l) \leq 2\Big[\binom{n}{l}/\binom{k}{l}\Big]\Big(1 + \log\binom{k}{l}\Big)$$

Hint. Use the alteration method. Choose a random k-uniform family $\mathcal{F}$, where each k-set is chosen with probability p independently. Argue that with positive probability $|\mathcal{F}| \leq 2p\binom{n}{k}$ AND fewer than $2\binom{n}{l}e^{-x}$ l-sets are not covered by $\mathcal{F}$, for an appropriate x.

(c) Can you get rid of the factor 2 in the r.h.s. of (b)?

4. Let $G(n, p)$ denote the standard random graph on n vertices.

(a) Prove that if $np \to 0$ as $n \to \infty$, then $\Pr[G(n, p) \text{ is a forest}] \to 1$. (A forest is a graph without cycles.)

(b) Prove that $p = (\log n)/n$ is a threshold probability for the disappearance of isolated vertices.

Remark. Parts (a) and (b) of this problem are not necessarily related to each other.